9 The equations of two straight lines l_1 and l_2 are

$$l_1: \quad \mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + a\mathbf{k}) \quad \text{and} \quad l_2: \quad \mathbf{r} = -\mathbf{i} - \mathbf{j} - \mathbf{k} + \mu(3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}),$$

where a is a constant.

The lines l_1 and l_2 are perpendicular.

(a) Show that $a = 4$. [1]

$l_1: \underline{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ a \end{pmatrix}$ $\quad l_2: \underline{r} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$

If perpendicular, the scalar product of the direction vectors $= 0$:

$$\begin{pmatrix} 2 \\ -1 \\ a \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} = 0$$

$2 \times 3 + -1 \times -2 + a \times -2 = 0$

$6 + 2 - 2a = 0$

$8 - 2a = 0$

$2a = 8$

$\underline{a = 4}$ QED

The lines l_1 and l_2 also intersect.

(b) Find the position vector of the point of intersection. [4]

$$\begin{pmatrix} 1 + 2\lambda \\ -2 - \lambda \\ 3 + 4\lambda \end{pmatrix} = \begin{pmatrix} -1 + 3\mu \\ -1 - 2\mu \\ -1 - 2\mu \end{pmatrix}$$

$\cancel{i}$: $1 + 2\lambda = -1 + 3\mu$

$2 = 3\mu - 2\lambda$ ①

$\cancel{j}$: $-2 - \lambda = -1 - 2\mu$

$-1 = -2\mu + \lambda$ $\times 2$

$-2 = -4\mu + 2\lambda$ ②

①+②: $0 = -\mu$

$\underline{\mu = 0}$

Sub. into l_2: $\underline{r} = \begin{pmatrix} -1 + 0 \\ -1 - 0 \\ -1 - 0 \end{pmatrix} = \underline{\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}}$

The point A has position vector $-5\mathbf{i}+\mathbf{j}-9\mathbf{k}$.

(c) Show that A lies on l_1. [2]

$$\begin{pmatrix}-5\\1\\-9\end{pmatrix} = \begin{pmatrix}1+2\lambda\\-2-\lambda\\3+4\lambda\end{pmatrix}$$

$\underline{\underline{i}}$: $-5 = 1+2\lambda$
$-6 = 2\lambda$
$\underline{\lambda = -3}$

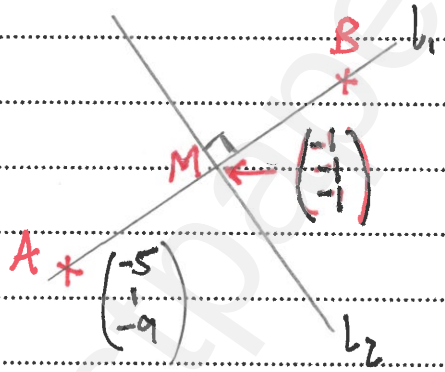
$\underline{\underline{j}}$: $1 = -2-\lambda$
$3 = -\lambda$
$\underline{\lambda = -3}$

$\underline{\underline{k}}$: $-9 = 3+4\lambda$
$-12 = 4\lambda$
$\underline{\lambda = -3}$

$\lambda = -3$ for all three components, therefore A lies on l_1.

The point B is the image of A after a reflection in the line l_2.

(d) Find the position vector of B. [2]



$= \vec{AM}$

$\underline{\vec{OB} = \vec{OM} + \vec{MB}}$

$\vec{OM} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$

$\vec{AM} = \underline{m} - \underline{a}$

$= \begin{pmatrix}-1\\-1\\-1\end{pmatrix} - \begin{pmatrix}-5\\1\\-9\end{pmatrix}$

$= \begin{pmatrix}4\\-2\\8\end{pmatrix}$

$\vec{OB} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix} + \begin{pmatrix}4\\-2\\8\end{pmatrix} = \underline{\begin{pmatrix}3\\-3\\7\end{pmatrix}}$

8 Two lines have equations $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

(a) Show that the lines are skew. [5]

① Show they're not parallel:

If parallel, $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = t \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

i // $2 = -t$, $t = -2$

j // $3 = -2t$, $t = -3/2$

k // $-1 = t$

No common solution for t, so not parallel.

