

- 3 The equation of a curve is such that $\frac{dy}{dx} = \frac{1}{2}x + \frac{72}{x^4}$. The curve passes through the point $P(2, 8)$.

(a) Find the equation of the normal to the curve at P .

[2]

gradient at P :

$$\begin{aligned} m &= \frac{1}{2}(2) + \frac{72}{2^4} \\ &= 1 + \frac{9}{2} \\ &= \frac{11}{2} \end{aligned}$$

gradient of normal: $= -\frac{2}{11}$

$$y - y_1 = m(x - x_1)$$

$$y - 8 = -\frac{2}{11}(x - 2)$$

$$y - 8 = -\frac{2}{11}x + \frac{4}{11}$$

$$\underline{y = -\frac{2}{11}x + \frac{92}{11}}$$

(b) Find the equation of the curve.

[4]

Integrate: $\frac{dy}{dx} = \frac{1}{2}x + 72x^{-4}$

$$y = \frac{1}{4}x^2 - 24x^{-3} + C$$

Sub. $(2, 8)$:

$$8 = \frac{1}{4}(2)^2 - 24(2)^{-3} + C$$

$$8 = \frac{1}{4} \times 4 - \frac{24}{8} + C$$

$$8 = 1 - 3 + C$$

$$8 = -2 + C$$

$$C = 10$$

$$\rightarrow \underline{y = \frac{1}{4}x^2 - 24x^{-3} + 10}$$

- 5 The equation of a curve is such that $\frac{dy}{dx} = 4x - 3\sqrt{x} + 1$.

(a) Find the x -coordinate of the point on the curve at which the gradient is $\frac{11}{2}$. [3]

$$\frac{11}{2} = 4x - 3x^{\frac{1}{2}} + 1$$

$$4x - 3x^{\frac{1}{2}} - \frac{9}{2} = 0$$

$$\begin{array}{cccc} \times 2 & \times 2 & \times 2 & \times 2 \\ 8x - 6x^{\frac{1}{2}} - 9 = 0 \end{array}$$

Let $Y = x^{\frac{1}{2}}$:

$$8Y^2 - 6Y - 9 = 0$$

ac = -72. Two numbers with product -72 and sum -6: -12 and 6

$$8Y^2 - 12Y + 6Y - 9 = 0$$

$$4Y(2Y - 3) + 3(2Y - 3) = 0$$

$$(4Y + 3)(2Y - 3) = 0$$

$$4Y + 3 = 0 \quad \text{or} \quad 2Y - 3 = 0$$

$$4Y = -3$$

$$2Y = 3$$

$$Y = -\frac{3}{4}$$

$$Y = \frac{3}{2}$$

$$\sqrt{x} = \frac{3}{4} \quad \text{or} \quad \sqrt{x} = \frac{3}{2}$$

$$x = \frac{9}{4}$$

means positive square root, so this can't be the correct answer.

- (b) Given that the curve passes through the point (4, 11), find the equation of the curve. [4]

Integrate:

$$\frac{dy}{dx} = 4x - 3x^{\frac{1}{2}} + 1$$

$$y = 2x^2 - \frac{2}{3} \times 3x^{\frac{3}{2}} + x + C$$

$$y = 2x^2 - 2x^{\frac{3}{2}} + x + C$$

Sub. (4, 11):

$$11 = 2(4)^2 - 2(4)^{\frac{3}{2}} + 4 + C$$

$$11 = 2 \times 16 - 2 \times 8 + 4 + C$$

$$11 = 32 - 16 + 4 + C$$

$$C = -9$$

$$\rightarrow y = 2x^2 - 2x^{\frac{3}{2}} + x - 9$$

- 11 The equation of a curve is such that $\frac{dy}{dx} = 6x^2 - 30x + 6a$, where a is a positive constant. The curve has a stationary point at $(a, -15)$.

(a) Find the value of a .

[2]

$$\text{SP: } \frac{dy}{dx} = 0: \quad 6x^2 - 30x + 6a = 0$$

$$x^2 - 5x + a = 0$$

Sub. $x = a$:

$$a^2 - 5a + a = 0$$

$$a^2 - 4a = 0$$

$$a(a - 4) = 0$$

$$a = 0$$

~~x~~

$$\text{or } \underline{a = 4}$$

(b) Determine the nature of this stationary point.

[2]

$$\frac{dy}{dx} = 6x^2 - 30x + 24$$

$$\frac{d^2y}{dx^2} = 12x - 30$$

$$\text{Sub. } x = 4: \quad 12(4) - 30$$

$$48 - 30 = 18$$

$$\frac{d^2y}{dx^2} > 0$$

so minimum point

(c) Find the equation of the curve.

