

- 4 The first term of an arithmetic progression is  $a$  and the common difference is  $-4$ . The first term of a geometric progression is  $5a$  and the common ratio is  $-\frac{1}{4}$ . The sum to infinity of the geometric progression is equal to the sum of the first eight terms of the arithmetic progression.

(a) Find the value of  $a$ .

[4]

$$\begin{aligned} \text{Sum of geometric: } S_{\infty} &= \frac{5a}{1 - (-\frac{1}{4})} = \frac{5a \times 4}{5/4 \times 4} \\ &= \frac{20a}{5} = 4a \end{aligned}$$

$$\text{Sum of arithmetic: } S_8 = \frac{8}{2} [2a + 7(-4)]$$

$$= 4(2a - 28)$$

$$= 8a - 112$$

$$S_{\infty} = S_8: 4a = 8a - 112$$

$$-4a = -112$$

$$\underline{a = 28}$$

The  $k$ th term of the arithmetic progression is zero.

(b) Find the value of  $k$ .

[2]

$$a + (n-1)d = 0$$

$$28 + (k-1)(-4) = 0$$

$$28 - 4k + 4 = 0$$

$$-4k + 32 = 0$$

$$4k = 32$$

$$\underline{k = 8}$$

- 6 The first three terms of an arithmetic progression are  $\frac{p^2}{6}$ ,  $2p - 6$  and  $p$ .

(a) Given that the common difference of the progression is not zero, find the value of  $p$ . [3]

$$u_2 - u_1: d = 2p - 6 - \frac{p^2}{6} \quad (1)$$

$$\begin{aligned} u_3 - u_2: d &= p - (2p - 6) \\ &= p - 2p + 6 \\ &= -p + 6 \quad (2) \end{aligned}$$

$$(1) = (2): 2p - 6 - \frac{p^2}{6} = -p + 6$$

$$\frac{p^2}{6} - 3p + 12 = 0$$

$$p^2 - 18p + 72 = 0$$

$$(p - 6)(p - 12) = 0$$

$$p = 6 \quad \text{or} \quad p = 12$$

$$\begin{aligned} \text{If } p = 6, d &= -p + 6 \\ &= -6 + 6 \\ &= 0 \end{aligned}$$

but common difference is not zero, so  $p = 12$

- (b) Using this value, find the sum to infinity of the geometric progression with first two terms  $\frac{p^2}{6}$  and  $2p - 6$ . [2]

$$u_1: \frac{12^2}{6} = 24$$

$$u_2: 2(12) - 6 = 18$$

$$u_2 \div u_1: r = \frac{18}{24} = \frac{3}{4}$$

$$= \frac{24}{1 - \frac{3}{4}}$$

$$= \underline{\underline{96}}$$

$$S_{\infty} = \frac{a}{1 - r}$$

- 9 The first term of a progression is  $\cos \theta$ , where  $0 < \theta < \frac{1}{2}\pi$ .

(a) For the case where the progression is geometric, the sum to infinity is  $\frac{1}{\cos \theta}$ .

(i) Show that the second term is  $\cos \theta \sin^2 \theta$ . [3]

$$u_1: a = \cos \theta$$

$$S_{\infty}: S_{\infty} = \frac{a}{1-r}$$

$$= \frac{\cos \theta}{1-r}$$

$$u_2: u_2 = ar$$

$$= \cos \theta \sin^2 \theta \quad \text{QED}$$

$$S_{\infty} = \frac{1}{\cos \theta}: \frac{\cos \theta}{1-r} = \frac{1}{\cos \theta}$$

$$\cos^2 \theta = 1-r$$

$$r = 1 - \cos^2 \theta$$

$$r = \sin^2 \theta$$

(ii) Find the sum of the first 12 terms when  $\theta = \frac{1}{3}\pi$ , giving your answer correct to 4 significant figures. [2]

$$u_1: a = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$r: r = \sin^2\left(\frac{\pi}{3}\right) = \frac{3}{4}$$

$$S_{12}: S_{12} = \frac{\frac{1}{2} \left(1 - \left(\frac{3}{4}\right)^{12}\right)}{1 - \frac{3}{4}}$$

$$= 1.937$$

- (b) For the case where the progression is arithmetic, the first two terms are again  $\cos \theta$  and  $\cos \theta \sin^2 \theta$  respectively.

Find the 85th term when  $\theta = \frac{1}{3}\pi$ .

