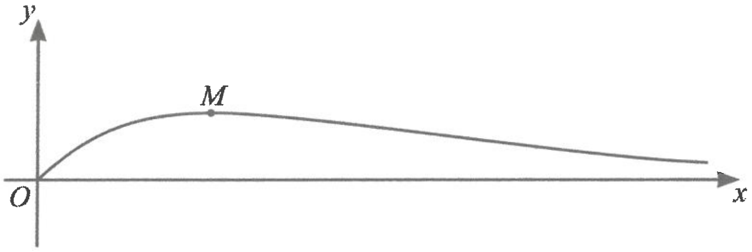
6



The diagram shows the curve $y = xe^{-ax}$, where a is a positive constant, and its maximum point M.

(a) Find the exact coordinates of M. **[4]**

$u \quad v$

$y = xe^{-ax}$

$u = x \qquad v = e^{-ax}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -ae^{-ax}$

$\frac{dy}{dx} = -axe^{-ax} + e^{-ax}$

$= e^{-ax}(1 - ax)$

SP, so $\frac{dy}{dx} = 0$:

$e^{-ax}(1 - ax) = 0$

$e^{-ax} = 0$ or $1 - ax = 0$

X No sol^s $\qquad ax = 1$

$\underline{x = \frac{1}{a}}$

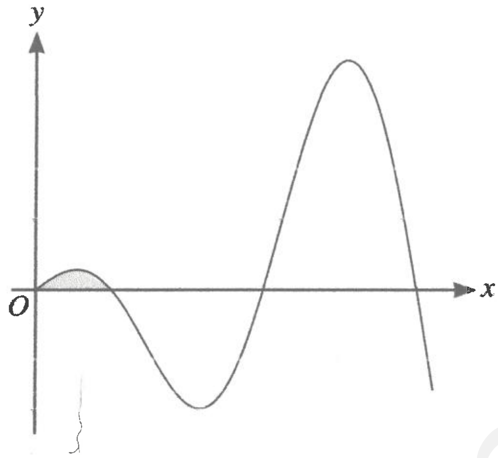
Sub. $x = \frac{1}{a}$ into y:

$y = \frac{1}{a}e^{-a \times \frac{1}{a}}$

$= \underline{\frac{1}{a}e^{-1}}$

$\rightarrow \underline{\left(\frac{1}{a}, \frac{1}{a}e^{-1}\right)}$

10



The diagram shows the curve $y = \overset{u}{x} \overset{v}{\cos 2x}$, for $x \geqslant 0$.

(a) Find the equation of the tangent to the curve at the point where $x = \frac{1}{2}\pi$. [4]

$u = x$ $\quad$ $v = \cos 2x$

$\frac{du}{dx} = 1$ $\quad$ $\frac{dv}{dx} = -2\sin 2x$

$\frac{dy}{dx} = -2x\sin 2x + \cos 2x$

gradient at $x = \frac{\pi}{2}$:

$m = -2\left(\frac{\pi}{2}\right)\sin(\pi) + \cos(\pi)$

$\underline{m = -1}$

y-coordinate at $x = \frac{\pi}{2}$:

$y = \frac{\pi}{2} \times \cos(\pi)$

$\underline{y = \frac{-\pi}{2}}$

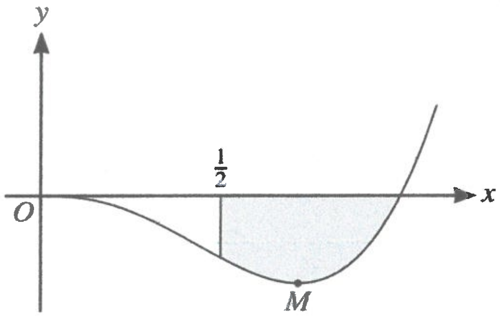
$\rightarrow y - y_1 = m(x - x_1)$

$y - -\frac{\pi}{2} = -1\left(x - \frac{\pi}{2}\right)$

$y + \frac{\pi}{2} = -x + \frac{\pi}{2}$

$\underline{y = -x}$

8



u v

The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M.

