

2 Find the exact solution of the equation

$$\cos \frac{1}{6}\pi + \tan 2x + \frac{\sqrt{3}}{2} = 0 \text{ for } -\frac{1}{4}\pi < x < \frac{1}{4}\pi.$$

[2]

$$\tan 2x = -\frac{\sqrt{3}}{2} - \cos\left(\frac{\pi}{6}\right)$$

$$-\frac{\pi}{4} < x < \frac{\pi}{4}$$

$$\tan 2x = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$$\frac{\pi}{2} < 2x < \frac{3\pi}{2}$$

$$\tan 2x = -\sqrt{3}$$

$$2x = -\frac{\pi}{3}$$

← only solution in range

$$x = -\frac{\pi}{6}$$

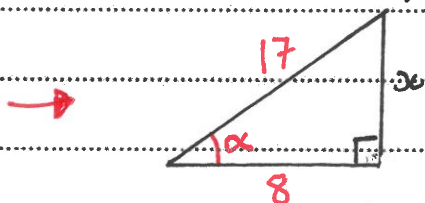


- 6 It is given that $\alpha = \cos^{-1}\left(\frac{8}{17}\right)$.

Find, without using the trigonometric functions on your calculator, the exact value of $\frac{1}{\sin \alpha} + \frac{1}{\tan \alpha}$. [5]

$$\alpha = \cos^{-1}\left(\frac{8}{17}\right)$$

$$\cos \alpha = \frac{8}{17}$$

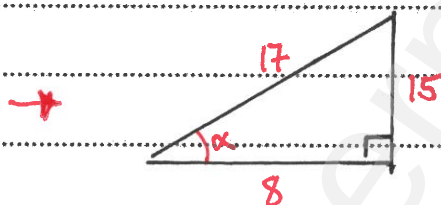


Pythagoras: $8^2 + x^2 = 17^2$

$$x^2 = 17^2 - 8^2$$

$$x = \sqrt{17^2 - 8^2}$$

$$= 15$$



$$\sin \alpha = \frac{15}{17} \quad \tan \alpha = \frac{15}{8}$$

$$\frac{1}{\sin \alpha} + \frac{1}{\tan \alpha} = \frac{17}{15} + \frac{8}{15}$$

$$= \frac{25}{15}$$

$$= \frac{5}{3}$$

- 2 (a) Express the equation $3 \cos \theta = 8 \tan \theta$ as a quadratic equation in $\sin \theta$.

[3]

$$3 \cos \theta = \frac{8 \sin \theta}{\cos \theta}$$

$1 - \sin^2 \theta$

$$3 \cos^2 \theta = 8 \sin \theta$$

$$3(1 - \sin^2 \theta) = 8 \sin \theta$$

$$3 - 3 \sin^2 \theta = 8 \sin \theta$$

$$\underline{3 \sin^2 \theta + 8 \sin \theta - 3 = 0}$$

- (b) Hence find the acute angle, in degrees, for which $3 \cos \theta = 8 \tan \theta$.

[2]

Let $Y = \sin \theta$:

$$3Y^2 + 8Y - 3 = 0$$

$$(3Y - 1)(Y + 3) = 0$$

$$3Y - 1 = 0$$

or

$$Y + 3 = 0$$

$$Y = \frac{1}{3}$$

or

$$Y = -3$$

$$\sin \theta = \frac{1}{3}$$

or

$$\sin \theta = -3$$

$$\underline{\theta = 19.5^\circ}$$

↑ no solutions

- 3 (a) Show that the equation $\frac{7 \tan \theta}{\cos \theta} + 12 = 0$ can be expressed as

$$12 \sin^2 \theta - 7 \sin \theta - 12 = 0.$$

[3]

$$7 \frac{\sin \theta}{\cos \theta} + 12 = 0$$

$$\frac{7 \sin \theta}{\cos^2 \theta} + 12 = 0$$

$$7 \sin \theta + 12 \cos^2 \theta = 0$$

$$7 \sin \theta + 12(1 - \sin^2 \theta) = 0$$

$$7 \sin \theta + 12 - 12 \sin^2 \theta = 0$$

$$-12 \sin^2 \theta + 7 \sin \theta + 12 = 0 \quad \times -1$$

$$12 \sin^2 \theta - 7 \sin \theta - 12 = 0$$

- (b) Hence solve the equation $\frac{7 \tan \theta}{\cos \theta} + 12 = 0$ for $0^\circ \leq \theta \leq 360^\circ$.

[3]

$$\text{Let } y = \sin \theta:$$

$$12y^2 - 7y - 12 = 0$$

$$y = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(12)(-12)}}{2(12)}$$

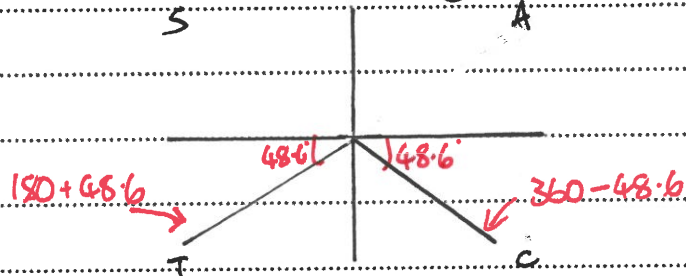
$$y = \frac{4}{3} \quad \text{or} \quad y = -\frac{3}{4}$$

$$\sin \theta = \frac{4}{3}$$

no solⁿs

$$\sin \theta = -\frac{3}{4}$$

$$\theta = -48.6^\circ$$



$$\theta = 278.6^\circ, 311.4^\circ$$

- 3 Solve the equation $\frac{\tan \theta + 2 \sin \theta}{\tan \theta - 2 \sin \theta} = 3$ for $0^\circ < \theta < 180^\circ$.

[4]

$$\tan \theta + 2 \sin \theta = 3(\tan \theta - 2 \sin \theta)$$

$$\tan \theta + 2 \sin \theta = 3 \tan \theta - 6 \sin \theta$$

$$8 \sin \theta - 2 \tan \theta = 0$$

$$\begin{array}{ccc} 8 \sin \theta & - & 2 \sin \theta \\ \times \cos \theta & & \cos \theta \\ & & \times \cos \theta \end{array} = 0$$

$$8 \sin \theta \cos \theta - 2 \sin \theta = 0$$

$$\sin \theta (8 \cos \theta - 2) = 0$$

$$\sin \theta = 0$$

or

$$8 \cos \theta - 2 = 0$$

$$\theta = 0^\circ$$

↑

$$\cos \theta = \frac{1}{4}$$

$$\theta = 75.5^\circ$$

outside range (and so is 180°)

$$\underline{\theta = 75.5^\circ}$$

- 3 Solve the equation $3 \tan^2 \theta + 1 = \frac{2}{\tan^2 \theta}$ for $0^\circ < \theta < 180^\circ$.