② Show they don't intersect:

$$\begin{pmatrix} -1+2\lambda \\ 3+3\lambda \\ -4-\lambda \end{pmatrix} = \begin{pmatrix} 2-\mu \\ -3-2\mu \\ -1+\mu \end{pmatrix}$$

i // $-1+2\lambda = 2-\mu$

$-3+2\lambda = -\mu$

$\mu = 3-2\lambda$ ①

j // $3+3\lambda = -3-2\mu$

$3\lambda = -6-2\mu$ ②

Sub. ① into ②:

$3\lambda = -6-2(3-2\lambda)$

$3\lambda = -6-6+4\lambda$

$-\lambda = -12$

$\underline{\lambda = 12}$

Sub into ①:

$\mu = 3-2\times 12$

$\underline{\mu = -21}$

See if these values work for k:

k // $-4-\lambda = -1+\mu$

$-4-12 = -1-21$

$-16 \neq -22$ → don't intersect ∴ Skew

(b) Find the obtuse angle between the directions of the two lines. [3]

$$\begin{pmatrix}2\\3\\-1\end{pmatrix} \cdot \begin{pmatrix}-1\\-2\\1\end{pmatrix} = \sqrt{2^2+3^2+(-1)^2}\sqrt{(-1)^2+(-2)^2+1^2}\cos\theta$$

direction vector of l_1 direction vector of l_2.

$$2\times -1 + 3\times -2 + -1\times 1 = \sqrt{14}\times\sqrt{6}\times\cos\theta$$

$$-2 + -6 + -1 = \sqrt{84}\cos\theta$$

$$-9 = \sqrt{84}\cos\theta$$

$$\cos\theta = \frac{-9}{\sqrt{84}}$$

$$\theta = 169.1^\circ$$

6 The lines l and m have vector equations

$$l: \quad \mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{k}) \quad \text{and} \quad m: \quad \mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + 5\mathbf{k}).$$

Lines l and m intersect at the point P.

(a) State the coordinates of P. [1]

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

$\mathbf{i}: \; 2 - \lambda = 2 + 2\mu$ $\qquad$ $\mathbf{j}: \; 1 = 1 - \mu$

$2\mu + \lambda = 0$ ① $\qquad$ $\mu = 0$ ②

② → ①: $\lambda = 0$

$P = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \rightarrow$ coordinates: $(2, 1, -3)$ (It's actually pretty obvious this is the case if you look at the equations carefully)

(b) Find the exact value of the cosine of the acute angle between l and m. [3]

Scalar product between direction vectors:

$$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \sqrt{(-1)^2 + 2^2} \times \sqrt{2^2 + (-1)^2 + 5^2} \cos\theta$$

$$-2 + 0 + 10 = \sqrt{5} \times \sqrt{30} \cos\theta$$

$$\cos\theta = \frac{8}{\sqrt{5} \times \sqrt{30}}$$

$$= \frac{8}{\sqrt{150}}$$

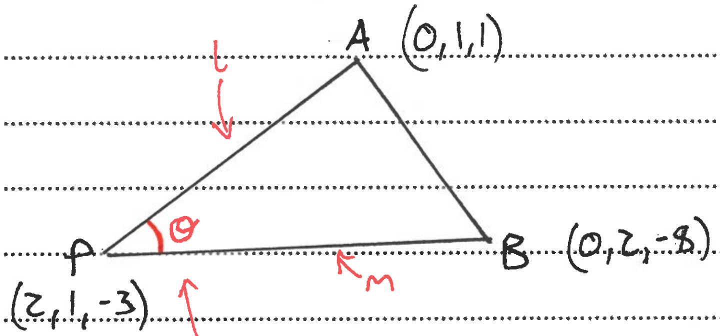
$$= \frac{8}{5\sqrt{6}}$$

or $= \dfrac{4\sqrt{6}}{15}$

(You can check it's acute by doing $\cos^{-1}\left(\frac{4\sqrt{6}}{15}\right)$)

(c) The point A on line l has coordinates $(0, 1, 1)$. The point B on line m has coordinates $(0, 2, -8)$.