(a) Find the exact coordinates of M. **[4]**

$u = x^3$ $v = \ln x$

$\frac{du}{dx} = 3x^2$ $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = x^3 \times \frac{1}{x} + 3x^2 \ln x$

$= x^2 + 3x^2 \ln x$

SP, so $\frac{dy}{dx} = 0$:

$x^2 + 3x^2 \ln x = 0$

$x^2(1 + 3\ln x) = 0$

$x^2 = 0$ or $1 + 3\ln x = 0$

$x = 0$ $\ln x = -\frac{1}{3}$

not the point we want $x = e^{-\frac{1}{3}}$

Sub. into eqn to find y:

$y = (e^{-\frac{1}{3}})^3 \times \ln(e^{-\frac{1}{3}})$

$= e^{-1} \times -\frac{1}{3}$

$= -\frac{1}{3}e^{-1}$

$\rightarrow \left(e^{-\frac{1}{3}}, -\frac{1}{3}e^{-1}\right)$

3 The equation of a curve is $y = \sin x \sin 2x$. The curve has a stationary point in the interval $0 < x < \frac{1}{2}\pi$.

Find the x-coordinate of this point, giving your answer correct to 3 significant figures. [6]

$$y = \overset{u}{\sin x}\ \overset{v}{\sin 2x}$$

$u = \sin x$ $\qquad$ $v = \sin 2x$

$\frac{du}{dx} = \cos x$ $\qquad$ $\frac{dv}{dx} = 2\cos 2x$

$$\frac{dy}{dx} = 2\sin x \cos 2x + \sin 2x \cos x$$

SP: $\frac{dy}{dx} = 0$:

$$2\sin x \underbrace{\cos 2x}_{1-2\sin^2 x} + \underbrace{\sin 2x}_{2\sin x\cos x} \cos x = 0$$

$$2\sin x(1 - 2\sin^2 x) + 2\sin x \underbrace{\cos^2 x}_{1-\sin^2 x} = 0$$

$$2\sin x - 4\sin^3 x + 2\sin x(1 - \sin^2 x) = 0$$

$$2\sin x - 4\sin^3 x + 2\sin x - 2\sin^3 x = 0$$

$$-6\sin^3 x + 4\sin x = 0$$

$$2\sin x(-3\sin^2 x + 2) = 0$$

$2\sin x = 0$ or $-3\sin^2 x + 2 = 0$

$\sin x = 0$ $\qquad$ $3\sin^2 x = 2$

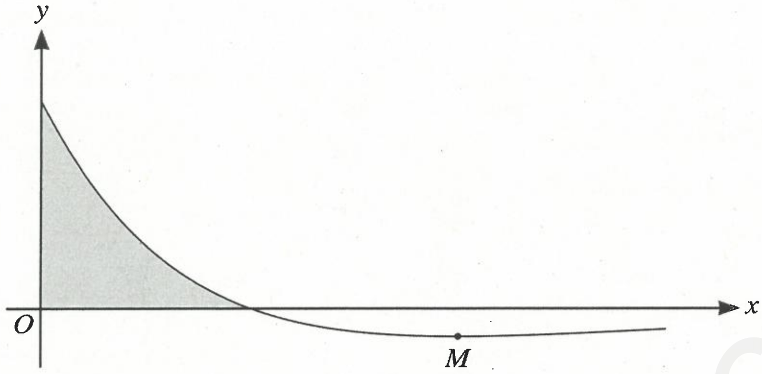
$x = 0, \pi$ ✗ outside domain $\qquad$ $\sin^2 x = \frac{2}{3}$

$\sin x = \pm\sqrt{\frac{2}{3}}$

$\sin x = \sqrt{\frac{2}{3}}$ or $\sin x = -\sqrt{\frac{2}{3}}$

$\underline{x = 0.955}$ $\qquad$ $x = -0.955$ ✗ outside domain

9



The diagram shows part of the curve $y = (3 - x)e^{-\frac{1}{3}x}$ for $x \geqslant 0$, and its minimum point M.