[5]

$$3 \tan^4 \theta + \tan^2 \theta = 2$$

$$3 \tan^4 \theta + \tan^2 \theta - 2 = 0$$

Let $y = \tan^2 \theta$:

$$3y^2 + y - 2 = 0$$

$$(3y - 2)(y + 1) = 0$$

$$3y - 2 = 0$$

or

$$y + 1 = 0$$

$$y = \frac{2}{3}$$

$$y = -1$$

$$\tan^2 \theta = \frac{2}{3}$$

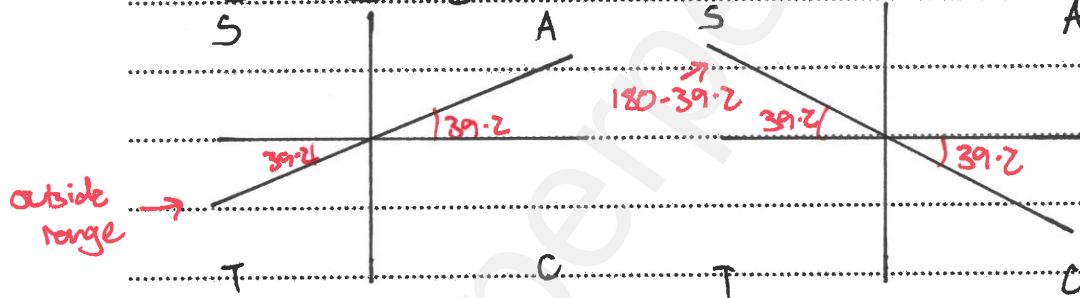
$$\tan^2 \theta = -1$$

↳ no solutions

$$\tan \theta = \sqrt{\frac{2}{3}} \quad \text{or} \quad \tan \theta = -\sqrt{\frac{2}{3}}$$

$$\theta = 39.2^\circ$$

$$\theta = -39.2^\circ$$



$$\underline{\underline{\theta = 39.2^\circ, 140.8^\circ}}$$

- 4 (a) Show that the equation $\cos \theta (7 \tan \theta - 5 \cos \theta) = 1$ can be written in the form $a \sin^2 \theta + b \sin \theta + c = 0$, where a , b and c are integers to be found. [3]

$$\cos \theta \left(7 \frac{\sin \theta}{\cos \theta} - 5 \cos \theta \right) = 1$$

$$7 \sin \theta - 5 \cos^2 \theta = 1$$

$$7 \sin \theta - 5(1 - \sin^2 \theta) = 1$$

$$7 \sin \theta - 5 + 5 \sin^2 \theta = 1$$

$$\underline{5 \sin^2 \theta + 7 \sin \theta - 6 = 0 \quad \text{QED}}$$

- (b) Hence solve the equation $\cos 2x (7 \tan 2x - 5 \cos 2x) = 1$ for $0^\circ < x < 180^\circ$. [3]

Replace θ with $2x$:

$$5 \sin^2(2x) + 7 \sin(2x) - 6 = 0$$

$$0 < x < 180$$

$$0 < 2x < 360$$

Let $Y = \sin 2x$:

$$5Y^2 + 7Y - 6 = 0$$

$$(5Y - 3)(Y + 2) = 0$$

$$5Y - 3 = 0 \quad \text{or} \quad Y + 2 = 0$$

$$Y = \frac{3}{5}$$

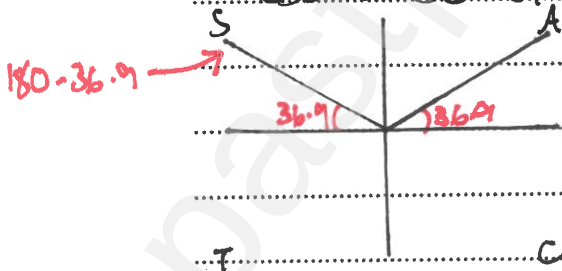
$$Y = -2$$

$$\sin 2x = \frac{3}{5}$$

$$\sin 2x = -2$$

$$2x = 36.87^\circ$$

$\times$ no sol^{ns}



$$\rightarrow 2x = 36.87^\circ, 143.13^\circ$$

$$\underline{x = 18.4^\circ, 71.6^\circ}$$

- 3 (a) Find the set of values of k for which the equation $8x^2 + kx + 2 = 0$ has no real roots. [2]

No real roots, so $b^2 - 4ac < 0$:

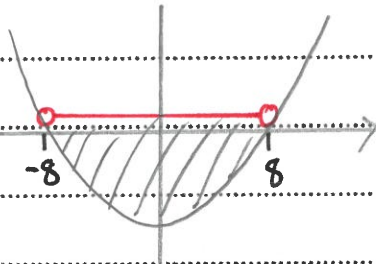
$$k^2 - 4(8)(2) < 0$$

$$k^2 - 64 < 0$$

$$(k+8)(k-8) < 0$$

critical values:

$$k = -8, k = 8$$



$$\underline{-8 < k < 8}$$

- (b) Solve the equation $8 \cos^2 \theta - 10 \cos \theta + 2 = 0$ for $0^\circ \leq \theta \leq 180^\circ$. [3]

$$4 \cos^2 \theta - 5 \cos \theta + 1 = 0$$

Let $Y = \cos \theta$: $4Y^2 - 5Y + 1 = 0$

$$(4Y - 1)(Y - 1) = 0$$

$$4Y - 1 = 0$$

or $Y - 1 = 0$

$$Y = \frac{1}{4}$$

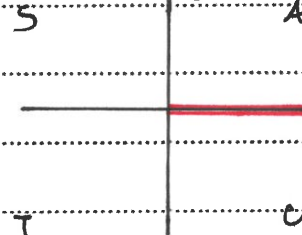
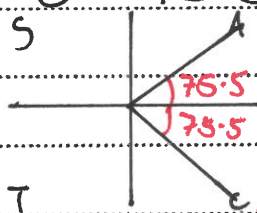
$$Y = 1$$

$$\cos \theta = \frac{1}{4}$$

$$\cos \theta = 1$$

$$\theta = 75.5^\circ$$

$$\theta = 0^\circ$$



$$\underline{\theta = 0^\circ, 75.5^\circ}$$

- 4 (a) Prove that $\frac{(\sin \theta + \cos \theta)^2 - 1}{\cos^2 \theta} \equiv 2 \tan \theta$. [3]