[3]

$$y = \int (6x^2 - 30x + 24) dx$$

$$= 2x^3 - 15x^2 + 24x + C$$

Sub. (4, -15):

$$-15 = 2(4)^3 - 15(4)^2 + 24(4) + C$$

$$-15 = 2 \times 64 - 15 \times 16 + 96 + C$$

$$-15 = -16 + C$$

$$C = 1$$

$$\rightarrow \underline{y = 2x^3 - 15x^2 + 24x + 1}$$

(d) Find the coordinates of any other stationary points on the curve.

[2]

$$\text{SP: } \frac{dy}{dx} = 0: 6x^2 - 30x + 24 = 0$$

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$x = 1 \quad \text{or} \quad x = 4$$

↪ from part (a) and (b)

Sub. $x=1$ into y :

$$\begin{aligned} y &= 2(1)^3 - 15(1)^2 + 24(1) + 1 \\ &= 2 - 15 + 24 + 1 \\ &= 12 \end{aligned}$$

$$\rightarrow \underline{(1, 12)}$$

10 The equation of a curve is $y = f(x)$, where $f(x) = (4x - 3)^{\frac{5}{3}} - \frac{20}{3}x$.

- (a) Find the x -coordinates of the stationary points of the curve and determine their nature. [6]

$$f'(x) = \frac{5}{3} \times 4(4x-3)^{\frac{2}{3}} - \frac{20}{3}$$

$$= \frac{20}{3}(4x-3)^{\frac{2}{3}} - \frac{20}{3}$$

SP, so $\frac{dy}{dx} = 0$:

$$\frac{20}{3}(4x-3)^{\frac{2}{3}} - \frac{20}{3} = 0$$

$$\frac{20}{3}(4x-3)^{\frac{2}{3}} = \frac{20}{3} \quad \div \frac{20}{3}$$

$$(4x-3)^{\frac{2}{3}} = 1 \quad \text{cube both sides}$$

$$(4x-3)^2 = 1$$

$$4x-3 = \pm 1$$

$$4x = 3 \pm 1$$

$$x = \frac{3 \pm 1}{4}$$

$$x = \frac{3+1}{4} \quad \text{or} \quad x = \frac{3-1}{4}$$

$$\underline{x = 1} \quad \text{or} \quad \underline{x = \frac{1}{2}}$$

Note: $f''(x) = \frac{2}{3} \times 4 \times \frac{20}{3}(4x-3)^{-\frac{1}{3}}$

$$= \frac{160}{9}(4x-3)^{-\frac{1}{3}} \quad \text{cont...}$$

$$\begin{aligned}
 x=1: f''(x) &= \frac{160}{9} (4(1)-3)^{-\frac{1}{3}} \\
 &= \frac{160}{9} (1)^{-\frac{1}{3}} \\
 &= \frac{160}{9} \quad f''(x) > 0 \text{ so } \underline{\text{minimum at } x=1}
 \end{aligned}$$

$$\begin{aligned}
 x=\frac{1}{2}: f''(x) &= \frac{160}{9} (4(\frac{1}{2})-3)^{-\frac{1}{3}} \\
 &= \frac{160}{9} (-1)^{-\frac{1}{3}} \\
 &= -\frac{160}{9} \quad f''(x) < 0 \text{ so } \underline{\text{maximum at } x=\frac{1}{2}}
 \end{aligned}$$

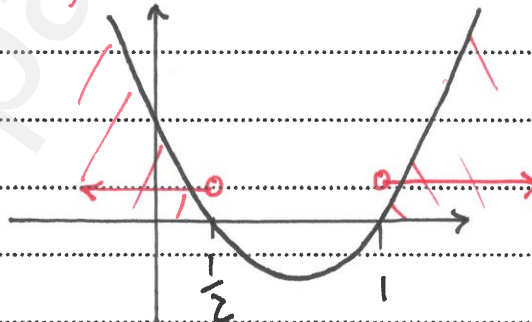
(b) State the set of values for which the function f is increasing.

[1]

Increasing when $f'(x) > 0$:

$$\frac{20}{3} (4x-3)^{\frac{2}{3}} - \frac{20}{3} > 0$$

from part (a), critical values: $x=1$, $x=\frac{1}{2}$



so $x < \frac{1}{2}$ or $x > 1$

- 10 A curve has a stationary point at $(2, -10)$ and is such that $\frac{d^2y}{dx^2} = 6x$.

(a) Find $\frac{dy}{dx}$.

[3]

Integrate $\frac{d^2y}{dx^2}$:

$$\frac{dy}{dx} = 3x^2 + C$$

$\frac{dy}{dx} = 0$ at $(2, -10)$:

$$0 = 3(2)^2 + C$$

$$0 = 3 \times 4 + C$$

$$C = -12$$

$$\rightarrow \underline{\underline{\frac{dy}{dx} = 3x^2 - 12}}$$

(b) Find the equation of the curve.