(a) Find the exact coordinates of M. [5]

u v

$y = (3-x)e^{-\frac{1}{3}x}$

$u = 3 - x$ $\quad$ $v = e^{-\frac{1}{3}x}$

$\frac{du}{dx} = -1$ $\quad$ $\frac{dv}{dx} = -\frac{1}{3}e^{-\frac{1}{3}x}$

$\frac{dy}{dx} = (3-x) \times \frac{1}{3}e^{-\frac{1}{3}x} - e^{-\frac{1}{3}x}$

$\frac{dy}{dx} = 0$:

$-\frac{1}{3}(3-x)e^{-\frac{1}{3}x} - e^{-\frac{1}{3}x} = 0$

$(-1 + \frac{x}{3})e^{-\frac{1}{3}x} - e^{-\frac{1}{3}x} = 0$

$e^{-\frac{1}{3}x}(-1 + \frac{x}{3} - 1) = 0$

$e^{-\frac{1}{3}x}(\frac{x}{3} - 2) = 0$

$e^{-\frac{1}{3}x} = 0$ or $\frac{x}{3} - 2 = 0$

X not possible $\quad$ $\frac{x}{3} = 2$

$x = 6$

→ Original equation:

$y = (3-6)e^{-\frac{1}{3}\times 6}$

$= -3e^{-2}$

$\rightarrow (6, -3e^{-2})$

4 A curve has equation $y = \cos x \sin 2x$.

Find the x-coordinate of the stationary point in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

$u \quad v$

$y = \cos x \sin 2x$

$u = \cos x \qquad v = \sin 2x$

$\frac{du}{dx} = -\sin x \qquad \frac{dv}{dx} = 2\cos 2x$

$\frac{dy}{dx} = 2\cos x \cos 2x + \sin 2x(-\sin x)$

$\frac{dy}{dx} = 0$:

$(2\cos^2 x - 1) \qquad 2\sin x \cos x$

$2\cos x \cos 2x - \sin x \sin 2x = 0$

$2\cos x(2\cos^2 x - 1) - \sin x(2\sin x \cos x) = 0$

$4\cos^3 x - 2\cos x - 2\sin^2 x \cos x = 0$

$1 - \cos^2 x$

$4\cos^3 x - 2\cos x - 2(1 - \cos^2 x)\cos x = 0$

$4\cos^3 x - 2\cos x - 2\cos x + 2\cos^3 x = 0$

$6\cos^3 x - 4\cos x = 0$

$2\cos x(3\cos^2 x - 2) = 0$

$2\cos x = 0 \quad$ or $\quad 3\cos^2 x - 2 = 0$

$\cos x = 0 \qquad \cos^2 x = \frac{2}{3}$

$x = \frac{\pi}{2} \qquad \cos x = \sqrt{\frac{2}{3}} \quad$ or $\quad \cos x = -\sqrt{\frac{2}{3}}$

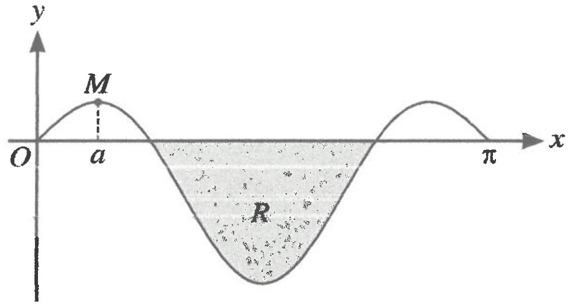
× outside domain

$x = 0.615 \qquad x = 2.53$

× outside domain

$\underline{x = 0.615}$

9



The diagram shows the curve $y = \overset{u}{\sin x}\,\overset{v}{\cos 2x}$, for $0 \leqslant x \leqslant \pi$, and a maximum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Find the value of a correct to 2 decimal places. [5]

$u = \sin x \qquad v = \cos 2x$

$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -2\sin 2x$

$\frac{dy}{dx} = -\underbrace{2\sin 2x}_{2\sin x\cos x}\sin x + \underbrace{\cos 2x}_{2\cos^2 x - 1}\cos x$