LHS:

$$\frac{\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta - 1}{\cos^2 \theta}$$

$$\frac{2 \sin \theta \cos \theta + 1 - 1}{\cos^2 \theta}$$

$$\frac{2 \sin \theta \cos \theta}{\cos^2 \theta} \div \cos \theta$$

$$= \frac{2 \sin \theta}{\cos \theta}$$

$$= \underline{2 \tan \theta}$$

- (b) Hence solve the equation $\frac{(\sin \theta + \cos \theta)^2 - 1}{\cos^2 \theta} = 5 \tan^3 \theta$ for $-90^\circ < \theta < 90^\circ$. [3]

from (a):

$$2 \tan \theta = 5 \tan^3 \theta$$

$$5 \tan^3 \theta - 2 \tan \theta = 0$$

$$\tan \theta (5 \tan^2 \theta - 2) = 0$$

$$\tan \theta = 0 \quad \text{or} \quad 5 \tan^2 \theta - 2 = 0$$

$$\underline{\theta = 0}$$

$$\tan^2 \theta = \frac{2}{5}$$

$$\tan \theta = \sqrt{\frac{2}{5}} \quad \text{or} \quad \tan \theta = -\sqrt{\frac{2}{5}}$$

$$\underline{\theta = 32.3^\circ}$$

$$\underline{\theta = -32.3^\circ}$$

No other solutions in given range, so no need for CAST diagram.

$$\underline{\underline{\theta = -32.3^\circ, 0^\circ, 32.3^\circ}}$$

7 (a) Show that $\frac{\tan \theta}{1 + \cos \theta} + \frac{\tan \theta}{1 - \cos \theta} \equiv \frac{2}{\sin \theta \cos \theta}$.

[4]

$$\begin{aligned}
 & \frac{\tan \theta \times (1 - \cos \theta)}{1 + \cos \theta \times (1 - \cos \theta)} + \frac{\tan \theta \times (1 + \cos \theta)}{1 - \cos \theta (1 + \cos \theta)} \\
 & \frac{\tan \theta - \tan \theta \cos \theta}{1 - \cos^2 \theta} + \frac{\tan \theta + \tan \theta \cos \theta}{1 - \cos^2 \theta} \\
 & \frac{\tan \theta - \tan \theta \cos \theta + \tan \theta + \tan \theta \cos \theta}{1 - \cos^2 \theta} \\
 & \frac{2 \tan \theta}{1 - \cos^2 \theta} \leftarrow \sin^2 \theta \\
 & \frac{2 \tan \theta}{\sin^2 \theta} \leftarrow \frac{\sin \theta}{\cos \theta} \\
 & \frac{2 \frac{\sin \theta}{\cos \theta} \times \cos \theta}{\sin^2 \theta \times \cos \theta} \\
 & \frac{2 \sin \theta}{\sin^2 \theta \cos \theta} \\
 & \frac{2}{\sin \theta \cos \theta} \quad \text{QED}
 \end{aligned}$$

(b) Hence solve the equation $\frac{\tan \theta}{1 + \cos \theta} + \frac{\tan \theta}{1 - \cos \theta} = \frac{6}{\tan \theta}$ for $0^\circ < \theta < 180^\circ$.

[4]

from (a): $\frac{2}{\sin \theta \cos \theta} = \frac{6}{\tan \theta}$

$$2 \tan \theta = 6 \sin \theta \cos \theta$$

$$2 \tan \theta - 6 \sin \theta \cos \theta = 0$$

$$2 \frac{\sin \theta}{\cos \theta} - 6 \sin \theta \cos \theta = 0$$

$\times \cos \theta$ $\times \cos \theta$

$$2 \sin \theta - 6 \sin \theta \cos^2 \theta = 0$$

$$2 \sin \theta (1 - 3 \cos^2 \theta) = 0$$

$$2 \sin \theta = 0$$

$$\text{or } 1 - 3 \cos^2 \theta = 0$$

$$\theta = 0$$

$$\cos^2 \theta = \frac{1}{3}$$

↑ outside range
(and so is 180°)

$$\cos \theta = \sqrt{\frac{1}{3}}$$

or

$$\cos \theta = -\sqrt{\frac{1}{3}}$$

$$\theta = 54.7^\circ$$

$$\theta = 125.3^\circ$$

$$\underline{\underline{\theta = 54.7^\circ, 125.3^\circ}}$$

- 7 (a) Prove the identity $\frac{1 + \sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} = \frac{2}{\cos \theta}$.

[3]

$$\frac{1 + \sin \theta \times (1 + \sin \theta)}{\cos \theta \times (1 + \sin \theta)} + \frac{\cos \theta \times \cos \theta}{1 + \sin \theta \times \cos \theta}$$

$$\frac{(1 + \sin \theta)^2}{\cos \theta (1 + \sin \theta)} + \frac{\cos^2 \theta}{\cos \theta (1 + \sin \theta)}$$

$$\frac{1 + 2\sin \theta + \sin^2 \theta + \cos^2 \theta}{\cos \theta (1 + \sin \theta)} = 1$$

$$\frac{2 + 2\sin \theta}{\cos \theta (1 + \sin \theta)}$$

$$\frac{2(1 + \sin \theta)}{\cos \theta (1 + \sin \theta)}$$

$$\frac{2}{\cos \theta}$$

$$\frac{2}{\cos \theta}$$

$$\frac{2}{\cos \theta}$$

$$\frac{2}{\cos \theta}$$

$$\frac{2}{\cos \theta} \quad \text{QED}$$

- (b) Hence solve the equation $\frac{1 + \sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} = \frac{3}{\sin \theta}$, for $0 \leq \theta \leq 2\pi$.

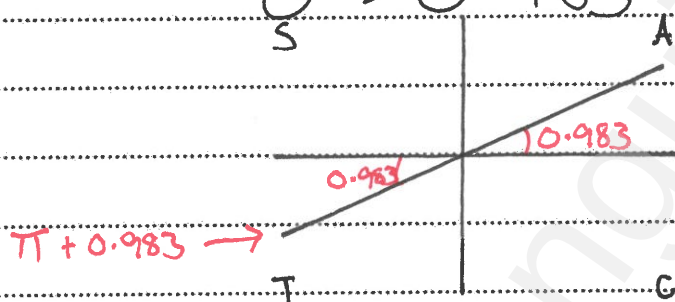
[3]

from (a): $\frac{2}{\cos \theta} = \frac{3}{\sin \theta}$

$$\frac{\sin \theta}{\cos \theta} = \frac{3}{2}$$

$$\tan \theta = \frac{3}{2}$$

$$\theta = 0.983$$



$$\underline{\underline{\theta = 0.983, 4.12}}$$

- 7 (a) Prove the identity $\frac{\sin \theta}{\sin \theta + \cos \theta} + \frac{\cos \theta}{\sin \theta - \cos \theta} \equiv \frac{\tan^2 \theta + 1}{\tan^2 \theta - 1}$. [3]