[4]

$$u_1: u_1 = \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}$$

$$u_2: u_2 = \cos\left(\frac{\pi}{3}\right) \times \sin^2\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2} \times \frac{3}{4}$$

$$= \frac{3}{8}$$

$$\text{Sequence: } \frac{1}{2}, \frac{3}{8}, \dots$$

$$\text{common difference: } d = \frac{3}{8} - \frac{1}{2}$$

$$= -\frac{1}{8}$$

$$a = \frac{1}{2}, d = -\frac{1}{8}$$

$$u_{85}: u_{85} = \frac{1}{2} + 84\left(-\frac{1}{8}\right)$$

$$= \underline{\underline{-10}}$$

- 8 (a) An arithmetic progression is such that its first term is 6 and its tenth term is 19.5.

Find the sum of the first 100 terms of this arithmetic progression.

[4]

$$u_1: a = 6$$

$$u_{10}: a + 9d = 19.5$$

$$6 + 9d = 19.5$$

$$9d = 13.5$$

$$d = 1.5$$

$S_{100}$ :

$$S_{100} = \frac{100}{2} [2(6) + (100-1)(1.5)]$$

$$= 50 [12 + 99 \times 1.5]$$

$$= \underline{\underline{8025}}$$

- (b) A geometric progression  $a_1, a_2, a_3, \dots$  is such that  $a_1 = 24$  and the common ratio is  $\frac{1}{2}$ .

The sum to infinity of this geometric progression is denoted by  $S$ . The sum to infinity of the even-numbered terms (i.e.  $a_2, a_4, a_6, \dots$ ) is denoted by  $S_E$ .

Find the values of  $S$  and  $S_E$ .

[4]

All terms:

$$S = \frac{a}{1-r}$$

$$= \frac{24}{1-\frac{1}{2}}$$

$$= \underline{\underline{48}}$$

(progression:  $24, 12, 6, 3, 1.5, 0.75, \dots$ )  
 $a_1, a_2, a_3, a_4, a_5, a_6$

Even numbered terms:

$12, 3, 0.75, \dots$   
 $\times \frac{1}{4} \quad \times \frac{1}{4}$

so  $a = 12$  and  $r = \frac{1}{4}$

$$S_E = \frac{12}{1-\frac{1}{4}}$$

$$= \frac{12}{\frac{3}{4}}$$

$$= \underline{\underline{16}}$$

- 4 The first term of a geometric progression and the first term of an arithmetic progression are both equal to  $a$ .

The third term of the geometric progression is equal to the second term of the arithmetic progression.

The fifth term of the geometric progression is equal to the sixth term of the arithmetic progression.

Given that the terms are all positive and not all equal, find the sum of the first twenty terms of the arithmetic progression in terms of  $a$ . [6]

$$u_3(\text{geo}) = u_2(\text{ari}):$$

$$ar^2 = a + d \quad (1)$$

$$u_5(\text{geo}) = u_6(\text{ari}):$$

$$ar^4 = a + 5d \quad (2)$$

$$(1): r^2 = \frac{a+d}{a}$$

$$r^4 = \frac{(a+d)^2}{a^2}$$

Sub. into (2):

$$a \times \frac{(a+d)^2}{a^2} = a + 5d$$

$$\frac{(a+d)^2}{a} = a + 5d$$

$$(a+d)^2 = a^2 + 5ad$$

$$a^2 + 2ad + d^2 = a^2 + 5ad$$

$$-3ad + d^2 = 0$$

$$d^2 - 3ad = 0$$

$$d(d - 3a) = 0$$

$$d=0 \text{ or } d=3a$$

terms would all be equal if  $d=0$ .

$$\begin{aligned} S_{20} &= \frac{20}{2} [2a + 19(3a)] \\ &= 10(2a + 57a) \\ &= 10 \times 59a \\ &= \underline{\underline{590a}} \end{aligned}$$

- 9 The first term of a geometric progression is 216 and the fourth term is 64.

(a) Find the sum to infinity of the progression.

[3]

$$u_1: a = 216 \quad (1)$$

$$u_4: ar^3 = 64 \quad (2)$$

Sub. (1) into (2):

$$216 \times r^3 = 64$$

$$r^3 = \frac{8}{27}$$

$$r = \frac{2}{3}$$

$$S_{\infty}: S_{\infty} = \frac{216}{1 - \frac{2}{3}}$$

$$= \frac{216}{\frac{1}{3}}$$

$$= \underline{\underline{648}}$$

The second term of the geometric progression is equal to the second term of an arithmetic progression.

The third term of the geometric progression is equal to the fifth term of the same arithmetic progression.

(b) Find the sum of the first 21 terms of the arithmetic progression.