$= -4\sin x\cos x \times \sin x + (2\cos^2 x - 1)\cos x$

$= -4\underbrace{\sin^2 x}_{1-\cos^2 x}\cos x + 2\cos^3 x - \cos x$

$= -4(1-\cos^2 x)\cos x + 2\cos^3 x - \cos x$

$= -4\cos x + 4\cos^3 x + 2\cos^3 x - \cos x$

$= 6\cos^3 x - 5\cos x$

SP, so $\frac{dy}{dx} = 0$:

$6\cos^3 x - 5\cos x = 0$

$\cos x(6\cos^2 x - 5) = 0$

$\cos x = 0$ or $6\cos^2 x = 5$

$x = \frac{\pi}{2}$ $\qquad \cos^2 x = \frac{5}{6}$

$\cos x = \pm\sqrt{\frac{5}{6}}$

$\cos x = \sqrt{\frac{5}{6}}$ or $\cos x = -\sqrt{\frac{5}{6}}$

$x = 0.421$ or $x = 2.72$

Of the three stationary points, a is furthest to the left, so $x = 0.42$

3 The curve with equation $y = xe^{1-2x}$ has one stationary point.

(a) Find the coordinates of this point. [4]

$$y = xe^{1-2x}$$

$u = x \qquad v = e^{1-2x}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -2e^{1-2x}$

$$\frac{dy}{dx} = -2xe^{1-2x} + e^{1-2x}$$

$\frac{dy}{dx} = 0$:

$$-2xe^{1-2x} + e^{1-2x} = 0$$

$$e^{1-2x}(-2x + 1) = 0$$

$e^{1-2x} = 0$ or $-2x + 1 = 0$

X not possible $\qquad 2x = 1$

$x = \frac{1}{2}$

→ original eqⁿ: $y = \frac{1}{2}e^{1-2(\frac{1}{2})}$

$= \frac{1}{2}e^{1-1}$

$= \frac{1}{2}e^{0} \rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$

$= \frac{1}{2}$

(b) Determine whether the stationary point is a maximum or a minimum. [2]

$$\frac{dy}{dx} = e^{1-2x}(1 - 2x)$$

$u = e^{1-2x} \qquad V = 1 - 2x$

$\frac{du}{dx} = -2e^{1-2x} \qquad \frac{dv}{dx} = -2$

$$\frac{d^2y}{dx^2} = -2e^{1-2x} - 2(1-2x)e^{1-2x}$$

$$= -2e^{1-2x}(1 + 1 - 2x)$$

$$= -2e^{1-2x}(2 - 2x)$$

$x = \frac{1}{2}$: $\frac{d^2y}{dx^2} = -2e^{0}\left(2 - 2\left(\frac{1}{2}\right)\right)$

$= -2(1) \rightarrow \frac{d^2y}{dx^2} < 0 \therefore$ Maximum

$= -2$

7 Let $f(x) = \dfrac{\cos x}{1 + \sin x}$.

(a) Show that $f'(x) < 0$ for all x in the interval $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$. [4]

$$f(x) = \frac{\cos x^{\,u}}{1 + \sin x_{\,v}}$$

$u = \cos x$ $\qquad$ $v = 1 + \sin x$

$\dfrac{du}{dx} = -\sin x$ $\qquad$ $\dfrac{dv}{dx} = \cos x$

$$f'(x) = \frac{(1+\sin x) \times -\sin x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \underbrace{(\sin^2 x + \cos^2 x)}_{=1}}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{(1+\sin x)}$$

$\sin x$ varies between -1 and 1, so for $-\frac{\pi}{2} < x < \frac{3\pi}{2}$, $-1 < \sin x < 1$.

This means $(1 + \sin x)$ is always > 0, so $\dfrac{-1}{(1+\sin x)}$ is always < 0.