$$\begin{aligned}
 & \frac{\sin \theta \times (\sin \theta - \cos \theta)}{\sin \theta + \cos \theta \times (\sin \theta - \cos \theta)} + \frac{\cos \theta \times (\sin \theta + \cos \theta)}{\sin \theta - \cos \theta \times (\sin \theta + \cos \theta)} \\
 & \frac{\sin^2 \theta - \sin \theta \cos \theta}{\sin^2 \theta - \cos^2 \theta} + \frac{\sin \theta \cos \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} \\
 & \frac{\sin^2 \theta - \sin \theta \cos \theta + \sin \theta \cos \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} \\
 & \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} \div \cos^2 \theta \\
 & \frac{\sin^2 \theta}{\cos^2 \theta} + 1 \\
 & \frac{\sin^2 \theta}{\cos^2 \theta} - 1 \\
 & \frac{\tan^2 \theta + 1}{\tan^2 \theta - 1} \quad \text{QED}
 \end{aligned}$$

- (b) Hence find the exact solutions of the equation $\frac{\sin \theta}{\sin \theta + \cos \theta} + \frac{\cos \theta}{\sin \theta - \cos \theta} = 2$ for $0 \leq \theta \leq \pi$. [4]

from (a): $\frac{\tan^2 \theta + 1}{\tan^2 \theta - 1} = 2$

$$\tan^2 \theta + 1 = 2\tan^2 \theta - 2$$

$$0 = \tan^2 \theta - 3$$

$$\tan^2 \theta = 3$$

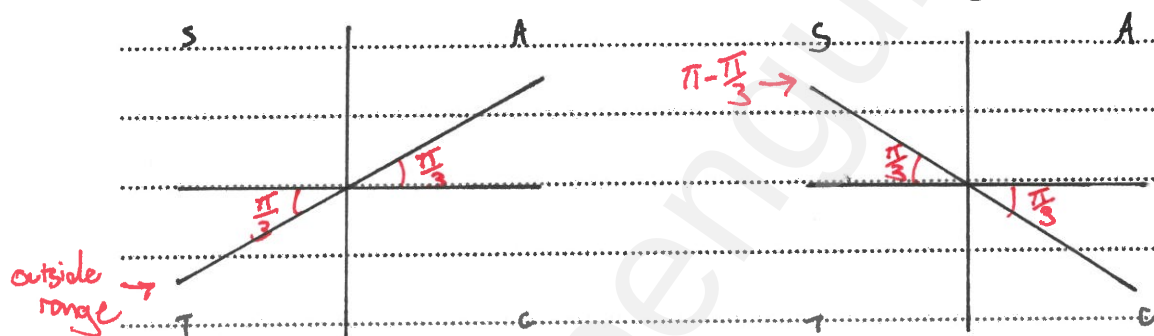
$$\tan \theta = \sqrt{3}$$

or

$$\tan \theta = -\sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

$$\theta = \frac{-\pi}{3}$$



$$\theta = \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$$

- 7 (a) Show that $\frac{\sin \theta + 2 \cos \theta}{\cos \theta - 2 \sin \theta} - \frac{\sin \theta - 2 \cos \theta}{\cos \theta + 2 \sin \theta} \equiv \frac{4}{5 \cos^2 \theta - 4}$. [4]

$$\begin{aligned}
 & \frac{\sin \theta + 2 \cos \theta}{\cos \theta - 2 \sin \theta} \times \frac{(\cos \theta + 2 \sin \theta)}{(\cos \theta + 2 \sin \theta)} - \frac{\sin \theta - 2 \cos \theta}{\cos \theta + 2 \sin \theta} \times \frac{(\cos \theta - 2 \sin \theta)}{(\cos \theta - 2 \sin \theta)} \\
 & \frac{\sin \theta \cos \theta + 2 \sin^2 \theta + 2 \cos^2 \theta + 4 \sin \theta \cos \theta}{\cos^2 \theta + 2 \sin \theta \cos \theta - 2 \sin \theta \cos \theta - 4 \sin^2 \theta} - \frac{\sin \theta \cos \theta - 2 \sin^2 \theta - 2 \cos^2 \theta + 4 \sin \theta \cos \theta}{\cos^2 \theta - 2 \sin \theta \cos \theta + 2 \sin \theta \cos \theta - 4 \sin^2 \theta} \\
 & \frac{\sin \theta \cos \theta + 2 \sin^2 \theta + 2 \cos^2 \theta + 4 \sin \theta \cos \theta - (\sin \theta \cos \theta - 2 \sin^2 \theta - 2 \cos^2 \theta + 4 \sin \theta \cos \theta)}{\cos^2 \theta - 4 \sin^2 \theta} \\
 & \frac{\sin \theta \cos \theta + 2 \sin^2 \theta + 2 \cos^2 \theta + 4 \sin \theta \cos \theta - \sin \theta \cos \theta + 2 \sin^2 \theta + 2 \cos^2 \theta - 4 \sin \theta \cos \theta}{\cos^2 \theta - 4 \sin^2 \theta} \\
 & \frac{4 \sin^2 \theta + 4 \cos^2 \theta}{\cos^2 \theta - 4 \sin^2 \theta} \\
 & \frac{4(\sin^2 \theta + \cos^2 \theta)}{\cos^2 \theta - 4 \sin^2 \theta} \quad \leftarrow = 1 \\
 & \frac{4}{\cos^2 \theta - 4 \sin^2 \theta} \quad \leftarrow = 1 - \cos^2 \theta \\
 & \frac{\cos^2 \theta - 4(1 - \cos^2 \theta)}{4} \\
 & \frac{\cos^2 \theta - 4 + 4 \cos^2 \theta}{4} \\
 & \frac{5 \cos^2 \theta - 4}{4} \quad \text{QED}
 \end{aligned}$$