[3]

Integrate $\frac{dy}{dx}$:

$$y = x^3 - 12x + C$$

Sub. $(2, -10)$:

$$-10 = 2^3 - 12(2) + C$$

$$-10 = 8 - 24 + C$$

$$-10 = -16 + C$$

$$C = 6$$

$$\rightarrow \underline{\underline{y = x^3 - 12x + 6}}$$

- (c) Find the coordinates of the other stationary point and determine its nature. [3]

At SP, $\frac{dy}{dx} = 0$:

$$3x^2 - 12 = 0$$

$$3x^2 = 12$$

$$x^2 = 4$$

$$x = \pm 2$$

$x = -2$:

$$\begin{aligned} y &= (-2)^3 - 12(-2) + 6 \\ &= -8 + 24 + 6 \\ &= 22 \end{aligned}$$

$\rightarrow (-2, 22)$

Sub. into $\frac{d^2y}{dx^2}$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 6(-2) \\ &= -12 \end{aligned}$$

$$\frac{d^2y}{dx^2} < 0 \text{ so}$$

$(-2, 22)$ is a maximum

- (d) Find the equation of the tangent to the curve at the point where the curve crosses the y-axis. [2]

At y-axis, $x = 0$:

$$y = 0^3 - 12(0) + 6$$

$$y = 6 \rightarrow (0, 6)$$

Gradient at $(0, 6)$:

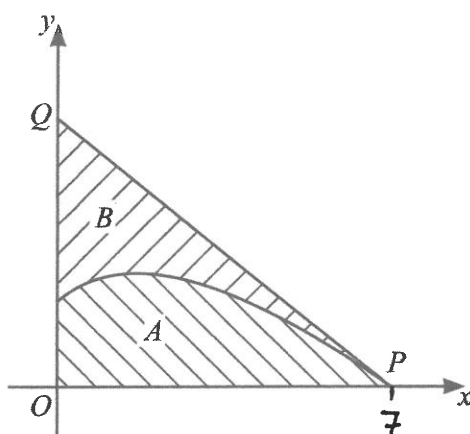
$$\begin{aligned} \frac{dy}{dx} &= 3(0)^2 - 12 \\ &= -12 \end{aligned}$$

$$y - y_1 = m(x - x_1)$$

$$y - 6 = -12(x - 0)$$

$$y - 6 = -12x$$

$$\underline{y = -12x + 6}$$



The diagram shows the curve with equation

$$y = 4(3x+4)^{\frac{1}{2}} - 2x - 6$$

for values of x such that $0 \leq x \leq 7$. The tangent to the curve at the point $P(7,0)$ meets the y -axis at the point Q . Region A is bounded by the curve and the two axes. Region B is bounded by the curve, the line segment PQ and the y -axis.

- (a) Find the area of region A . [4]

$$\text{Area} = \int_0^7 [4(3x+4)^{\frac{1}{2}} - 2x - 6] dx$$

$$= \left[4 \times \frac{2}{3} \times \frac{1}{3} (3x+4)^{\frac{3}{2}} - x^2 - 6x \right]_0^7$$

$$= \left[\frac{8}{9} (3x+4)^{\frac{3}{2}} - x^2 - 6x \right]_0^7$$

$$= \left[\frac{8}{9} (21+4)^{\frac{3}{2}} - 7^2 - 42 \right] - \left[\frac{8}{9} (4)^{\frac{3}{2}} - 0 - 0 \right]$$

$$= \left[\frac{8}{9} (25)^{\frac{3}{2}} - 49 - 42 \right] - \left[\frac{8}{9} \times 8 \right]$$

$$= \left[\frac{8}{9} \times 125 - 91 \right] - \left[\frac{64}{9} \right]$$

$$= \frac{181}{9} - \frac{64}{9}$$

$$= \underline{\underline{13}}$$



(b) Find the area of region B.