4 The curve $y = e^{-4x} \tan x$ has two stationary points in the interval $0 \leqslant x < \frac{1}{2}\pi$.

(a) Obtain an expression for $\frac{dy}{dx}$ and show it can be written in the form $\sec^2 x(a + b \sin 2x)e^{-4x}$, where a and b are constants. [4]

$u \qquad v$

$y = e^{-4x} \tan x$

$u = e^{-4x} \qquad v = \tan x$

$\frac{du}{dx} = -4e^{-4x} \qquad \frac{dv}{dx} = \sec^2 x$

$\frac{dy}{dx} = e^{-4x} \sec^2 x + \tan x \times -4e^{-4x}$

$\frac{dy}{dx}$:

$e^{-4x} \sec^2 x - 4e^{-4x} \tan x$

$\frac{dy}{dx} = e^{-4x} \left(\sec^2 x - 4 \tan x \right)$

$= e^{-4x} \left(\frac{1}{\cos^2 x} - \frac{4 \sin x}{\cos x} \right)$

$= e^{-4x} \left(\frac{1}{\cos^2 x} - \frac{4 \sin x \cos x}{\cos^2 x} \right)$

$= e^{-4x} \left(\frac{1 - 4 \sin x \cos x}{\cos^2 x} \right)$ $\sin 2x = 2 \sin x \cos x$

$= e^{-4x} \left(\frac{1 - 2 \sin 2x}{\cos^2 x} \right)$

$= e^{-4x} \sec^2 x (1 - 2 \sin 2x)$

(b) Hence find the exact x-coordinates of the two stationary points. [3]

$\frac{dy}{dx} = 0$: $\quad e^{-4x}\sec^2 x(1 - 2\sin 2x) = 0$

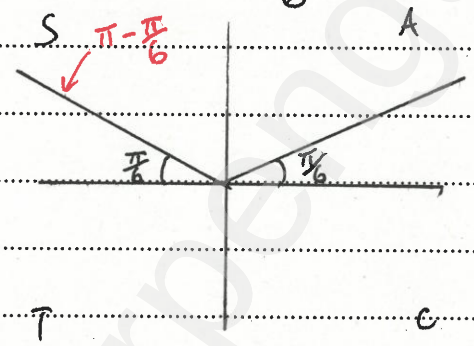
$e^{-4x} = 0 \quad$ or $\quad \sec^2 x = 0 \quad$ or $\quad 1 - 2\sin 2x = 0$

X not possible $\qquad \frac{1}{\cos^2 x} = 0$

X not possible

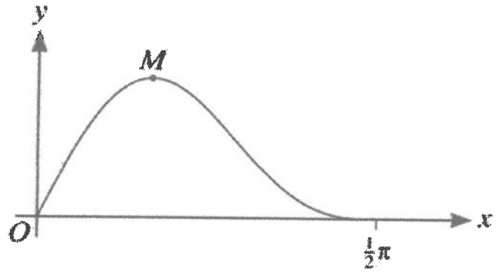
$$1 - 2\sin 2x = 0$$
$$2\sin 2x = 1$$
$$\sin 2x = \frac{1}{2}$$
$$2x = \frac{\pi}{6}$$

$$\boxed{\begin{gathered} 0 \leqslant x < \frac{\pi}{2} \\ 0 \leqslant 2x < \pi \end{gathered}}$$



$$2x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$



The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(b) Find the exact x-coordinate of M. **[6]**

u v

$y = \sin 2x \cos^2 x$

$u = \sin 2x$ $\quad\quad$ $v = \cos^2 x$

$\frac{du}{dx} = 2\cos 2x$ $\quad\quad$ $\frac{dv}{dx} = -2\cos x \sin x$

$\frac{dy}{dx} = -2\sin 2x \sin x \cos x + 2\cos 2x \cos^2 x$

$2\sin x\cos x$ $\quad\quad$ $2\cos^2 x - 1$

$\frac{dy}{dx} = 0$: $-2(2\sin x \cos x)\sin x \cos x + 2(2\cos^2 x - 1)\cos^2 x = 0$