[6]

$$u_2(\text{geo}): u_2 = 216 \times \frac{2}{3}$$

$$= \underline{144}$$

$$u_3(\text{geo}): u_3 = 216 \times \left(\frac{2}{3}\right)^2$$

$$= \underline{96}$$

$$u_2(\text{ari}) = 144: a + d = 144 \quad (1)$$

$$u_5(\text{ari}) = 96: a + 4d = 96 \quad (2)$$

$$(2) - (1): 3d = -48$$

$$\underline{d = -16}$$

$$\text{sub into (1): } a + -16 = 144$$

$$a - 16 = 144$$

$$\underline{a = 160}$$

$$S_{21}: S_{21} = \frac{21}{2} [2(160) + 20(-16)]$$

$$= \frac{21}{2} [320 - 320]$$

$$= \frac{21}{2} \times 0$$

$$= \underline{0}$$

- 10 An arithmetic progression has first term 5 and common difference  $d$ , where  $d > 0$ . The second, fifth and eleventh terms of the arithmetic progression, in that order, are the first three terms of a geometric progression.

(a) Find the value of  $d$ .

[3]

Sequence:  $5, 5+d, 5+2d, \dots$

Arithmetic sequence:

$$u_2 = 5 + d$$

$$u_5 = 5 + 4d$$

$$u_{11} = 5 + 10d$$

Geometric sequence:

$$5 + d, 5 + 4d, 5 + 10d, \dots$$

Find common ratio:

$$r = \frac{5+4d}{5+d} \quad \text{but also: } r = \frac{5+10d}{5+4d}$$

So:  $\frac{5+4d}{5+d} = \frac{5+10d}{5+4d}$

$$(5+4d)(5+4d) = (5+d)(5+10d)$$

$$25 + 20d + 20d + 16d^2 = 25 + 50d + 5d + 10d^2$$

$$25 + 40d + 16d^2 = 25 + 55d + 10d^2$$

$$6d^2 - 15d = 0$$

$$2d^2 - 5d = 0$$

$$d(2d - 5) = 0$$

$$d=0 \quad \text{or} \quad 2d-5=0$$

$\times$   
 $d > 0$

$$2d = 5$$

$$d = \frac{5}{2}$$

- (b) The sum of the first 77 terms of the arithmetic progression is denoted by  $S_{77}$ . The sum of the first 10 terms of the geometric progression is denoted by  $G_{10}$ .

Find the value of  $S_{77} - G_{10}$ .

[5]

Arithmetic: 5, 7.5, 10, 12.5...

$$S_{77} = \frac{77}{2} [2(5) + 76(2.5)]$$

$$= \frac{77}{2} [10 + 190]$$

$$= \frac{77}{2} \times 200$$

$$= \underline{7700}$$

Geometric: 7.5, 15, 30, ...

$$S_{10} = \frac{7.5(1-2^{10})}{1-2}$$

$$= \frac{7.5 \times -1023}{-1}$$

$$= \underline{7672.5}$$

$$\rightarrow 7700 - 7672.5 = \underline{\underline{27.5}}$$



- 8 The first, second and third terms of an arithmetic progression are  $a$ ,  $\frac{3}{2}a$  and  $b$  respectively, where  $a$  and  $b$  are positive constants. The first, second and third terms of a geometric progression are  $a$ , 18 and  $b+3$  respectively.

(a) Find the values of  $a$  and  $b$ .

[5]

arithmetic progression:

$$u_2 - u_1: d = \frac{3}{2}a - a$$

$$d = \frac{a}{2} \quad (1)$$

$$u_3 - u_2: d = b - \frac{3}{2}a \quad (2)$$

$$(1) = (2): \frac{a}{2} = b - \frac{3}{2}a$$

$$2a = b$$

geometric progression:

$$u_2 \div u_1: r = \frac{18}{a} \quad (3)$$

$$u_3 \div u_2: r = \frac{b+3}{18} \quad (4)$$

$$(3) = (4): \frac{18}{a} = \frac{b+3}{18}$$

$$324 = ab + 3a$$

Sub.  $b = 2a$  into this

$$324 = a(2a) + 3a$$

$$324 = 2a^2 + 3a$$

$$2a^2 + 3a - 324 = 0$$

$$a = \frac{-3 \pm \sqrt{3^2 - 4(2)(-324)}}{2(2)}$$

$$a = 12 \quad \text{or} \quad a = -\frac{27}{2}$$

Sub. into  $2a = b$ :

$$b = 2 \times 12$$

$$b = 24$$

(b) Find the sum of the first 20 terms of the arithmetic progression.