[5]

Find equation of tangent to curve to be able to find Q:

$$\text{gradient: } \frac{dy}{dx} = 4 \times \frac{1}{2} \times 3(3x+4)^{-\frac{1}{2}} - 2$$

$$= 6(3x+4)^{-\frac{1}{2}} - 2$$

Sub. $x=7$:

$$\text{gradient} = 6(21+4)^{-\frac{1}{2}} - 2$$

$$= 6(25)^{-\frac{1}{2}} - 2$$

$$= 6 \times \frac{1}{5} - 2$$

$$= \underline{\underline{-\frac{4}{5}}}$$

eqn of line:

$$y - y_1 = m(x - x_1)$$

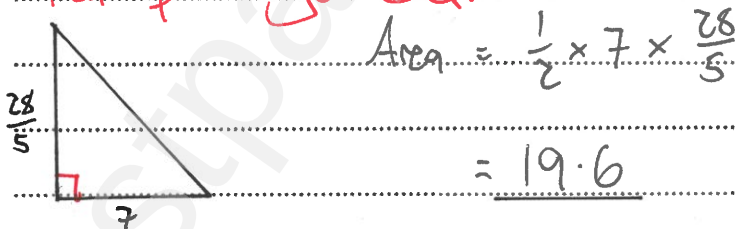
$$y - 0 = -\frac{4}{5}(x - 7)$$

$$y = -\frac{4}{5}x + \frac{28}{5}$$

y-intercept of this line is at Q, so:

$$Q = (0, \frac{28}{5})$$

Area of triangle OQP:



$$\text{Area} = \frac{1}{2} \times 7 \times \frac{28}{5}$$

$$= \underline{\underline{19.6}}$$

Region B = Area of triangle - answer from (a)

$$= 19.6 - 13$$

$$= \underline{\underline{6.6}}$$



- 10 The equation of a curve is such that $\frac{d^2y}{dx^2} = 6x^2 - \frac{4}{x^3}$. The curve has a stationary point at $(-1, \frac{9}{2})$.

(a) Determine the nature of the stationary point at $(-1, \frac{9}{2})$.

[1]

Sub. $x = -1$ into $\frac{d^2y}{dx^2}$: $6(-1)^2 - \frac{4}{(-1)^3}$

$$= 6 \times 1 - \frac{4}{-1}$$

$$= 6 + 4$$

$$= 10$$

$\frac{d^2y}{dx^2} > 0$ so minimum point

(b) Find the equation of the curve.

[5]

$$\frac{dy}{dx} = \int (6x^2 - 4x^{-3}) dx$$

$$= 2x^3 + 2x^{-2} + C$$

SP at $(-1, \frac{9}{2})$, so $\frac{dy}{dx} = 0$ when $x = -1$:

$$0 = 2(-1)^3 + 2(-1)^{-2} + C$$

$$0 = -2 + 2 + C$$

$$\underline{C = 0}$$

$$\rightarrow \frac{dy}{dx} = 2x^3 + 2x^{-2}$$

$$y = \int (2x^3 + 2x^{-2}) dx$$

$$y = \frac{1}{2}x^4 - 2x^{-1} + C$$

Sub. $(-1, \frac{9}{2})$:

$$\frac{9}{2} = \frac{1}{2}(-1)^4 - 2(-1)^{-1} + C$$

$$\frac{9}{2} = \frac{1}{2} + 2 + C$$

$$\frac{9}{2} = \frac{5}{2} + C$$

$$\underline{C = 2}$$

$$\underline{y = \frac{1}{2}x^4 - 2x^{-1} + 2}$$

- (c) Show that the curve has no other stationary points.

[3]

$$\text{SP: } \frac{dy}{dx} = 0: 2x^3 + 2x^{-2} = 0$$

$$2x^3 + \frac{2}{x^2} = 0$$

$$2x^3 = -\frac{2}{x^2}$$

$$2x^5 = -2$$

$$x^5 = -1$$

$$\underline{x = -1} \rightarrow x = -1 \text{ is only solution}$$

- (d) A point A is moving along the curve and the y-coordinate of A is increasing at a rate of 5 units per second.

①

Find the rate of increase of the x-coordinate of A at the point where $x = 1$.

[3]

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\text{①: } \frac{dy}{dt} = 5$$

$$2x^3 + 2x^{-2} = 5 \times \frac{dt}{dx}$$

$x=1$:

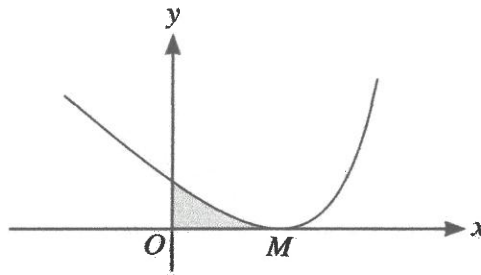
$$2(1)^3 + 2(1)^{-2} = 5 \frac{dt}{dx}$$

$$4 = 5 \times \frac{dt}{dx}$$

$$\frac{dt}{dx} = \frac{4}{5}$$

$$\underline{\underline{\frac{dx}{dt} = \frac{5}{4}}}$$

10



The diagram shows part of the curve $y = \frac{2}{(3-2x)^2} - x$ and its minimum point M , which lies on the x -axis.

- (a) Find expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\int y \, dx$.