- (b) Hence solve the equation $\frac{\sin \theta + 2 \cos \theta}{\cos \theta - 2 \sin \theta} - \frac{\sin \theta - 2 \cos \theta}{\cos \theta + 2 \sin \theta} = 5$ for $0^\circ < \theta < 180^\circ$. [3]

from (a): $\frac{4}{5 \cos^2 \theta - 4} = 5$

$$4 = 5(5 \cos^2 \theta - 4)$$

$$4 = 25 \cos^2 \theta - 20$$

$$25 \cos^2 \theta - 24 = 0$$

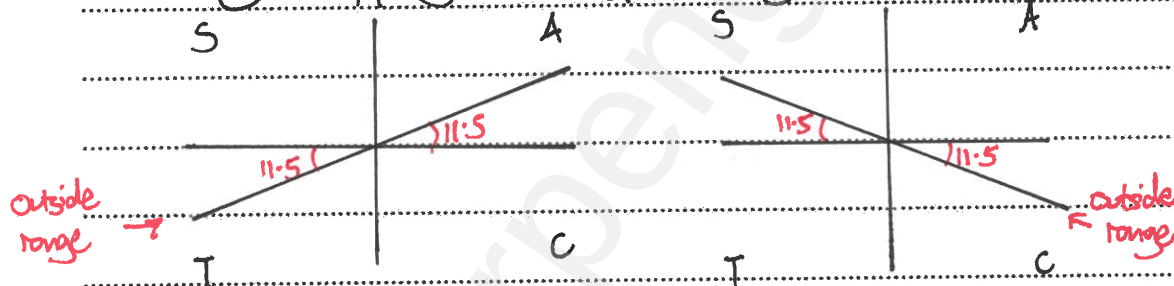
$$\cos^2 \theta = \frac{24}{25}$$

$$\cos \theta = \sqrt{\frac{24}{25}} \quad \text{or} \quad \cos \theta = -\sqrt{\frac{24}{25}}$$

$$\theta = 11.5^\circ$$

or

$$\theta = 168.5^\circ$$



$$\theta = 11.5^\circ, 168.5^\circ$$

3 Solve, by factorising, the equation

$$6 \cos \theta \tan \theta - 3 \cos \theta + 4 \tan \theta - 2 = 0,$$

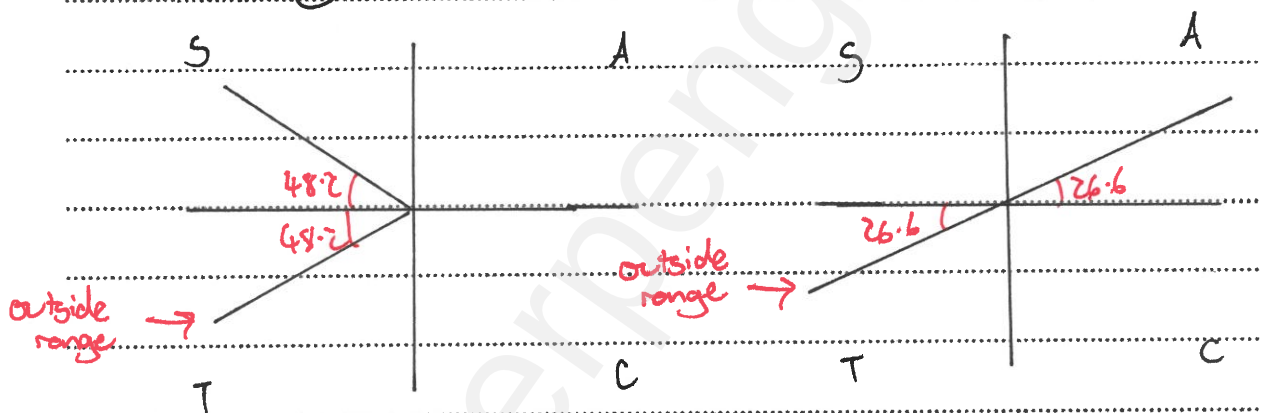
for $0^\circ \leq \theta \leq 180^\circ$.

[4]

$$\begin{aligned} 6 \cos \theta \tan \theta - 3 \cos \theta + 4 \tan \theta - 2 &= 0 \\ 3 \cos \theta (2 \tan \theta - 1) + 2(2 \tan \theta - 1) &= 0 \\ (3 \cos \theta + 2)(2 \tan \theta - 1) &= 0 \end{aligned}$$

$$\begin{aligned} 3 \cos \theta + 2 &= 0 & \text{or} & & 2 \tan \theta - 1 &= 0 \\ \cos \theta &= -\frac{2}{3} & & & \tan \theta &= \frac{1}{2} \end{aligned}$$

$$\theta = 131.8^\circ \qquad \theta = 26.6^\circ$$



$$\theta = 26.6^\circ, 131.8^\circ$$

- 7 (a) By first obtaining a quadratic equation in $\cos \theta$, solve the equation

$$\tan \theta \sin \theta = 1$$

for $0^\circ < \theta < 360^\circ$.

[5]

$$\frac{\sin \theta}{\cos \theta} \times \sin \theta = 1$$

$$\frac{\sin^2 \theta}{\cos \theta} = 1$$

$1 - \cos^2 \theta \rightarrow$

$$\sin^2 \theta = \cos \theta$$

$$1 - \cos^2 \theta = \cos \theta$$

$$\cos^2 \theta + \cos \theta - 1 = 0$$

Let $y = \cos \theta$:

$$y^2 + y - 1 = 0$$

$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$

$$y = \frac{-1 + \sqrt{5}}{2}$$

or

$$y = \frac{-1 - \sqrt{5}}{2}$$

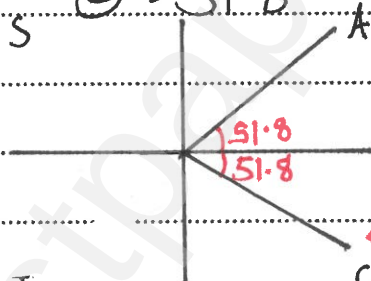
$$\cos \theta = \frac{-1 + \sqrt{5}}{2}$$

or

$$\cos \theta = \frac{-1 - \sqrt{5}}{2}$$

$$\theta = 51.8^\circ$$

$\hookrightarrow$ no solutions



$$\theta = \underline{51.8^\circ, 308.2^\circ}$$

(b) Show that $\frac{\tan \theta}{\sin \theta} - \frac{\sin \theta}{\tan \theta} = \tan \theta \sin \theta$.

[3]

$$\frac{\tan \theta}{\sin \theta} - \frac{\sin \theta}{\tan \theta}$$

$$\frac{\frac{\sin \theta}{\cos \theta}}{\sin \theta} \times \cos \theta - \frac{\sin \theta}{\frac{\sin \theta}{\cos \theta}} \times \cos \theta$$

$$\frac{\sin \theta}{\sin \theta \cos \theta} - \frac{\sin \theta \cos \theta}{\sin \theta} \times \cos \theta$$

$$\frac{\sin \theta}{\sin \theta \cos \theta} - \frac{\sin \theta \cos^2 \theta}{\sin \theta \cos \theta}$$

$$\frac{\sin \theta - \sin \theta \cos^2 \theta}{\sin \theta \cos \theta}$$

$$\frac{\sin \theta (1 - \cos^2 \theta)}{\sin \theta \cos \theta}$$

$$\frac{1 - \cos^2 \theta}{\cos \theta}$$

$$\frac{\sin^2 \theta}{\cos \theta}$$

$$\frac{\sin^2 \theta}{\cos \theta}$$

$$\frac{\sin \theta}{\cos \theta} \times \sin \theta$$

$$\underline{\tan \theta \sin \theta} \text{ QED}$$

- 7 (a) Verify the identity $(2x-1)(4x^2+2x-1) \equiv 8x^3-4x+1$. [1]