[3]

Sub.  $a=12, b=24$  into sequence: 12, 18, 24, ...

$$\rightarrow a = 12, d = 6$$

$$S_{20} = \frac{20}{2} [2(12) + 19(6)]$$

$$= 10(24 + 114)$$

$$= 10 \times 138 = 1380$$

- 4 The circumference round the trunk of a large tree is measured and found to be 5.00 m. After one year the circumference is measured again and found to be 5.02 m.

- (a) Given that the circumferences at yearly intervals form an arithmetic progression, find the circumference 20 years after the first measurement. [2]

Common difference:  $d = 5.02 - 5.00$   
 $= 0.02$

Let  $u_1$  be the measurement after 1 year:

$$u_1 = a = 5.02$$

$u_{20}$ :  $u_{20} = a + 19d$   
 $= 5.02 + 19(0.02)$   
 $= 5.02 + 0.38$   
 $= \underline{5.40 \text{ m}}$

- (b) Given instead that the circumferences at yearly intervals form a geometric progression, find the circumference 20 years after the first measurement. [3]

Common ratio:  $r = \frac{5.02}{5.00} = \underline{1.004}$

Let  $u_1$  be the measurement after 1 year:

$$u_1 = a = 5.02$$

$u_{20}$ :  $u_{20} = ar^{19}$   
 $= 5.02 \times (1.004)^{19}$   
 $= \underline{5.42 \text{ m}}$

- 8 A progression has first term  $a$  and second term  $\frac{a^2}{a+2}$ , where  $a$  is a positive constant.

- (a) For the case where the progression is geometric and the sum to infinity is 264, find the value of  $a$ . [5]

$$u_2 \div u_1: \quad r = \frac{\frac{a^2}{a+2}}{a} \times \frac{(a+2)}{(a+2)}$$

$$= \frac{a^2}{a(a+2)}$$

$$r = \frac{a}{a+2} \quad (1)$$

$$S_{\infty} = 264: \quad \frac{a}{1-r} = 264$$

$$a = 264(1-r)$$

$$a = 264 - 264r \quad (2)$$

sub. (1) into (2):

$$a = 264 - 264 \times \left( \frac{a}{a+2} \right)$$

$$a = 264 - \frac{264a}{a+2}$$

$$a(a+2) = 264(a+2) - 264a$$

$$a^2 + 2a = 264a + 528 - 264a$$

$$a^2 + 2a - 528 = 0$$

$$(a+24)(a-22) = 0$$

$$a = -24 \quad \text{or} \quad a = 22$$

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$$\rightarrow \underline{a = 22}$$

- (b) For the case where the progression is arithmetic and  $a = 6$ , determine the least value of  $n$  required for the sum of the first  $n$  terms to be less than  $-480$ . [5]

sub.  $a = 6$ :  $u_1$ :  $u_1 = 6$

$$u_2: \frac{6^2}{6+2} = \frac{36}{8} = 4.5$$

Sequence:  $6, 4.5, \dots$

common difference:  $d = 4.5 - 6 = -1.5$

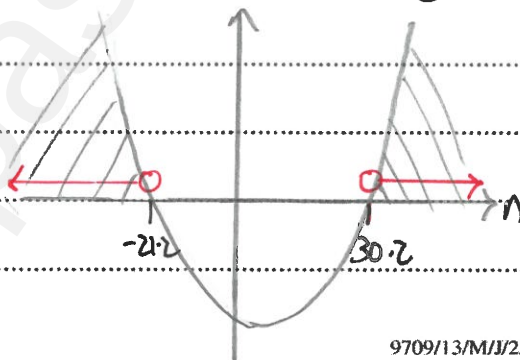
$$S_n: S_n = \frac{n}{2} [2(6) + (n-1)(-1.5)]$$

$$\begin{aligned} S_n < -480: \quad \frac{n}{2} [12 - 1.5n + 1.5] &< -480 \\ n(13.5 - 1.5n) &< -960 \\ -1.5n^2 + 13.5n &< -960 \\ -3n^2 + 27n &< -1920 \\ -n^2 + 9n &< -640 \\ n^2 - 9n &> 640 \\ n^2 - 9n - 640 &> 0 \end{aligned}$$

Critical values:

$$n = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-640)}}{2(1)}$$

$$n = 30.2 \quad \text{or} \quad n = -21.2$$



$n < -21.2$  or  $n > 30.2$   
but number of terms can't be negative, so  $n > 30.2$

$$n = 31$$