[6]

$$y = 2(3-2x)^{-2} - x$$

$$\frac{dy}{dx} = -2x - 2 \times 2(3-2x)^{-3} - 1$$

$$= \underline{\underline{8(3-2x)^{-3} - 1}}$$

$$\frac{d^2y}{dx^2} = -3x - 2 \times 8(3-2x)^{-4}$$

$$= \underline{\underline{48(3-2x)^{-4}}}$$

$$\int y \, dx = \int [2(3-2x)^{-2} - x] \, dx$$

$$= -1 \times \frac{-1}{2} \times 2(3-2x)^{-1} - \frac{1}{2}x^2 + C$$

$$= \underline{\underline{(3-2x)^{-1} - \frac{1}{2}x^2 + C}}$$

- (b) Find, by calculation, the
- x
- coordinate of
- M
- .

[2]

$$\text{SP: } \frac{dy}{dx} = 0: \quad 8(3-2x)^{-3} - 1 = 0$$

$$\frac{8}{(3-2x)^3} = 1$$

$$8 = (3-2x)^3$$

$$2 = 3-2x$$

$$-1 = -2x$$

$$\underline{\underline{x = \frac{1}{2}}}$$

- (c) Find the area of the shaded region bounded by the curve and the coordinate axes.

[2]

$$\int_0^{\frac{1}{2}} [2(3-2x)^{-2} - x] dx$$

$$= \left[(3-2x)^{-1} - \frac{1}{2}x^2 \right]_0^{\frac{1}{2}}$$

$$= \left[(3-2(\frac{1}{2}))^{-1} - \frac{1}{2}(\frac{1}{2})^2 \right] - \left[(3)^{-1} - 0 \right]$$

$$= \left[(2)^{-1} - \frac{1}{2} \times \frac{1}{4} \right] - \left[\frac{1}{3} \right]$$

$$= \left[\frac{1}{2} - \frac{1}{8} \right] - \frac{1}{3}$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{3} = \underline{\underline{\frac{1}{24}}}$$

- 9 A curve has equation $y = f(x)$, and it is given that $f'(x) = 2x^2 - 7 - \frac{4}{x^2}$.

(a) Given that $f(1) = -\frac{1}{3}$, find $f(x)$.

[4]

$$f(x) = \int (2x^2 - 7 - 4x^{-2}) dx$$

$$= \frac{2}{3}x^3 - 7x + 4x^{-1} + C$$

$$f(1) = -\frac{1}{3} :$$

$$-\frac{1}{3} = \frac{2}{3}(1)^3 - 7(1) + 4(1)^{-1} + C$$

$$-\frac{1}{3} = \frac{2}{3} - 7 + 4 + C$$

$$-\frac{1}{3} = \frac{-7}{3} + C$$

$$\underline{C = 2}$$

$$\rightarrow \underline{f(x) = \frac{2}{3}x^3 - 7x + 4x^{-1} + 2}$$

(b) Find the coordinates of the stationary points on the curve.

[5]

$$\text{SP: } f'(x) = 0: \quad 2x^2 - 7 - \frac{4}{x^2} = 0$$

$\times x^2 \quad \times x^2 \quad \times x^2 \quad \times x^2$

$$2x^4 - 7x^2 - 4 = 0$$

$$\text{Let } Y = x^2:$$

$$2Y^2 - 7Y - 4 = 0$$

$$(2Y + 1)(Y - 4) = 0$$

$$2Y + 1 = 0 \quad \text{or} \quad Y - 4 = 0$$

$$Y = -\frac{1}{2}$$

$$Y = 4$$

$$x^2 = -\frac{1}{2}$$

$$x^2 = 4$$

$\times$ no sol's

$$x = 2 \quad \text{or} \quad x = -2$$

$$\begin{aligned} \text{Sub. } x=2: \quad y &= \frac{2}{3}(2)^3 - 7(2) + 4(2)^{-1} + 2 \\ &= -\frac{14}{3} \rightarrow \underline{\underline{(2, -\frac{14}{3})}} \\ \text{Sub. } x=-2: \quad y &= \frac{2}{3}(-2)^3 - 7(-2) + 4(-2)^{-1} + 2 \\ &= \frac{26}{3} \rightarrow \underline{\underline{(-2, \frac{26}{3})}} \end{aligned}$$

[1]

(c) Find $f''(x)$.

$$f'(x) = 2x^2 - 7 - 4x^{-2}$$

$$f''(x) = \underline{\underline{4x + 8x^{-3}}}$$

(d) Hence, or otherwise, determine the nature of each of the stationary points.

