

- 5 Express $\frac{6x^2 - 2x + 2}{(x-1)(2x+1)}$ in partial fractions.

[5]

↑ improper fraction → algebraic division first

$$(x-1)(2x+1) = 2x^2 - x - 1$$

$$\begin{array}{r|l} 2x^2 - x - 1 & 3 \\ \hline & 6x^2 - 2x + 2 \\ & \underline{6x^2 - 3x - 3} \\ & x + 5 \end{array}$$

$$\rightarrow 3 + \frac{x+5}{(x-1)(2x+1)}$$

$$\frac{x+5}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$$

$$x+5 = A(2x+1) + B(x-1)$$

x=1:

$$1+5 = 3A + 0$$

$$6 = 3A$$

$$A = 2$$

x = -1/2:

$$-\frac{1}{2} + 5 = 0 + B\left(-\frac{1}{2} - 1\right)$$

$$\frac{9}{2} = -\frac{3}{2}B$$

$$9 = -3B$$

$$B = -3$$

$$\rightarrow \underline{\underline{3 + \frac{2}{x-1} - \frac{3}{2x+1}}}$$

- 2 Express $\frac{6x^2 - 9x - 16}{2x^2 - 5x - 12}$ in partial fractions.

[5]

Improper fraction $\rightarrow$ long division first.

$$\begin{array}{r} 3 \\ 2x^2 - 5x - 12 \overline{) 6x^2 - 9x - 16} \\ \underline{6x^2 - 15x - 36} \\ 6x + 20 \end{array}$$

$$f(x) = 3 + \frac{6x + 20}{2x^2 - 5x - 12}$$

factorise: $\rightarrow ac = -24 \rightarrow -8 \text{ and } +3$

$$2x^2 - 8x + 3x - 12$$

$$2x(x-4) + 3(x-4)$$

$$(2x+3)(x-4)$$

Partial Fractions:

$$\frac{6x+20}{(2x+3)(x-4)} = \frac{A}{2x+3} + \frac{B}{x-4}$$

$$6x+20 = A(x-4) + B(2x+3)$$

$x=4$:

$$6(4) + 20 = 0 + B(2(4) + 3)$$

$$44 = 11B$$

$$\underline{B = 4}$$

$x = -\frac{3}{2}$:

$$6\left(-\frac{3}{2}\right) + 20 = A\left(-\frac{3}{2} - 4\right) + 0$$

$$11 = -\frac{11}{2}A$$

$$22 = -11A$$

$$\underline{A = -2}$$

$$\rightarrow 3 - \frac{2}{2x+3} + \frac{4}{x-4}$$

9 Let $f(x) = \frac{2 + 11x - 10x^2}{(1 + 2x)(1 - 2x)(2 + x)}$.

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{2 + 11x - 10x^2}{(1 + 2x)(1 - 2x)(2 + x)} = \frac{A}{1 + 2x} + \frac{B}{1 - 2x} + \frac{C}{2 + x}$$

$$2 + 11x - 10x^2 = A(1 - 2x)(2 + x) + B(1 + 2x)(2 + x) + C(1 + 2x)(1 - 2x)$$

$x = -2$:

$$2 + 11(-2) - 10(-2)^2 = 0 + 0 + C(-3)(5)$$

$$2 - 22 - 40 = -15C$$

$$-60 = -15C$$

$$C = 4$$

$x = \frac{1}{2}$:

$$2 + 11\left(\frac{1}{2}\right) - 10\left(\frac{1}{2}\right)^2 = 0 + B(2)\left(\frac{5}{2}\right) + 0$$

$$2 + \frac{11}{2} - \frac{5}{2} = 5B$$

$$5 = 5B$$

$$B = 1$$

$x = -\frac{1}{2}$:

$$2 + 11\left(-\frac{1}{2}\right) - 10\left(-\frac{1}{2}\right)^2 = A(2)\left(\frac{3}{2}\right) + 0 + 0$$

$$2 - \frac{11}{2} - \frac{5}{2} = 3A$$

$$-6 = 3A$$

$$A = -2$$

$$f(x) = \frac{-2}{1 + 2x} + \frac{1}{1 - 2x} + \frac{4}{2 + x}$$

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [5]

$$f(x) = \overset{\textcircled{A}}{-2(1+2x)^{-1}} + \overset{\textcircled{B}}{(1-2x)^{-1}} + \overset{\textcircled{C}}{4(2+x)^{-1}}$$

$$\begin{aligned} \textcircled{A} \quad -2(1+2x)^{-1} &= -2 \left[1 + (-1)(2x) + \frac{(-1)(-2)}{2!} (2x)^2 + \dots \right] \\ &= -2 \left[1 - 2x + 4x^2 \right] \\ &= -2 + 4x - 8x^2 \end{aligned}$$