Find the exact area of triangle APB. [3]



θ is the same angle as in part (b)

$|PA| = \sqrt{(0-2)^2 + (1-1)^2 + (1--3)^2}$
$= \sqrt{2^2 + 4^2}$
$= \sqrt{20}$

$|PB| = \sqrt{(0-2)^2 + (2-1)^2 + (-8--3)^2}$
$= \sqrt{2^2 + 1^2 + 5^2}$
$= \sqrt{30}$

$\text{Area} = \frac{1}{2} ab\sin C$

$= \frac{1}{2} \times \sqrt{20} \times \sqrt{30} \times \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\cos\theta = \frac{4\sqrt{6}}{15}$ from (b)

$= \frac{1}{2} \times \sqrt{20} \times \sqrt{30} \times \sqrt{\frac{43}{75}}$

$\sin^2\theta + \left(\frac{4\sqrt{6}}{15}\right)^2 = 1$

$= \frac{1}{2} \times \sqrt{\frac{25800}{75}}$

$\sin^2\theta + \frac{32}{75} = 1$

$\sin^2\theta = \frac{43}{75}$

$= \frac{1}{2} \times \sqrt{344}$

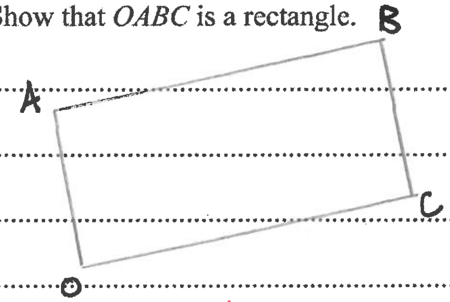
$\sin\theta = \sqrt{\frac{43}{75}}$

$= \frac{1}{2} \times 2\sqrt{86} = \underline{\sqrt{86}}$

9 Relative to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 8\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}.$$

(a) Show that $OABC$ is a rectangle. [4]



① Show that AOC is a right angle by doing scalar product with $\vec{OA}$ and $\vec{OC}$:

$\vec{OA} \cdot \vec{OC} \; (= |\vec{OA}||\vec{OC}|\cos\theta)$

$\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} = 5\times 3 + -2\times 4 + 1\times -7$

$= 15 - 8 - 7$

$= \underline{0}$ so $\vec{OA}$ and $\vec{OC}$ are perpendicular.

② Show that $\vec{OC}$ and $\vec{AB}$ are parallel and same length:

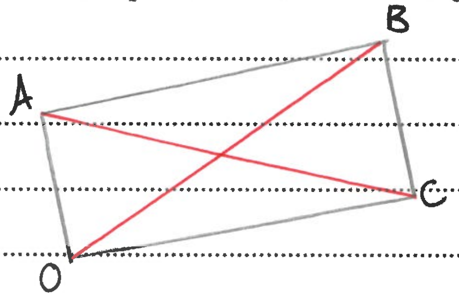
$\vec{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}$

$\vec{AB} = \vec{OC} = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}$ so the two sides are parallel and equal in length, so OABC is a rectangle.

(b) Use a scalar product to find the acute angle between the diagonals of *OABC*. [4]



diagonals are $\vec{OB}$ and $\vec{AC}$

$\vec{AC} = \underline{c} - \underline{a}$

$= \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 6 \\ -8 \end{pmatrix}$

$\vec{OB} \cdot \vec{AC} = |\vec{OB}||\vec{AC}| \cos\theta$

$\begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 6 \\ -8 \end{pmatrix} = \sqrt{8^2 + 2^2 + (-6)^2}\sqrt{(-2)^2 + 6^2 + (-8)^2} \cos\theta$

$-16 + 12 + 48 = \sqrt{104}\sqrt{104} \cos\theta$

$44 = 104 \cos\theta$

$\cos\theta = \frac{44}{104}$

$\theta = 65.0^\circ$

9 The position vector of point A relative to the origin O is $\overrightarrow{OA} = 8\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$.
The line l passes through A and is parallel to the vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

(a) State a vector equation for l. [2]

$$\underline{r} = \begin{pmatrix} 8 \\ -5 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

point on line

direction vector of line

(b) The position vector of point B relative to the origin O is $\overrightarrow{OB} = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}$, where t is a constant. The line l also passes through B.