[2]

$$\text{Sub. } x=2: \quad 4(2) + 8(2)^{-3}$$

$$= 8 + 8 \times \frac{1}{8} = 9$$

$\rightarrow f''(x) > 0$ so $\underline{\underline{(2, -\frac{14}{3})}}$ is a minimum point.

$$\text{Sub. } x=-2: \quad 4(-2) + 8(-2)^{-3}$$

$$= -8 + 8 \times \frac{-1}{8} = -9$$

$\rightarrow f''(x) < 0$ so $\underline{\underline{(-2, \frac{26}{3})}}$ is a maximum point.

- 10 A curve has equation $y = f(x)$ and it is given that

$$f'(x) = \left(\frac{1}{2}x + k\right)^{-2} - (1+k)^{-2},$$

↙ this is just a constant
(no x) So disappears
when differentiated.

where k is a constant. The curve has a minimum point at $x = 2$.

- (a) Find $f''(x)$ in terms of k and x , and hence find the set of possible values of k . [3]

$$\begin{aligned} f''(x) &= -2 \times \frac{1}{2} \left(\frac{1}{2}x + k\right)^{-3} \\ &= -\left(\frac{1}{2}x + k\right)^{-3} \end{aligned}$$

Sub. $x=2$:

$$\begin{aligned} f''(2) &= -\left(\frac{1}{2}(2) + k\right)^{-3} \\ &= -(1+k)^{-3} \end{aligned}$$

minimum point, so $f''(x) > 0$:

$$-(1+k)^{-3} > 0$$

$$(1+k)^{-3} < 0$$

$$\frac{1}{(1+k)^3} < 0$$

$$(1+k)^3 < 0$$

cube root:

$$1+k < 0$$

$$\underline{k < -1}$$

reciprocal of both sides

It is now given that $k = -3$ and the minimum point is at $(2, 3\frac{1}{2})$.

- (b) Find $f(x)$. [4]

$$\begin{aligned} k=-3: f'(x) &= \left(\frac{1}{2}x - 3\right)^{-2} - (-2)^{-2} \\ &= \left(\frac{1}{2}x - 3\right)^{-2} - \frac{1}{4} \end{aligned}$$

$$f(x) = \int \left[\left(\frac{1}{2}x - 3\right)^{-2} - \frac{1}{4} \right] dx$$

$$= -1 \times 2 \left(\frac{1}{2}x - 3\right)^{-1} - \frac{1}{4}x + C$$

$$= -2 \left(\frac{1}{2}x - 3\right)^{-1} - \frac{1}{4}x + C$$

continued...

Sub. $(2, \frac{7}{2})$: $\frac{7}{2} = -2\left(\frac{1}{2}(2) - 3\right)^{-1} - \frac{1}{4}(2) + C$

$$\frac{7}{2} = -2(-2)^{-1} - \frac{1}{2} + C$$

$$\frac{7}{2} = -2 \times \frac{-1}{2} - \frac{1}{2} + C$$

$$\frac{7}{2} = 1 - \frac{1}{2} + C \quad \rightarrow C = 3$$

$$\frac{7}{2} = \frac{1}{2} + C \quad \rightarrow \underline{f(x) = -2\left(\frac{1}{2}x - 3\right)^{-1} - \frac{1}{4}x + 3}$$

(c) Find the coordinates of the other stationary point and determine its nature.

[4]

SP: $f'(x) = 0$:

$$\left(\frac{1}{2}x - 3\right)^{-2} - (1 - 3)^{-2} = 0$$

$$\left(\frac{1}{2}x - 3\right)^{-2} - \frac{1}{4} = 0$$

$$\frac{1}{\left(\frac{1}{2}x - 3\right)^2} = \frac{1}{4}$$

$$4 = \left(\frac{1}{2}x - 3\right)^2$$

$$\frac{1}{2}x - 3 = \pm 2$$

$$\frac{1}{2}x - 3 = -2 \quad \text{or} \quad \frac{1}{2}x - 3 = 2$$

$$\frac{1}{2}x = 1$$

$$\frac{1}{2}x = 5$$

$$x = 2$$

$$x = 10$$

↑ from part (a) & (b)

Sub. $x = 10$ into $f(x)$:

$$f(10) = -2\left(\frac{1}{2}(10) - 3\right)^{-1} - \frac{1}{4}(10) + 3$$

$$= -2(2)^{-1} - \frac{10}{4} + 3$$

$$= \frac{-2}{2} - \frac{5}{2} + 3$$

$$= -\frac{1}{2}$$

$$\rightarrow \underline{(10, -\frac{1}{2})}$$

Sub. $x = 10$ into $f''(x)$:

$$-\left(\frac{1}{2}(10) - 3\right)^{-3}$$

$$= -(2)^{-3}$$

$$= -\frac{1}{8} \rightarrow f''(x) < 0 \text{ so}$$

$$(10, -\frac{1}{2}) \text{ is a}$$

maximum point

- 10 A curve has equation $y = \frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2}$ where $x > 0$ and k is a positive constant.