$$\begin{aligned} \textcircled{B} \quad (1-2x)^{-1} &= 1 + (-1)(-2x) + \frac{(-1)(-2)}{2!} (-2x)^2 + \dots \\ &= 1 + 2x + 4x^2 \end{aligned}$$

$$\begin{aligned} \textcircled{C} \quad 4(2+x)^{-1} &= 4 \times 2^{-1} \left(1 + \frac{x}{2} \right)^{-1} \\ &= 4 \times \frac{1}{2} \left(1 + \frac{x}{2} \right)^{-1} \\ &= 2 \left(1 + \frac{x}{2} \right)^{-1} \\ 2 \left(1 + \frac{x}{2} \right)^{-1} &= 2 \left[1 + (-1) \left(\frac{x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2} \right)^2 + \dots \right] \\ &= 2 \left[1 - \frac{x}{2} + \frac{x^2}{4} \right] \\ &= 2 - x + \frac{x^2}{2} \end{aligned}$$

$$f(x) = \overset{\textcircled{A}}{-2 + 4x - 8x^2} + \overset{\textcircled{B}}{1 + 2x + 4x^2} + \overset{\textcircled{C}}{2 - x + \frac{x^2}{2}}$$

$$= \underline{\underline{1 + 5x - \frac{7}{2}x^2}}$$

10 Let $f(x) = \frac{21 - 8x - 2x^2}{(1 + 2x)(3 - x)^2}$.

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{21 - 8x - 2x^2}{(1 + 2x)(3 - x)^2} = \frac{A}{1 + 2x} + \frac{B}{3 - x} + \frac{C}{(3 - x)^2}$$

$$21 - 8x - 2x^2 = A(3 - x)^2 + B(1 + 2x)(3 - x) + C(1 + 2x)$$

$x = 3$:

$$21 - 8(3) - 2(3)^2 = 0 + 0 + C(7)$$

$$21 - 24 - 18 = 7C$$

$$-21 = 7C$$

$$C = -3$$

$x = -\frac{1}{2}$:

$$21 - 8\left(-\frac{1}{2}\right) - 2\left(-\frac{1}{2}\right)^2 = A(3 - (-\frac{1}{2}))^2 + 0 + 0$$

$$21 + 4 - \frac{1}{2} = \left(\frac{7}{2}\right)^2 A$$

$$\frac{49}{2} = \frac{49}{4} A$$

$$A = 2$$

$x = 0$:

$$21 - 0 - 0 = A(3)^2 + B(1)(3) + C(1)$$

$$21 = 9A + 3B + C$$

$$21 = 9(2) + 3B + (-3)$$

$$21 = 15 + 3B$$

$$3B = 6$$

$$B = 2$$

$$\rightarrow \frac{2}{1 + 2x} + \frac{2}{3 - x} - \frac{3}{(3 - x)^2}$$

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [5]

$$\begin{aligned} \textcircled{A} \quad 2(1+2x)^{-1} &= 2\left[1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \dots\right] \\ &= 2[1 - 2x + 4x^2 + \dots] \\ &= 2 - 4x + 8x^2 \end{aligned}$$

$$\begin{aligned} \textcircled{B} \quad 2(3-x)^{-1} &= 2 \times 3^{-1} \left(1 - \frac{x}{3}\right)^{-1} \\ &= \frac{2}{3} \left(1 - \frac{x}{3}\right)^{-1} \\ &= \frac{2}{3} \left[1 + (-1)\left(-\frac{x}{3}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{3}\right)^2 + \dots\right] \\ &= \frac{2}{3} \left[1 + \frac{x}{3} + \frac{x^2}{9} + \dots\right] \\ &= \frac{2}{3} + \frac{2}{9}x + \frac{2}{27}x^2 \end{aligned}$$

$$\begin{aligned} \textcircled{C} \quad -3(3-x)^{-2} &= -3 \times 3^{-2} \left(1 - \frac{x}{3}\right)^{-2} \\ &= -\frac{1}{3} \left(1 - \frac{x}{3}\right)^{-2} \\ &= -\frac{1}{3} \left[1 + (-2)\left(-\frac{x}{3}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{x}{3}\right)^2 + \dots\right] \\ &= -\frac{1}{3} \left[1 + \frac{2}{3}x + 3 \times \frac{x^2}{9}\right] \\ &= -\frac{1}{3} \left[1 + \frac{2}{3}x + \frac{1}{3}x^2\right] \\ &= -\frac{1}{3} - \frac{2}{9}x - \frac{1}{9}x^2 \end{aligned}$$