Find the value of t. [3]

$$\begin{pmatrix} -t \\ 4t \\ 3t \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$\underline{i}$: $-t = 8 + 2\lambda$ ①

$\underline{j}$: $4t = -5 + \lambda$ $\times 2$

$8t = -10 + 2\lambda$ ②

② - ①: $9t = -18$

$\underline{\underline{t = -2}}$

Optional: Check by finding λ:

→ ①: $-(-2) = 8 + 2\lambda$

$2 = 8 + 2\lambda$

$-6 = 2\lambda$

$\lambda = -3$

and check in $\underline{k}$:

$3(-2) = 6 + 4(-3)$

$-6 = 6 + -12$

$-6 = -6$ ✓

DO NOT WRITE IN THIS MARGIN

(c) The line m has vector equation $\mathbf{r} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(a\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. The acute angle between the directions of l and m is θ, where $\cos\theta = \frac{1}{\sqrt{6}}$.

Find the possible values of a. [5]

$$\begin{pmatrix} a \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \sqrt{a^2 + (-1)^2 + 3^2} \times \sqrt{2^2 + 1^2 + 4^2} \cos\theta$$

direction vector of m direction vector of l $= \frac{1}{\sqrt{6}}$

$$2a - 1 + 12 = \sqrt{a^2 + 1 + 9} \times \sqrt{4 + 1 + 16} \times \frac{1}{\sqrt{6}}$$

$$2a + 11 = \sqrt{a^2 + 10} \times \sqrt{21} \times \frac{1}{\sqrt{6}}$$

$$2a + 11 = \sqrt{(a^2 + 10) \times 21 \times \frac{1}{6}}$$

Square both sides

$$(2a + 11)^2 = \frac{21}{6}(a^2 + 10)$$

$$4a^2 + 44a + 121 = \frac{7}{2}(a^2 + 10)$$

$\times 2$ $\times 2$

$$8a^2 + 88a + 242 = 7a^2 + 70$$

$$a^2 + 88a + 172 = 0$$

$$(a + 86)(a + 2) = 0$$

$a = -86$ or $a = -2$

10 The equations of two straight lines are

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + 2a\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} + a\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}),$$

where a is a constant.

(a) Given that the acute angle between the directions of these lines is $\frac{1}{4}\pi$, find the possible values of a. [6]

Scalar product with direction vectors:

$$\begin{pmatrix}3\\4\\a\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\2\end{pmatrix} = \sqrt{3^2+4^2+a^2} \times \sqrt{(-1)^2+2^2+2^2} \cos\theta$$

$$3\times -1 + 4\times 2 + 2a = \sqrt{25+a^2} \times \sqrt{9} \cos\theta$$

$$5 + 2a = 3\sqrt{25+a^2} \cos\theta$$

* Although the acute angle between the lines is $\frac{\pi}{4}$, the scalar product could give the answer $\frac{3\pi}{4}$. This doesn't matter because $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and we are going to square this:

$$5 + 2a = 3\sqrt{25+a^2} \times \left(\pm\frac{\sqrt{2}}{2}\right)$$

$$(5+2a)^2 = \left(3\sqrt{25+a^2} \times \left(\pm\frac{\sqrt{2}}{2}\right)\right)^2$$