Expand LHS:

$$\begin{aligned} & 8x^3 + 4x^2 - 2x - 4x^2 - 2x + 1 \\ &= \underline{8x^3 - 4x + 1} \quad \text{QED} \end{aligned}$$

- (b) Prove the identity $\frac{\tan^2 \theta + 1}{\tan^2 \theta - 1} \equiv \frac{1}{1 - 2 \cos^2 \theta}$. [3]

$$\text{LHS: } \frac{\sin^2 \theta}{\cos^2 \theta} + 1 \quad \times \cos^2 \theta$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} - 1 \quad \times \cos^2 \theta$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta}$$

$1 - \cos^2 \theta \rightarrow$

$$= \frac{1}{(1 - \cos^2 \theta) - \cos^2 \theta}$$

$$= \underline{\underline{\frac{1}{1 - 2 \cos^2 \theta}}}$$

(c) Using the results of (a) and (b), solve the equation

$$\frac{\tan^2 \theta + 1}{\tan^2 \theta - 1} = 4 \cos \theta,$$

for $0^\circ \leq \theta \leq 180^\circ$.

[5]

Part (b):
$$\frac{1}{1 - 2\cos^2 \theta} = 4 \cos \theta$$

$$1 = 4 \cos \theta (1 - 2\cos^2 \theta)$$

$$1 = 4 \cos \theta - 8 \cos^3 \theta$$

$$8 \cos^3 \theta - 4 \cos \theta + 1 = 0$$

Let $x = \cos \theta$:

$$8x^3 - 4x + 1 = 0$$

part (a):

$$(2x - 1)(4x^2 + 2x - 1) = 0$$

$$2x - 1 = 0 \quad \text{or} \quad 4x^2 + 2x - 1 = 0$$

$$x = \frac{1}{2} \quad \text{or} \quad x = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$$

$$x = \frac{-1 + \sqrt{5}}{4} \quad \text{or} \quad x = \frac{-1 - \sqrt{5}}{4}$$

$$\rightarrow \cos \theta = \frac{1}{2} \quad \cos \theta = \frac{-1 + \sqrt{5}}{4} \quad \cos \theta = \frac{-1 - \sqrt{5}}{4}$$

$$\theta = 60^\circ$$

$$\theta = 72^\circ$$

$$\theta = 144^\circ$$

(any other solutions are outside range)

$$\underline{\underline{\theta = 60^\circ, 72^\circ, 144^\circ}}$$

- 4 (a) Show that the equation

$$\frac{\tan x + \sin x}{\tan x - \sin x} = k,$$

where k is a constant, may be expressed as

$$\frac{1 + \cos x}{1 - \cos x} = k. \quad [2]$$

$\tan x = \frac{\sin x}{\cos x} :$

$$\frac{\frac{\sin x}{\cos x} + \sin x}{\frac{\sin x}{\cos x} - \sin x} = k$$

$$\frac{\sin x + \sin x \cos x}{\sin x - \sin x \cos x} = k$$

$$\frac{\sin x(1 + \cos x)}{\sin x(1 - \cos x)} = k$$

$$\frac{1 + \cos x}{1 - \cos x} = k \quad \text{QED}$$

- (b) Hence express $\cos x$ in terms of k . [2]

$$1 + \cos x = k(1 - \cos x)$$

$$1 + \cos x = k - k \cos x$$

$$k \cos x + \cos x = k - 1$$

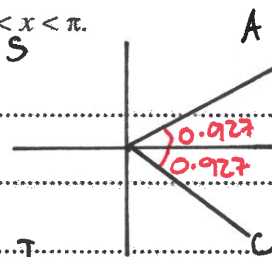
$$\cos x(k + 1) = k - 1$$

$$\cos x = \frac{k - 1}{k + 1}$$

- (c) Hence solve the equation $\frac{\tan x + \sin x}{\tan x - \sin x} = 4$ for $-\pi < x < \pi$. [2]

$k=4:$ $\cos x = \frac{4-1}{4+1}$

$$\cos x = \frac{3}{5}$$



$$x = 0.927$$

$$x = 0.927, -0.927$$

- 5 (a) Solve the equation $6\sqrt{y} + \frac{2}{\sqrt{y}} - 7 = 0$.

[4]

$$\begin{array}{cccc} \times \sqrt{y} & \times \sqrt{y} & \times \sqrt{y} & \times \sqrt{y} \\ 6\sqrt{y} + 2 - 7\sqrt{y} = 0 \end{array}$$

$$6y + 2 - 7\sqrt{y} = 0$$

$$6y - 7\sqrt{y} + 2 = 0$$

$$6y - 7y^{\frac{1}{2}} + 2 = 0$$

Let $X = y^{\frac{1}{2}}$:

$$6X^2 - 7X + 2 = 0$$

$$(3X - 2)(2X - 1) = 0$$

$$3X - 2 = 0$$

$$\text{or } 2X - 1 = 0$$

$$X = \frac{2}{3}$$

$$X = \frac{1}{2}$$

$$y^{\frac{1}{2}} = \frac{2}{3}$$

$$y^{\frac{1}{2}} = \frac{1}{2}$$

$$\underline{y = \frac{4}{9}}$$

$$\underline{y = \frac{1}{4}}$$

- (b) Hence solve the equation $6\sqrt{\tan x} + \frac{2}{\sqrt{\tan x}} - 7 = 0$ for $0^\circ \leq x \leq 360^\circ$.

[3]

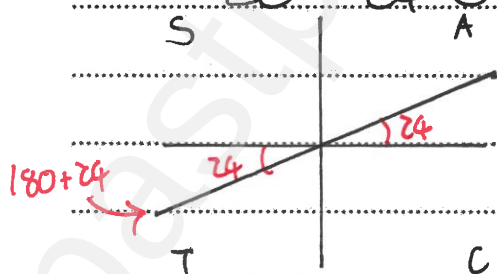
$y = \tan x$:

$$\tan x = \frac{4}{9}$$

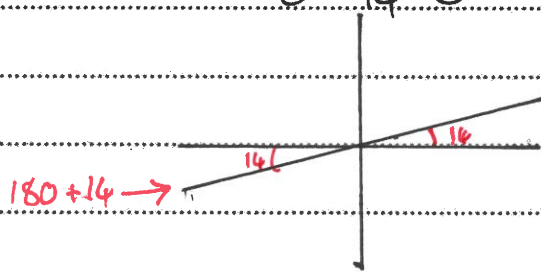
$$\text{or } \tan x = \frac{1}{4}$$

$$x = 24^\circ, 204^\circ$$

$$x = 14^\circ, 194^\circ$$



$$x = 24^\circ, 204^\circ$$



$$x = 14^\circ, 194^\circ$$

$$\underline{x = 14^\circ, 24^\circ, 194^\circ, 204^\circ}$$

11 The function f is given by $f(x) = 4\cos^4 x + \cos^2 x - k$ for $0 \leq x \leq 2\pi$, where k is a constant.

(a) Given that $k = 3$, find the exact solutions of the equation $f(x) = 0$.