$\textcircled{A} + \textcircled{B} + \textcircled{C}$:

$$\begin{aligned} &2 - 4x + 8x^2 + \frac{2}{3} + \frac{2}{9}x + \frac{2}{27}x^2 - \frac{1}{3} - \frac{2}{9}x - \frac{1}{9}x^2 \\ &= \underline{\underline{\frac{7}{3} - 4x + \frac{25}{27}x^2}} \end{aligned}$$

10 Let $f(x) = \frac{24x + 13}{(1 - 2x)(2 + x)^2}$.

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{24x + 13}{(1 - 2x)(2 + x)^2} \equiv \frac{A}{1 - 2x} + \frac{B}{2 + x} + \frac{C}{(2 + x)^2}$$

$$24x + 13 \equiv A(2 + x)^2 + B(1 - 2x)(2 + x) + C(1 - 2x)$$

$x = -2$:

$$24(-2) + 13 = 0 + 0 + C(5)$$

$$-35 = 5C$$

$$\underline{C = -7}$$

$x = \frac{1}{2}$:

$$24\left(\frac{1}{2}\right) + 13 = A\left(\frac{5}{2}\right)^2 + 0 + 0$$

$$12 + 13 = \frac{25}{4}A$$

$$25 = \frac{25}{4}A$$

$$\underline{A = 4}$$

$x = 0$:

$$24(0) + 13 = A(2)^2 + B(1)(2) + C(1)$$

$$13 = 4A + 2B + C$$

$$13 = 4(4) + 2B - 7$$

$$13 = 16 + 2B - 7$$

$$13 = 9 + 2B$$

$$2B = 4$$

$$\underline{B = 2}$$

$$\rightarrow \underline{\underline{\frac{4}{1 - 2x} + \frac{2}{2 + x} - \frac{7}{(2 + x)^2}}}$$

- (b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [5]

$$\begin{aligned} \textcircled{A} \quad 4(1-2x)^{-1} &= 4 \left[1 + (-1)(-2x) + \frac{(-1)(-2)}{2!} (-2x)^2 + \dots \right] \\ &= 4 [1 + 2x + 4x^2 + \dots] \\ &= 4 + 8x + 16x^2 \end{aligned}$$

$$\begin{aligned} \textcircled{B} \quad 2(2+x)^{-1} &= 2 \times 2^{-1} \left(1 + \frac{x}{2} \right)^{-1} \\ &= \left(1 + \frac{x}{2} \right)^{-1} \\ &= 1 + (-1) \left(\frac{x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2} \right)^2 \\ &= 1 - \frac{x}{2} + \frac{x^2}{4} \end{aligned}$$

$$\begin{aligned} \textcircled{C} \quad -7(2+x)^{-2} &= -7 \times 2^{-2} \left(1 + \frac{x}{2} \right)^{-2} \\ &= \frac{-7}{4} \left[1 + (-2) \left(\frac{x}{2} \right) + \frac{(-2)(-3)}{2!} \left(\frac{x}{2} \right)^2 + \dots \right] \\ &= \frac{-7}{4} \left[1 - x + \frac{3}{4} x^2 + \dots \right] \\ &= \frac{-7}{4} + \frac{7}{4} x - \frac{21}{16} x^2 \end{aligned}$$

$\textcircled{A} + \textcircled{B} + \textcircled{C}$:

$$4 + 8x + 16x^2 + 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{7}{4} + \frac{7}{4}x - \frac{21}{16}x^2$$

$$= \underline{\underline{\frac{13}{4} + \frac{37}{4}x + \frac{239}{16}x^2}}$$

- (c) State the set of values of x for which the expansion in (b) is valid. [1]

$$| -2x | < 1 \quad \text{and} \quad \left| \frac{x}{2} \right| < 1$$

$$|x| < \frac{1}{2} \quad \rightarrow \quad |x| < 2$$

$$\text{so } \underline{\underline{|x| < \frac{1}{2}}}$$



7

$$\text{Let } f(x) = \frac{5x^2 + 8x + 5}{(1+2x)(2+x^2)}.$$

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{5x^2 + 8x + 5}{(1+2x)(2+x^2)} = \frac{A}{1+2x} + \frac{Bx + C}{2+x^2}$$

$$5x^2 + 8x + 5 = A(2+x^2) + (Bx + C)(1+2x)$$

$$x = -\frac{1}{2}:$$

$$5\left(-\frac{1}{2}\right)^2 + 8\left(-\frac{1}{2}\right) + 5 = A\left(2 + \left(-\frac{1}{2}\right)^2\right) + \left(-\frac{1}{2}B + C\right) \times 0$$