$$25 + 20a + 4a^2 = 9(25+a^2)\left(\frac{2}{4}\right)$$

$$25 + 20a + 4a^2 = (225 + 9a^2) \times \frac{1}{2}$$

$$50 + 40a + 8a^2 = 225 + 9a^2$$

$$a^2 - 40a + 175 = 0$$

$$(a-5)(a-35) = 0$$

$$\underline{a = 5} \quad \text{or} \quad \underline{a = 35}$$

(b) Given instead that the lines intersect, find the value of a and the position vector of the point of intersection. [5]

$$\begin{pmatrix} 1 + 3\lambda \\ 1 + 4\lambda \\ 2a + a\lambda \end{pmatrix} = \begin{pmatrix} -3 - \mu \\ -1 + 2\mu \\ 4 + 2\mu \end{pmatrix}$$

$\underline{i}$: $1 + 3\lambda = -3 - \mu$ $\quad$ $\underline{j}$: $1 + 4\lambda = -1 + 2\mu$

$3\lambda + \mu = -4$ ① $\quad$ $4\lambda - 2\mu = -2$ $\div 2$

$2\lambda - \mu = -1$ ②

①+②: $5\lambda = -5$

$\lambda = -1$

$\rightarrow$ ①: $3(-1) + \mu = -4$

$\mu = -1$

Sub. $\mu = 5$ to find point of intersection:

$$\underline{r} = \begin{pmatrix} -3 - -1 \\ -1 + -2 \\ 4 + -2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}}}$$

$\underline{k}$: $2a + a\lambda = 4 + 2\mu$

$\lambda = -1, \mu = -1$:

$2a - a = 4 - 2$

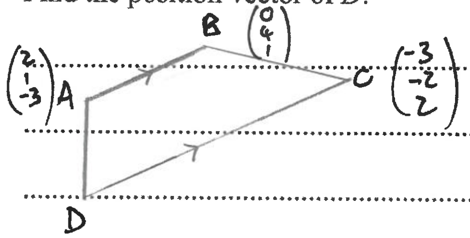
$\underline{\underline{a = 2}}$

9 With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}.$$

(a) The point D is such that $ABCD$ is a trapezium with $\overrightarrow{DC} = 3\overrightarrow{AB}$.

Find the position vector of D. [2]



$\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$

$\overrightarrow{DC} = 3\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \\ 12 \end{pmatrix}$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$= \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ -9 \\ -12 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 \\ -11 \\ -10 \end{pmatrix}}}$

(b) The diagonals of the trapezium intersect at the point P.

Find the position vector of P. [5]

Eqⁿ of line through A and C:

$\overrightarrow{AC} = \underline{c} - \underline{a}$

$= \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -5 \\ -3 \\ 5 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -3 \\ 5 \end{pmatrix}$

point on line (A) direction vector $\overrightarrow{AC}$

Eq^n of line through B and D:

$\vec{BD} = \underline{d} - \underline{b}$

$= \begin{pmatrix} 3 \\ -11 \\ -10 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -15 \\ -11 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -15 \\ -11 \end{pmatrix}$

Solve simultaneously: $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -15 \\ -11 \end{pmatrix}$

$\underline{i}: \; 2 - 5\lambda = 3\mu \;\; \times 5 \qquad \underline{j}: \; 1 - 3\lambda = 4 - 15\mu$ (2)

$10 - 25\lambda = 15\mu$ (1)

(1) + (2): $11 - 28\lambda = 4$

$28\lambda = 7$

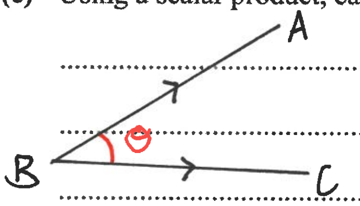
$\lambda = \frac{1}{4}$

Sub into eq^n of line:

$\underline{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} -5 \\ -3 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -5/4 \\ -3/4 \\ 5/4 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \\ -7/4 \end{pmatrix}$

(c) Using a scalar product, calculate angle ABC. [4]



$\vec{BA} = \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} \qquad \vec{BC} = \underline{c} - \underline{b}$