$1 - \cos^2 x$

$-4\sin^2 x \cos^2 x + 4\cos^4 x - 2\cos^2 x = 0$

$-4(1-\cos^2 x)\cos^2 x + 4\cos^4 x - 2\cos^2 x = 0$

$-4\cos^2 x + 4\cos^4 x + 4\cos^4 x - 2\cos^2 x = 0$

$8\cos^4 x - 6\cos^2 x = 0$

$2\cos^2 x(4\cos^2 x - 3) = 0$

$2\cos^2 x = 0$ $\quad\quad$ $4\cos^2 x - 3 = 0$

$\cos^2 x = 0$ $\quad\quad$ $4\cos^2 x = 3$

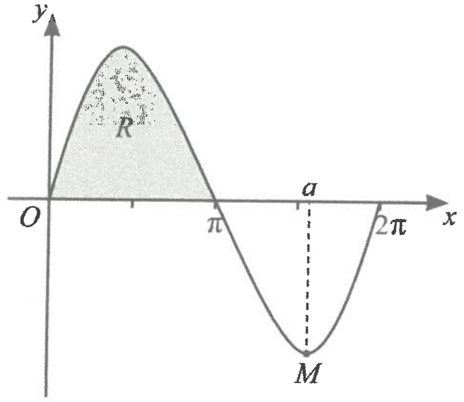
$\cos x = 0$ $\quad\quad$ $\cos^2 x = 3/4$

$x = \frac{\pi}{2}$ $\quad\quad$ $\cos x = \sqrt{\frac{3}{4}}$ or $\cos x = -\sqrt{\frac{3}{4}}$

X this is the min. pt. as seen on diagram

$\underline{x = \frac{\pi}{6}}$ $\quad\quad$ $x = \frac{5\pi}{6}$ × outside domain

11



The diagram shows the curve $y = 2\sin x\sqrt{2+\cos x}$, for $0 \leqslant x \leqslant 2\pi$, and its minimum point M, where $x = a$.

u v

(a) Find the value of a correct to 2 decimal places. [5]

$u = 2\sin x$ $\quad V = (2+\cos x)^{\frac{1}{2}}$

$\frac{du}{dx} = 2\cos x$ $\quad \frac{dv}{dx} = \frac{1}{2} \times -\sin x(2+\cos x)^{-\frac{1}{2}}$

$\frac{dy}{dx} = 2\sin x \times -\frac{1}{2}\sin x(2+\cos x)^{-\frac{1}{2}} + 2\cos x(2+\cos x)^{\frac{1}{2}}$

$= \frac{-\sin^2 x}{\sqrt{2+\cos x}} + 2\cos x\sqrt{2+\cos x}$

SP, so $\frac{dy}{dx} = 0$:

$\frac{-\sin^2 x}{\sqrt{2+\cos x}} + 2\cos x\sqrt{2+\cos x} = 0$ $\quad \times\sqrt{2+\cos x}$

$-\sin^2 x + 2\cos x(2+\cos x) = 0$

$\sin^2 x = 1-\cos^2 x$

$-(1-\cos^2 x) + 4\cos x + 2\cos^2 x = 0$

$-1 + \cos^2 x + 4\cos x + 2\cos^2 x = 0$

$3\cos^2 x + 4\cos x - 1 = 0$

Let $Y = \cos x$:

$3Y^2 + 4Y - 1 = 0$

cont...

$$y = \frac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(3)}$$

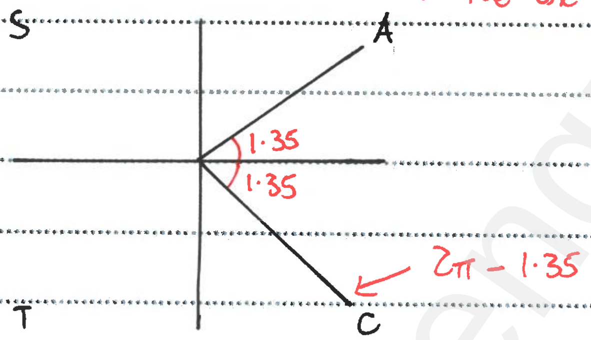
$$y = \frac{-2 + \sqrt{7}}{3} \quad \text{or} \quad y = \frac{-2 - \sqrt{7}}{3}$$

$$\cos x = \frac{-2 + \sqrt{7}}{3} \quad \text{or} \quad \cos x = \frac{-2 - \sqrt{7}}{3}$$