$$\frac{5}{4} - 4 + 5 = \frac{9}{4}A + 0$$

$$\frac{9}{4} = \frac{9}{4}A$$

$$A = 1$$

$$x = 0:$$

$$0 + 0 + 5 = A(2) + C(1)$$

$$5 = 2A + C$$

$$\leftarrow A=1$$

$$5 = 2 + C$$

$$C = 3$$

$$x = 1:$$

$$5 + 8 + 5 = A(2+1^2) + (B+C)(1+2)$$

$$18 = 3A + (B+C)(3)$$

$$18 = 3A + 3B + 3C$$

$$\leftarrow A=1 \quad \leftarrow C=3$$

$$18 = 3 + 3B + 9$$

$$6 = 3B$$

$$B = 2$$

$$f(x) = \frac{1}{1+2x} + \frac{2x+3}{2+x^2}$$



(b) Hence find the coefficient of x^3 in the expansion of $f(x)$.

[4]

$$f(x) = (1 + 2x)^{-1} + (2x + 3)(2 + x^2)^{-1}$$

$$\textcircled{A}: (1 + 2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \frac{(-1)(-2)(-3)}{3!}(2x)^3 + \dots$$

$$x^3 \text{ term: } \frac{(-1)(-2)(-3)}{3!}(2x)^3 = -1 \times 8x^3$$

$$= \underline{-8x^3}$$

$$\textcircled{B}: (2x + 3)(2 + x^2)^{-1} = (2x + 3) \times 2^{-1} \left(1 + \frac{x^2}{2}\right)^{-1}$$

$$= \frac{1}{2}(2x + 3) \left(1 + \frac{x^2}{2}\right)^{-1}$$

$$= \frac{1}{2}(2x + 3) \left[1 + (-1)\left(\frac{x^2}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x^2}{2}\right)^2 + \dots \right]$$

$$= \left(x + \frac{3}{2}\right) \left[1 - \frac{x^2}{2} + \frac{x^4}{4} + \dots \right]$$

$$x^3 \text{ term: } x \times \frac{-x^2}{2} = \underline{-\frac{1}{2}x^3}$$

x^3 term for whole thing:

$$-8x^3 - \frac{1}{2}x^3 = \underline{-\frac{17}{2}x^3}$$

$$\text{coeff only: } \underline{\underline{-\frac{17}{2}}}$$



9 Let $f(x) = \frac{8+5x+12x^2}{(1-x)(2+3x)^2}$.

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{8+5x+12x^2}{(1-x)(2+3x)^2} = \frac{A}{1-x} + \frac{B}{2+3x} + \frac{C}{(2+3x)^2}$$

$$8+5x+12x^2 = A(2+3x)^2 + B(1-x)(2+3x) + C(1-x)$$

$x=1$:

$$8+5+12 = A(5)^2 + 0 + 0$$

$$25 = 25A$$

$$A = 1$$

$x = -\frac{2}{3}$:

$$8+5\left(-\frac{2}{3}\right)+12\left(-\frac{2}{3}\right)^2 = 0 + 0 + C\left(1-\frac{-2}{3}\right)$$

$$8 - \frac{10}{3} + 12 \times \frac{4}{9} = \frac{5}{3}C$$

$$8 - \frac{10}{3} + \frac{16}{3} = \frac{5}{3}C$$

$$10 = \frac{5}{3}C$$

$$C = 6$$

$x=0$:

$$8+0+0 = A(2)^2 + B(1)(2) + C(1)$$

$$8 = 1 \times 4 + 2B + 6 \times 1$$

$$8 = 4 + 2B + 6$$

$$8 = 10 + 2B$$

$$-2 = 2B$$

$$B = -1$$

$$\rightarrow \frac{1}{1-x} + \frac{-1}{2+3x} + \frac{6}{(2+3x)^2}$$

- (b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [5]

$$f(x) = \frac{1}{1-x} + \frac{-1}{2+3x} + \frac{6}{(2+3x)^2}$$

$$\begin{aligned} \textcircled{A}: (1-x)^{-1} &= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 \\ &= 1 + x + x^2 \end{aligned}$$

$$\textcircled{B}: -1(2+3x)^{-1} = -1 \times 2^{-1} \left(1 + \frac{3x}{2}\right)^{-1} = -1 \times \frac{1}{2} \left(1 + \frac{3x}{2}\right)^{-1}$$

$$= \frac{-1}{2} \left(1 + \frac{3x}{2}\right)^{-1} = \frac{-1}{2} \left[1 + (-1)\left(\frac{3x}{2}\right) + \frac{(-1)(-2)}{2!} \left(\frac{3x}{2}\right)^2 + \dots\right]$$