$= \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 1 \end{pmatrix}$

$\vec{BA} \cdot \vec{BC} = |\vec{BA}||\vec{BC}| \cos\theta$

$\begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -6 \\ 1 \end{pmatrix} = \sqrt{2^2 + (-3)^2 + (-4)^2} \times \sqrt{(-3)^2 + (-6)^2 + 1^2} \cos\theta$

$-6 + 18 - 4 = \sqrt{29}\sqrt{46}\cos\theta$

$8 = \sqrt{29} \times \sqrt{46} \cos\theta$

$\cos\theta = \dfrac{8}{\sqrt{29} \times \sqrt{46}}$

$\theta = 77.3^\circ$

8 The points A, B and C have position vectors $\overrightarrow{OA} = -2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, $\overrightarrow{OB} = 5\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OC} = 8\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, where O is the origin. The line l_1 passes through B and C.

(a) Find a vector equation for l_1. [3]

direction vector: $\overrightarrow{BC} = \underline{c} - \underline{b}$

$$= \begin{pmatrix} 8 \\ 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

$$l_1: \underline{r} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

point on line (A) Other solutions are possible.

The line l_2 has equation $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + 4\mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.

(b) Find the coordinates of the point of intersection of l_1 and l_2. [4]

$$\begin{pmatrix} 5 + 3\lambda \\ 2 + 3\lambda \\ -3\lambda \end{pmatrix} = \begin{pmatrix} -2 + 3\mu \\ 1 + \mu \\ 4 - 2\mu \end{pmatrix}$$

$\underline{i}$: $5 + 3\lambda = -2 + 3\mu$ $\quad$ $\underline{j}$: $2 + 3\lambda = 1 + \mu$

$7 = 3\mu - 3\lambda$ (1) $\quad$ $1 = \mu - 3\lambda$ (2)

(1) − (2): $6 = 2\mu$

$\mu = 3$

Sub. into l_2: $\underline{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$$= \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 9 \\ 3 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix}$$

coordinates: $\underline{(7, 4, -2)}$

(c) The point D on l_2 is such that $AB = BD$.

Find the position vector of D. ← this means $|AB| = |BD|$ **[5]**

$|AB|$:

$$|AB| = \sqrt{(5--2)^2 + (2-1)^2 + (-4)^2}$$
$$= \sqrt{49 + 1 + 16}$$
$$= \sqrt{66}$$

D lies on l_2, so use equation of line to represent $\underline{d}$:

$$\underline{d} = \begin{pmatrix} -2+3\mu \\ 1+\mu \\ 4-2\mu \end{pmatrix}$$

$$\overrightarrow{BD} = \underline{d} - \underline{b} = \begin{pmatrix} -2+3\mu \\ 1+\mu \\ 4-2\mu \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -7+3\mu \\ -1+\mu \\ 4-2\mu \end{pmatrix}$$

$$|\overrightarrow{BD}| = \sqrt{(-7+3\mu)^2 + (-1+\mu)^2 + (4-2\mu)^2}$$
$$= \sqrt{49 - 42\mu + 9\mu^2 + 1 - 2\mu + \mu^2 + 16 - 16\mu + 4\mu^2}$$
$$= \sqrt{14\mu^2 - 60\mu + 66}$$

$|BD| = \sqrt{66}$:

$$\sqrt{14\mu^2 - 60\mu + 66} = \sqrt{66}$$
$$14\mu^2 - 60\mu + 66 = 66$$
$$14\mu^2 - 60\mu = 0$$
$$7\mu^2 - 30\mu = 0$$
$$\mu(7\mu - 30) = 0$$
$$\mu = 0 \quad \text{or} \quad \mu = \frac{30}{7} ✓$$

× ← this gives point A

sub. into l_2:

$$\underline{d} = \begin{pmatrix} -2+\frac{90}{7} \\ 1+30/7 \\ 4-60/7 \end{pmatrix} = \begin{pmatrix} \frac{76}{7} \\ 37/7 \\ -32/7 \end{pmatrix}$$