(b) It is given instead that $\int_{\frac{1}{4}k^2}^{k^2} \left(\frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2} \right) dx = \frac{13}{12}$.

Find the value of k .

[5]

$$\begin{aligned} & \int_{\frac{1}{4}k^2}^{k^2} \left(\frac{1}{k}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + \frac{1}{k^2} \right) dx \\ &= \left[\frac{2}{3} \times \frac{1}{k} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + \frac{1}{k^2} x \right]_{\frac{1}{4}k^2}^{k^2} \\ &= \left[\frac{2}{3k} (k^2)^{\frac{3}{2}} + 2(k^2)^{\frac{1}{2}} + \frac{1}{k^2} (k^2) \right] - \left[\frac{2}{3k} \left(\frac{k^2}{4} \right)^{\frac{3}{2}} + 2 \left(\frac{k^2}{4} \right)^{\frac{1}{2}} + \frac{1}{k^2} \left(\frac{k^2}{4} \right) \right] \\ &= \left[\frac{2}{3k} \times k^3 + 2k + 1 \right] - \left[\frac{2}{3k} \left(\frac{k}{2} \right)^3 + 2 \left(\frac{k}{2} \right) + \frac{1}{4} \right] \\ &= \left[\frac{2k^2}{3} + 2k + 1 \right] - \left[\frac{2}{3k} \times \frac{k^3}{8} + k + \frac{1}{4} \right] \\ &= \left[\frac{2k^2}{3} + 2k + 1 \right] - \left[\frac{k^2}{12} + k + \frac{1}{4} \right] \\ &= \frac{2k^2}{3} + 2k + 1 - \frac{k^2}{12} - k - \frac{1}{4} \end{aligned}$$

$$= \frac{7}{12} k^2 + k + \frac{3}{4}$$

integral = $\frac{13}{12}$:

$$\frac{7}{12} k^2 + k + \frac{3}{4} = \frac{13}{12}$$

$$7k^2 + 12k + 9 = 13$$

$$7k^2 + 12k - 4 = 0$$

$$(7k - 2)(k + 2) = 0$$

$$k = \frac{2}{7} \text{ or } k = -2$$

✓ k is positive

$$\underline{\underline{k = \frac{2}{7}}}$$



- 10 A function f with domain $x > 0$ is such that $f'(x) = 8(2x-3)^{\frac{1}{3}} - 10x^{\frac{2}{3}}$. It is given that the curve with equation $y = f(x)$ passes through the point $(1, 0)$.

- (a) Find the equation of the normal to the curve at the point $(1, 0)$.

[3]

gradient of f at $(1, 0)$: $f'(1) = 8(2-3)^{\frac{1}{3}} - 10 \times (1)^{\frac{2}{3}}$
 $= 8 \times -1 - 10 = -18$

gradient of normal: $m = \frac{1}{18} \rightarrow y - y_1 = m(x - x_1)$
 $y - 0 = \frac{1}{18}(x - 1)$

$y = \frac{1}{18}x - \frac{1}{18}$

- (b) Find $f(x)$.

[4]

Integrate:

$f(x) = \frac{3}{4} \times \frac{1}{2} \times 8(2x-3)^{\frac{4}{3}} - \frac{3}{5} \times 10x^{\frac{5}{3}}$
 $= 3(2x-3)^{\frac{4}{3}} - 6x^{\frac{5}{3}} + C$

Sub. $(1, 0)$:

$0 = 3(2(1)-3)^{\frac{4}{3}} - 6(1)^{\frac{5}{3}} + C$

$0 = 3(-1)^{\frac{4}{3}} - 6(1)^{\frac{5}{3}} + C$

$0 = 3 \times 1 - 6 \times 1 + C$

$0 = -3 + C$

$C = 3$

$f(x) = 3(2x-3)^{\frac{4}{3}} - 6x^{\frac{5}{3}} + 3$



It is given that the equation $f'(x) = 0$ can be expressed in the form

$$125x^2 - 128x + 192 = 0.$$

- (c) Determine, making your reasoning clear, whether f is an increasing function, a decreasing function or neither. [3]

!! Careful, this doesn't mean $f'(x) = 125x^2 - 128x + 192$!!
It could mean $-f'(x) = 125x^2 - 128x + 192$

Check if it could be neither increasing or decreasing:

If it's neither, then it will change from increasing to decreasing at some point, so there must be a stationary point:

So let's see if $f'(x) = 0$ has any solutions:

$$125x^2 - 128x + 192 = 0$$

$b^2 - 4ac$:

$$(-128)^2 - 4(125)(192) = -79616$$

$b^2 - 4ac < 0$, so this has no solutions.