$$= \frac{-1}{2} \left[1 - \frac{3x}{2} + \frac{9x^2}{4}\right]$$

$$= \frac{-1}{2} + \frac{3x}{4} - \frac{9x^2}{8}$$

$$\textcircled{C}: 6(2+3x)^{-2} = 6 \times 2^{-2} \left(1 + \frac{3x}{2}\right)^{-2} = 6 \times \frac{1}{4} \left(1 + \frac{3x}{2}\right)^{-2}$$

$$= \frac{3}{2} \left(1 + \frac{3x}{2}\right)^{-2} = \frac{3}{2} \left[1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-3)}{2!} \left(\frac{3x}{2}\right)^2 + \dots\right]$$

$$= \frac{3}{2} \left[1 - 3x + \frac{27}{4}x^2\right]$$

$$= \frac{3}{2} - \frac{9}{2}x + \frac{81}{8}x^2$$

$$\textcircled{A} + \textcircled{B} + \textcircled{C}: 1 + x + x^2 - \frac{1}{2} + \frac{3x}{4} - \frac{9x^2}{8} + \frac{3}{2} - \frac{9}{2}x + \frac{81}{8}x^2$$

$$= \underline{\underline{2 - \frac{11}{4}x + 10x^2}}$$

9 Let $f(x) = \frac{14 - 3x + 2x^2}{(2+x)(3+x^2)}$.

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{14 - 3x + 2x^2}{(2+x)(3+x^2)} = \frac{A}{2+x} + \frac{Bx+C}{3+x^2}$$

$$14 - 3x + 2x^2 = A(3+x^2) + (Bx+C)(2+x)$$

$x = -2$:

$$14 - 3(-2) + 2(-2)^2 = A(3 + (-2)^2) + 0$$

$$14 + 6 + 8 = 7A$$

$$28 = 7A$$

$$\underline{A = 4}$$

$x = 0$:

$$14 - 0 + 0 = A(3+0) + (B \times 0 + C)(2+0)$$

$$14 = 3A + 2C$$

$\leftarrow A=4$

$$14 = 12 + 2C$$

$$2 = 2C$$

$$\underline{C = 1}$$

$x = 1$:

$$14 - 3 + 2 = A(3+1) + (B+C)(3)$$

$$13 = 4A + 3B + 3C$$

$\leftarrow A=4$

$\leftarrow C=1$

$$13 = 16 + 3B + 3$$

$$13 = 19 + 3B$$

$$-6 = 3B$$

$$\underline{B = -2}$$

$$\rightarrow \frac{4}{2+x} + \frac{-2x+1}{3+x^2}$$

- (b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [5]

$$f(x) = \frac{4}{2+x} + \frac{-2x+1}{3+x^2}$$

$$\textcircled{A}: 4(2+x)^{-1} = 4 \times 2^{-1} \left(1 + \frac{x}{2}\right)^{-1} = 4 \times \frac{1}{2} \left(1 + \frac{x}{2}\right)^{-1} = 2 \left(1 + \frac{x}{2}\right)^{-1}$$

$$\begin{aligned} 2 \left(1 + \frac{x}{2}\right)^{-1} &= 2 \left[1 + (-1) \left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2}\right)^2 + \dots \right] \\ &= 2 \left[1 - \frac{x}{2} + \frac{x^2}{4} \right] \\ &= 2 - x + \frac{x^2}{2} \end{aligned}$$

$$\textcircled{B}: (-2x+1)(3+x^2)^{-1} = (-2x+1) \times 3^{-1} \left(1 + \frac{x^2}{3}\right)^{-1} = (-2x+1) \times \frac{1}{3} \left(1 + \frac{x^2}{3}\right)^{-1}$$

$$(-2x+1) \times \frac{1}{3} \left(1 + \frac{x^2}{3}\right)^{-1} = (-2x+1) \times \frac{1}{3} \left[1 + (-1) \left(\frac{x^2}{3}\right) + \dots \right]$$

$$\begin{aligned} &= (-2x+1) \times \frac{1}{3} \left(1 - \frac{x^2}{3}\right) \\ &= (-2x+1) \times \left(\frac{1}{3} - \frac{x^2}{9}\right) \end{aligned}$$

$$= \frac{-2x}{3} + \frac{\cancel{2x^3}}{9} + \frac{1}{3} - \frac{x^2}{9} \quad \text{--- } x^2$$

$$\textcircled{A} + \textcircled{B}: 2 - x + \frac{x^2}{2} + \frac{1}{3} - \frac{2x}{3} - \frac{x^2}{9}$$