✗ no sol^ns

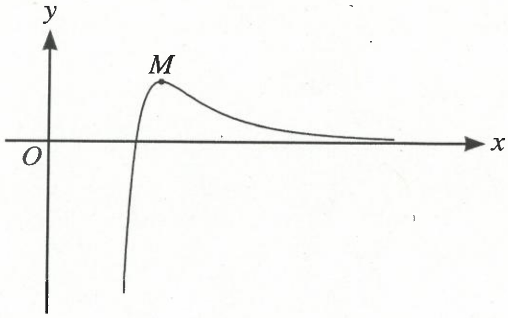
$$x = 1.35$$

this is the stationary point between 0 and π, so not the one we want.



$$a = 2\pi - 1.35$$
$$= \underline{\underline{4.93}}$$

8



The diagram shows the curve $y = \dfrac{\ln x}{x^4}$ and its maximum point M.

(a) Find the exact coordinates of M. [4]

$y = \dfrac{\ln x}{x^4}$ u v

$u = \ln x$ $\quad v = x^4$

$\dfrac{du}{dx} = \dfrac{1}{x}$ $\quad \dfrac{dv}{dx} = 4x^3$

$\dfrac{dy}{dx} = \dfrac{x^4 \times \frac{1}{x} - 4x^3 \ln x}{(x^4)^2}$

$\dfrac{dy}{dx} = 0: \quad \dfrac{x^3 - 4x^3 \ln x}{x^8} = 0$

$x^3 - 4x^3 \ln x = 0$

$x^3(1 - 4\ln x) = 0$

$x^3 = 0$ or $1 - 4\ln x = 0$

$x = 0$ $\quad 4\ln x = 1$

X not the point we want (see diagram) $\quad \ln x = \frac{1}{4}$

$x = e^{\frac{1}{4}}$

→ original eqn: $y = \dfrac{\ln(e^{\frac{1}{4}})}{(e^{\frac{1}{4}})^4}$

$= \dfrac{\frac{1}{4}}{e^1}$ ×4 ×4

$= \dfrac{1}{4e} = \frac{1}{4}e^{-1}$ → $(e^{\frac{1}{4}}, \frac{1}{4}e^{-1})$

11 Let $f(x) = \dfrac{2e^{2x}}{e^{2x} - 3e^{x} + 2}$ u v

(a) Find $f'(x)$ and hence find the exact coordinates of the stationary point of the curve with equation $y = f(x)$. [5]

$u = 2e^{2x}$ $\quad$ $v = e^{2x} - 3e^{x} + 2$

$\frac{du}{dx} = 4e^{2x}$ $\quad$ $\frac{dv}{dx} = 2e^{2x} - 3e^{x}$

$$\frac{dy}{dx} = \frac{(e^{2x} - 3e^{x} + 2) \times 4e^{2x} - 2e^{2x}(2e^{2x} - 3e^{x})}{(e^{2x} - 3e^{x} + 2)^2}$$

$$= \frac{4e^{4x} - 12e^{3x} + 8e^{2x} - 4e^{4x} + 6e^{3x}}{(e^{2x} - 3e^{x} + 2)^2}$$

$$= \frac{-6e^{3x} + 8e^{2x}}{(e^{2x} - 3e^{x} + 2)^2}$$

SP, so $\frac{dy}{dx} = 0$:

$$\frac{-6e^{3x} + 8e^{2x}}{(e^{2x} - 3e^{x} + 2)^2} = 0 \qquad \times (e^{2x} - 3e^{x} + 2)^2$$

$$-6e^{3x} + 8e^{2x} = 0$$

$$2e^{2x}(-3e^{x} + 4) = 0$$

$2e^{2x} = 0$ $\quad$ or $\quad$ $-3e^{x} + 4 = 0$

× No sol^s $\quad$ $3e^{x} = 4$

$e^{x} = 4/3$

$x = \ln 4/3$

sub. into $f(x)$:

$$y = \frac{2e^{2\ln 4/3}}{e^{2\ln 4/3} - 3e^{\ln 4/3} + 2}$$

$= -16$ $\rightarrow$ $\left(\ln \frac{4}{3}, -16\right)$