* Therefore $f'(x)$ is either always positive or always negative *

To find out which, we can substitute any value from the domain ($x > 0$) into $f'(x) = 8(2x-3)^{\frac{1}{3}} - 10x^{\frac{2}{3}}$

e.g. $x = 1$:

$$\begin{aligned} f'(1) &= 8(2-3)^{\frac{1}{3}} - 10(1)^{\frac{2}{3}} \\ &= 8 \times -1 - 10 \times 1 \\ &= -8 - 10 \\ &= -18 \end{aligned}$$

↑ gradient is negative

* So gradient is negative for the whole function, so it is a decreasing function *



- 11 The equation of a curve is $y = kx^{\frac{1}{2}} - 4x^2 + 2$, where k is a constant.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of k .

[2]

$$\frac{dy}{dx} = \frac{1}{2}kx^{-\frac{1}{2}} - 8x$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}kx^{-\frac{3}{2}} - 8$$

(b) It is given that $k = 2$.

Find the coordinates of the stationary point and determine its nature.

[4]

$$k=2: \frac{dy}{dx} = x^{-\frac{1}{2}} - 8x$$

$$\text{SP, so } \frac{dy}{dx} = 0:$$

$$x^{-\frac{1}{2}} - 8x = 0$$

$$\frac{1}{\sqrt{x}} - 8x = 0$$

$$1 - 8x^{\frac{3}{2}} = 0$$

$$8x^{\frac{3}{2}} = 1$$

$$x^{\frac{3}{2}} = \frac{1}{8}$$

$$x^{\frac{1}{2}} = \frac{1}{2}$$

$$x = \frac{1}{4}$$

$$\begin{aligned} \text{Sub. into } y: y &= 2x^{\frac{1}{2}} - 4x^2 + 2 \\ &= 2\left(\frac{1}{4}\right)^{\frac{1}{2}} - 4\left(\frac{1}{4}\right)^2 + 2 \\ &= 2 \times \frac{1}{2} - 4 \times \frac{1}{16} + 2 \\ &= \frac{11}{4} \end{aligned}$$

→ coordinates of SP: $\left(\frac{1}{4}, \frac{11}{4}\right)$

$$\text{Nature: } \frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}} - 8$$

$$\text{Sub. } x = \frac{1}{4}:$$

$$-\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{3}{2}} - 8 = -12 \rightarrow \frac{d^2y}{dx^2} < 0 \text{ therefore maximum}$$

- (c) Points A and B on the curve have x -coordinates 0.25 and 1 respectively. For a different value of k , the tangents to the curve at the points A and B meet at a point with x -coordinate 0.6.

Find this value of k .

[6]

Find eqn of tangent at A :

Find y -coord: $y = k(0.25)^{\frac{1}{2}} - 4(0.25)^2 + 2$

$$= k \times \frac{1}{2} - 4 \times \frac{1}{16} + 2$$

$$= \frac{k}{2} + \frac{7}{4}$$

Find gradient:

$$m = \frac{1}{2}k(0.25)^{-\frac{1}{2}} - 8(0.25)$$

$$= \frac{1}{2}k \times 2 - 2$$

$$= k - 2$$

Eqn: $y - y_1 = m(x - x_1)$

$$y - \frac{k}{2} - \frac{7}{4} = (k - 2)(x - \frac{1}{4})$$

$$y - \frac{k}{2} - \frac{7}{4} = kx - 2x - \frac{k}{4} + \frac{1}{2}$$

$$y = kx - 2x + \frac{k}{4} + \frac{9}{4} \quad (1)$$

Find eqn of tangent at B :

Find y -coord: $y = k(1)^{\frac{1}{2}} - 4(1)^2 + 2$

$$= k - 4 + 2$$

$$= k - 2$$

Find gradient: $m = \frac{1}{2}k(1)^{-\frac{1}{2}} - 8(1)$

$$= \frac{1}{2}k - 8$$

Eqn: $y - k + 2 = (\frac{1}{2}k - 8)(x - 1)$

$$y - k + 2 = \frac{1}{2}kx - 8x - \frac{1}{2}k + 8$$

$$y = \frac{1}{2}kx - 8x + \frac{1}{2}k + 6 \quad (2)$$

(1) = (2) when $x = 0.6$:

$$0.6k - 2(0.6) + \frac{k}{4} + \frac{9}{4} = \frac{1}{2}(0.6)k - 8(0.6) + \frac{k}{2} + 6$$

$$0.85k + 1.05 = 0.8k + 1.2$$

$$0.05k = 0.15$$

$$k = 3$$