$$= \frac{7}{3} - \frac{5x}{3} + \frac{7x^2}{18}$$

10 Let $f(x) = \frac{2x^2 + 7x + 8}{(1+x)(2+x)^2}$.

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{2x^2 + 7x + 8}{(1+x)(2+x)^2} = \frac{A}{1+x} + \frac{B}{2+x} + \frac{C}{(2+x)^2}$$

$$2x^2 + 7x + 8 = A(2+x)^2 + B(1+x)(2+x) + C(1+x)$$

$x = -1$:

$$2(-1)^2 + 7(-1) + 8 = A(1)^2 + 0 + 0$$

$$2 - 7 + 8 = A$$

$$A = 3$$

$x = -2$:

$$2(-2)^2 + 7(-2) + 8 = 0 + 0 + C(-1)$$

$$2 \times 4 - 14 + 8 = -C$$

$$2 = -C$$

$$C = -2$$

$x = 0$:

$$0 + 0 + 8 = A(2)^2 + B(1)(2) + C(1)$$

$$8 = 4A + 2B + C$$

$$\quad \quad \quad \nwarrow A=3 \quad \quad \quad \nwarrow C=-2$$

$$8 = 12 + 2B - 2$$

$$8 = 10 + 2B$$

$$-2 = 2B$$

$$B = -1$$

$$\rightarrow \frac{3}{1+x} + \frac{-1}{2+x} + \frac{-2}{(2+x)^2}$$

- (b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [5]

$$f(x) = \textcircled{A} \frac{3}{1+x} + \textcircled{B} \frac{-1}{2+x} + \textcircled{C} \frac{-2}{(2+x)^2}$$

$$\begin{aligned} \textcircled{A}: 3(1+x)^{-1} &= 3 \left[1 + (-1)(x) + \frac{(-1)(-2)}{2!}(x)^2 + \dots \right] \\ &= 3[1 - x + x^2] \\ &= \underline{3 - 3x + 3x^2} \end{aligned}$$

$$\begin{aligned} \textcircled{B}: -1(2+x)^{-1} &= -1 \times 2^{-1} \left(1 + \frac{x}{2}\right)^{-1} = -1 \times \frac{1}{2} \left(1 + \frac{x}{2}\right)^{-1} \\ \frac{-1}{2} \left(1 + \frac{x}{2}\right)^{-1} &= \frac{-1}{2} \left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2}\right)^2 + \dots \right] \\ &= \frac{-1}{2} \left[1 - \frac{x}{2} + \frac{x^2}{2} \right] \\ &= \underline{\frac{-1}{2} + \frac{x}{4} - \frac{x^2}{8}} \end{aligned}$$

$$\begin{aligned} \textcircled{C}: -2(2+x)^{-2} &= -2 \times 2^{-2} \left(1 + \frac{x}{2}\right)^{-2} = -2 \times \frac{1}{4} \left(1 + \frac{x}{2}\right)^{-2} \\ \frac{-1}{2} \left(1 + \frac{x}{2}\right)^{-2} &= \frac{-1}{2} \left[1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!} \left(\frac{x}{2}\right)^2 + \dots \right] \\ &= \frac{-1}{2} \left[1 - x + \frac{3}{2}x^2 \right] \\ &= \underline{\frac{-1}{2} + \frac{1}{2}x - \frac{3}{8}x^2} \end{aligned}$$

$$\begin{aligned} \textcircled{A} + \textcircled{B} + \textcircled{C}: 3 - 3x + 3x^2 - \frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} - \frac{1}{2} + \frac{1}{2}x - \frac{3}{8}x^2 \\ = \underline{2 - \frac{9}{4}x + \frac{5}{2}x^2} \end{aligned}$$

9 Let $f(x) = \frac{17x^2 - 7x + 16}{(2 + 3x^2)(2 - x)}$.

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{17x^2 - 7x + 16}{(2 + 3x^2)(2 - x)} = \frac{Ax + B}{(2 + 3x^2)} + \frac{C}{2 - x}$$

$$17x^2 - 7x + 16 = (Ax + B)(2 - x) + C(2 + 3x^2)$$

$x = 2$:

$$17(2)^2 - 7(2) + 16 = 0 + C(2 + 3(2)^2)$$

$$68 - 14 + 16 = C(2 + 12)$$

$$70 = 14C$$

$$\underline{C = 5}$$

$x = 0$:

$$17(0)^2 - 7(0) + 16 = (0 + B)(2) + C(2)$$

$$16 = 2B + 2C$$

$$16 = 2B + 10$$

$$2B = 6$$

$$\underline{B = 3}$$

$x = 1$:

$$17(1)^2 - 7(1) + 16 = (A + B)(1) + C(5)$$

$$17 - 7 + 16 = A + B + 5C$$

$$26 = A + 3 + 25$$

$$26 = A + 28$$

$$\underline{A = -2}$$

$$\rightarrow \underline{\underline{\frac{-2x + 3}{(2 + 3x^2)} + \frac{5}{2 - x}}}$$