[5]

$$4\cos^4 x + \cos^2 x - 3 = 0$$

Let $Y = \cos^2 x$:

$$4Y^2 + Y - 3 = 0$$

$$(4Y - 3)(Y + 1) = 0$$

$$4Y - 3 = 0$$

$$\text{or } Y + 1 = 0$$

$$Y = \frac{3}{4}$$

$$Y = -1$$

$$\cos^2 x = \frac{3}{4}$$

$$\cos^2 x = -1$$

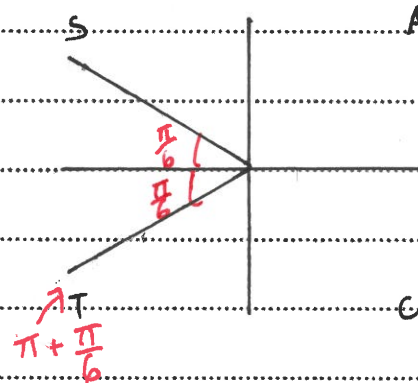
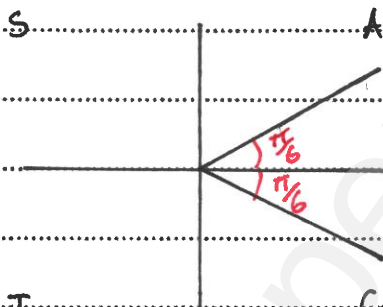
no solutions

$$\cos x = \sqrt{\frac{3}{4}}$$

$$\text{or } \cos x = -\sqrt{\frac{3}{4}}$$

$$x = \frac{\pi}{6}$$

$$x = \frac{5\pi}{6}$$



$$x = \frac{\pi}{6}, \frac{11\pi}{6}$$

$$x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

$$\underline{\underline{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}}$$

- (b) Use the quadratic formula to show that, when $k > 5$, the equation $f(x) = 0$ has no solutions. [5]

$$4 \cos^4 x + \cos^2 x - k = 0$$

Let $Y = \cos^2 x$:

$$4Y^2 + Y - k = 0$$

$$Y = \frac{-1 \pm \sqrt{1^2 - 4(4)(-k)}}{2(4)}$$

$$= \frac{-1 \pm \sqrt{1 + 16k}}{8}$$

$$\cos^2 x = \frac{-1 + \sqrt{1 + 16k}}{8} \quad \text{or} \quad \cos^2 x = \frac{-1 - \sqrt{1 + 16k}}{8}$$

if $k > 5$: $\sqrt{1 + 16k} = \sqrt{1 + 16(5)}$
 $= \sqrt{81}$
 $= 9$

$k=5$:

$$\rightarrow \cos^2 x = \frac{-1 + 9}{8} \quad \text{or} \quad \cos^2 x = \frac{-1 - 9}{8}$$

$$= \frac{8}{8} \quad \quad \quad = \frac{-10}{8}$$

$$\cos^2 x = 1$$

$$\cos^2 x = -\frac{5}{4}$$

if $k > 5$:

$$\cos^2 x > 1$$

$$\text{or} \quad \cos^2 x < -\frac{5}{4}$$

↑
no solutions
(cos is between -1 and 1)

↑
no solutions (can't $\sqrt{\text{negative}}$)

- [4]

Let $Y = \tan x$:

$$3y^2 - 5y - 2 = 0$$

$$(3Y + 1)(Y - 2) = 0$$

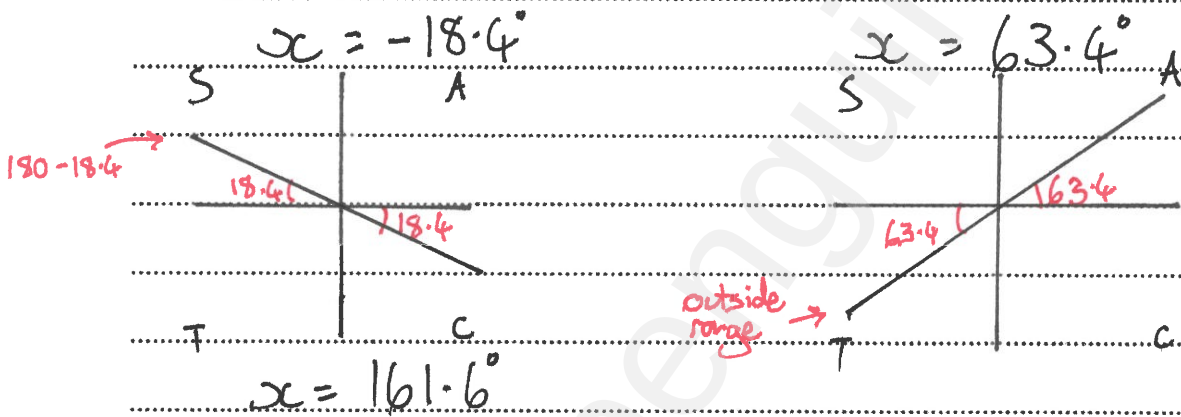
or

$$y - z = 0$$

$$Y = Z$$

or

$$\tan x = 2$$



$$\rightarrow x = 63.4^\circ, 161.6^\circ$$

- (b) Find the set of values of k for which the equation $3 \tan^2 x - 5 \tan x + k = 0$ has no solutions. [2]

If no solutions, $b^2 - 4ac < 0$:

$$(-5)^2 - 4(3)(k) < 0$$

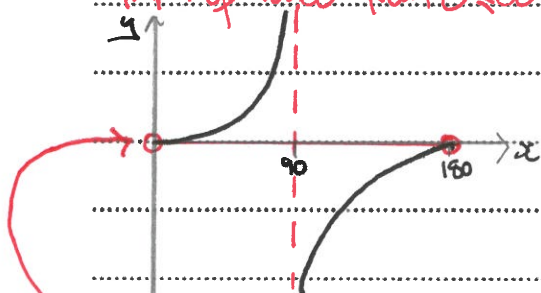
$$25 - 12k < 0$$

$$25 < 126$$

$$k > \frac{25}{12}$$

- (c) For the equation $3 \tan^2 x - 5 \tan x + k = 0$, state the value of k for which there are three solutions in the interval $0^\circ \leq x \leq 180^\circ$, and find these solutions. [3]

Graph of $\tan x$ from $0 \leq x \leq 180^\circ$:



For there to be 3 solutions, one of the factors must have 2 answers. In the range from 0 to 180, this only happens at $y=0$. A horizontal line anywhere else only intersects the graph once.