- (b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^3 . [5]

$$\begin{aligned}
 \textcircled{A} \quad (-2x+3)(2+3x^2)^{-1} &= (-2x+3) \times 2^{-1} \left(1 + \frac{3x^2}{2}\right)^{-1} \\
 &= \frac{1}{2}(-2x+3) \left(1 + \frac{3x^2}{2}\right)^{-1} \\
 &= \frac{1}{2}(-2x+3) \left[1 + (-1)\left(\frac{3x^2}{2}\right) + \dots\right] \\
 &= \frac{1}{2}(-2x+3) \left[1 - \frac{3x^2}{2}\right] \\
 &= \frac{1}{2}(-2x + 3x^3 + 3 - \frac{9}{2}x^2) \\
 &= \frac{3}{2} - x - \frac{9}{4}x^2 + \frac{3}{2}x^3
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{B} \quad 5(2-x)^{-1} &= 5 \times 2^{-1} \left(1 - \frac{x}{2}\right)^{-1} \\
 &= \frac{5}{2} \left(1 - \frac{x}{2}\right)^{-1} \\
 &= \frac{5}{2} \left[1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!} \left(-\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(-\frac{x}{2}\right)^3 + \dots\right] \\
 &= \frac{5}{2} \left[1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8}\right] \\
 &= \frac{5}{2} + \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{16}x^3
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{A} + \textcircled{B}: \quad &\frac{3}{2} - x - \frac{9}{4}x^2 + \frac{3}{2}x^3 + \frac{5}{2} + \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{16}x^3 \\
 &= \underline{\underline{4 + \frac{1}{4}x - \frac{13}{8}x^2 + \frac{29}{16}x^3}}
 \end{aligned}$$

- (c) State the set of values of x for which the expansion in (b) is valid. Give your answer in an exact form. [1]

$$\begin{aligned}
 \left|\frac{3x^2}{2}\right| &< 1 & \text{and} & \left|\frac{-x}{2}\right| < 1 \\
 |3x^2| &< 2 & & |x| < 2 \\
 |x^2| &< \frac{2}{3} & & \downarrow \\
 |x| &< \sqrt{\frac{2}{3}} & & \downarrow \\
 & \rightarrow \text{so } |x| &< \sqrt{\frac{2}{3}}
 \end{aligned}$$

7 Let $f(x) = \frac{5x^2 + 8x - 3}{(x-2)(2x^2+3)}$.

(a) Express $f(x)$ in partial fractions.

[5]

$$\frac{5x^2 + 8x - 3}{(x-2)(2x^2+3)} = \frac{A}{x-2} + \frac{Bx+C}{2x^2+3}$$

$$5x^2 + 8x - 3 = A(2x^2+3) + (Bx+C)(x-2)$$

$x=2$:

$$5(2)^2 + 8(2) - 3 = A(2(2)^2 + 3) + 0$$

$$20 + 16 - 3 = A(8 + 3)$$

$$33 = 11A$$

$$A = 3$$

$x=0$:

$$0 + 0 - 3 = A(3) + C(-2)$$

$$-3 = 9 - 2C$$

$$-12 = -2C$$

$$C = 6$$

$x=1$:

$$5 + 8 - 3 = A(2+3) + (B+C)(-1)$$

$$10 = 3 \times 5 + (B+6) \times -1$$

$$10 = 15 - B - 6$$

$$10 = 9 - B$$

$$1 = -B$$

$$B = -1$$

$$\rightarrow \frac{3}{x-2} + \frac{-x+6}{2x^2+3}$$

- (b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [5]

$$f(x) = \frac{3}{x-2} + \frac{-x+6}{2x^2+3}$$

$$\textcircled{A}: 3(x-2)^{-1} = 3(-2+x)^{-1} = 3 \times (-2)^{-1} \left(1 - \frac{x}{2}\right)^{-1} = 3 \times \frac{-1}{2} \left(1 - \frac{x}{2}\right)^{-1}$$

$$\frac{-3}{2} \left(1 - \frac{x}{2}\right)^{-1} = \frac{-3}{2} \left[1 + (-1) \left(\frac{-x}{2}\right) + \frac{(-1)(-2)}{2!} \left(\frac{-x}{2}\right)^2 + \dots \right]$$

$$= \frac{-3}{2} \left[1 + \frac{x}{2} + \frac{x^2}{4} \right]$$

$$= \frac{-3}{2} - \frac{3x}{4} - \frac{3x^2}{8}$$

$$\textcircled{B}: (-x+6)(2x^2+3)^{-1} = (-x+6)(3+2x^2)^{-1} = (-x+6) \times 3^{-1} \left(1 + \frac{2x^2}{3}\right)^{-1}$$

$$(-x+6) \times \frac{1}{3} \left(1 + \frac{2x^2}{3}\right)^{-1} = (-x+6) \times \frac{1}{3} \left[1 + (-1) \left(\frac{2x^2}{3}\right) + \dots \right]$$

$$= (-x+6) \times \frac{1}{3} \left[1 - \frac{2x^2}{3} \right]$$

$$= (-x+6) \times \left(\frac{1}{3} - \frac{2x^2}{9} \right)$$

$$= -\frac{1}{3}x + \frac{2x^2}{9} + 2 - \frac{4x^2}{3}$$

$$\textcircled{A} + \textcircled{B}: \frac{-3}{2} - \frac{3x}{4} - \frac{3x^2}{8} + 2 - \frac{x}{3} - \frac{4x^2}{3}$$

$$= \frac{1}{2} - \frac{13}{12}x - \frac{41}{24}x^2$$



- 8 Let $f(x) = \frac{7a^2}{(a-2x)(3a+x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions.

[3]

$$\frac{7a^2}{(a-2x)(3a+x)} \equiv \frac{A}{a-2x} + \frac{B}{3a+x}$$

$$7a^2 \equiv A(3a+x) + B(a-2x)$$

$$x = \frac{a}{2}$$

$$7a^2 = A\left(3a + \frac{a}{2}\right) + 0$$

$$7a^2 = A \times \frac{7a}{2}$$

$$14a^2 = A \times 7a$$

$$14a = A \times 7$$

$$A = 2a$$

$$x = -3a$$

$$7a^2 = 0 + B(a - 2(-3a))$$

$$7a^2 = B(7a)$$

$$7a^2 = B \times 7a$$

$$7a = B \times 7$$

$$B = a$$

$$\rightarrow f(x) = \frac{2a}{a-2x} + \frac{a}{3a+x}$$



- (b) Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [4]

$$f(x) = 2a(a-2x)^{-1} + a(3a+x)^{-1}$$

$$\begin{aligned} \textcircled{A}: 2a(a-2x)^{-1} &= 2a \times a^{-1} \left(1 - \frac{2x}{a}\right)^{-1} = 2 \left(1 - \frac{2x}{a}\right)^{-1} \\ &= 2 \left[1 + (-1) \left(-\frac{2x}{a}\right) + \frac{(-1)(-2)}{2!} \left(-\frac{2x}{a}\right)^2 + \dots \right] \\ &= 2 \left[1 + \frac{2x}{a} + \frac{4x^2}{a^2} + \dots \right] \\ &= 2 + \frac{4x}{a} + \frac{8x^2}{a^2} + \dots \end{aligned}$$

$$\begin{aligned} \textcircled{B}: a(3a+x)^{-1} &= a \times (3a)^{-1} \times \left(1 + \frac{x}{3a}\right)^{-1} = \frac{1}{3} \left(1 + \frac{x}{3a}\right)^{-1} \\ &= \frac{1}{3} \left[1 + (-1) \left(\frac{x}{3a}\right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{3a}\right)^2 + \dots \right] \\ &= \frac{1}{3} \left[1 - \frac{x}{3a} + \frac{x^2}{9a^2} + \dots \right] \\ &= \frac{1}{3} - \frac{x}{9a} + \frac{x^2}{27a^2} + \dots \end{aligned}$$

$$\textcircled{A} + \textcircled{B}: 2 + \frac{4x}{a} + \frac{8x^2}{a^2} + \frac{1}{3} - \frac{x}{9a} + \frac{x^2}{27a^2}$$

$$= 2 + \frac{1}{3} + \frac{4x \times 9}{a \times 9} - \frac{x}{9a} + \frac{8x^2 \times 27}{a^2 \times 27} + \frac{x^2}{27a^2}$$

$$= \frac{7}{3} + \frac{36x}{9a} - \frac{x}{9a} + \frac{216x^2}{27a^2} + \frac{x^2}{27a^2}$$

$$= \frac{7}{3} + \frac{35x}{9a} + \frac{217x^2}{27a^2}$$

- (c) State the set of values of x for which the expansion in part (b) is valid. [1]

$$\textcircled{A}: \left| \frac{-2x}{a} \right| < 1 \quad \text{and} \quad \textcircled{B}: \left| \frac{x}{3a} \right| < 1$$

$$|2x| < a$$

$$|x| < 3a$$

$$|x| < \frac{a}{2}$$

$$\text{so } |x| < \frac{a}{2}$$