Therefore one of the solutions must be $\tan x = 0$.

This only happens when it factorises as a single bracket: $\tan x(\dots) = 0$, meaning $k=0$:

$$3 \tan^2 x - 5 \tan x = 0$$

$$\tan x (3 \tan x - 5) = 0$$

$$\tan x = 0$$

$$x = 0^\circ, 180^\circ$$

$$\text{or } 3 \tan x - 5 = 0$$

$$\tan x = \frac{5}{3}$$

$$x = 59.0^\circ$$

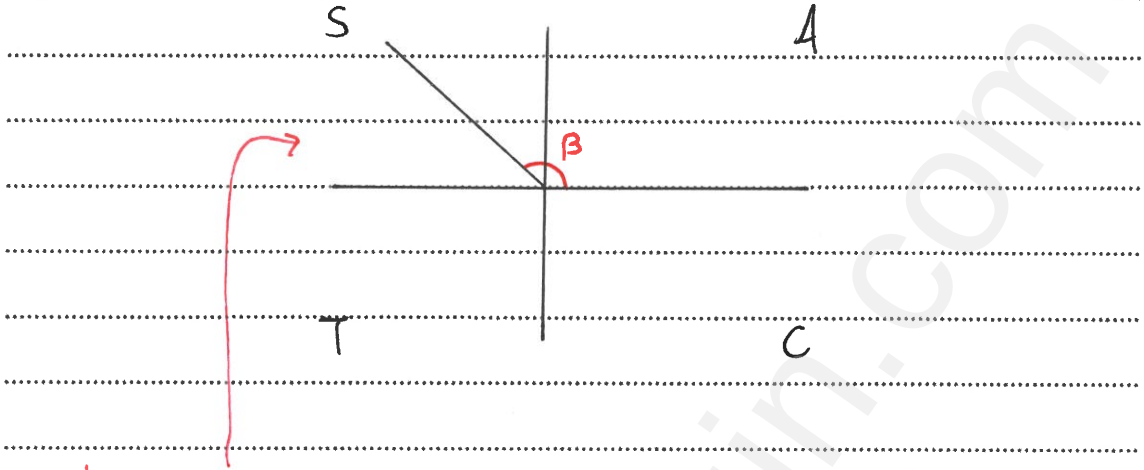
$$\underline{x = 0^\circ, 59.0^\circ, 180^\circ}$$



- 8 (a) It is given that β is an angle between 90° and 180° such that $\sin \beta = a$.

Express $\tan^2 \beta - 3 \sin \beta \cos \beta$ in terms of a .

[3]



In this quadrant, sine is positive, cosine and tangent are negative

$\cos \beta$:

$$\sin^2 \beta + \cos^2 \beta = 1$$

$$\cos^2 \beta = 1 - \sin^2 \beta$$

$$\cos \beta = \pm \sqrt{1 - \sin^2 \beta}$$

$\cos \beta$ is negative, so:

$$\cos \beta = -\sqrt{1 - \sin^2 \beta}$$

$\uparrow = a^2$

$$\cos \beta = -\sqrt{1 - a^2}$$

$\tan^2 \beta$:

$$\tan \beta = \frac{\sin \beta}{\cos \beta}$$

$$\tan^2 \beta = \frac{\sin^2 \beta}{\cos^2 \beta}$$

$$\tan^2 \beta = \frac{a^2}{1 - a^2}$$

$\tan^2 \beta - 3 \sin \beta \cos \beta$:

$$\frac{a^2}{1 - a^2} - 3a(-\sqrt{1 - a^2})$$

$$= \frac{a^2}{1 - a^2} + 3a\sqrt{1 - a^2}$$



(b) Solve the equation $\sin^2\theta + 2\cos^2\theta = 4\sin\theta + 3$ for $0^\circ < \theta < 360^\circ$.

[5]

$$\uparrow \cos^2\theta = 1 - \sin^2\theta$$

$$\sin^2\theta + 2(1 - \sin^2\theta) = 4\sin\theta + 3$$

$$\sin^2\theta + 2 - 2\sin^2\theta = 4\sin\theta + 3$$

$$-\sin^2\theta + 2 = 4\sin\theta + 3$$

$$\sin^2\theta + 4\sin\theta + 1 = 0$$

Let $Y = \sin\theta$:

$$Y^2 + 4Y + 1 = 0$$

$$Y = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)}$$

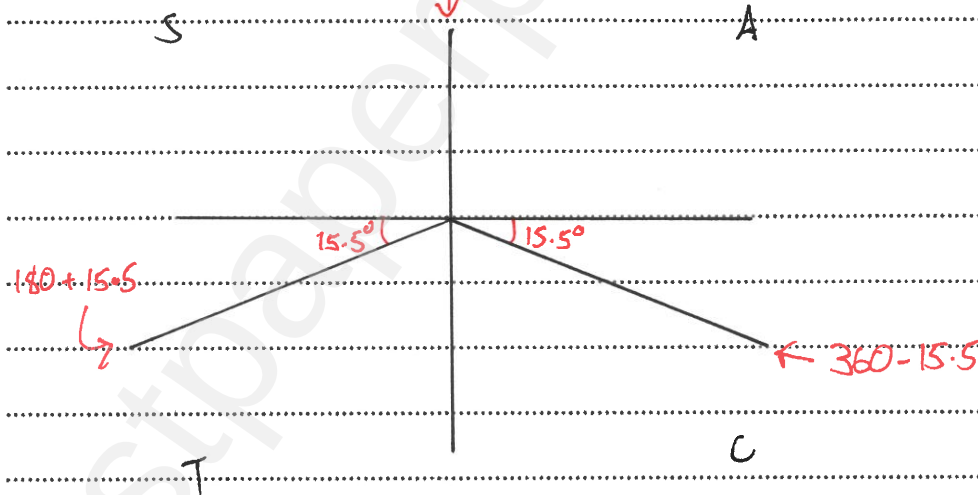
$$Y = -2 + \sqrt{3} \quad \text{or} \quad Y = -2 - \sqrt{3}$$

$$\sin\theta = -2 + \sqrt{3}$$

$$\sin\theta = -2 - \sqrt{3}$$

$$\theta = -15.5^\circ$$

X No solutions



$$\theta = \underline{195.5^\circ, 344.5^\circ}$$