

The diagram shows the part of the curve $y = x^2 \cos 3x$ for $0 \leq x \leq \frac{1}{6}\pi$, and its maximum point M , where $x = a$.

(a) Show that a satisfies the equation $a = \frac{1}{3} \tan^{-1}\left(\frac{2}{3a}\right)$. [3]

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$v = \cos 3x$$

$$\frac{dv}{dx} = -3 \sin 3x$$

$$\frac{dy}{dx} = -3x^2 \sin 3x + 2x \cos 3x$$

SP, so $\frac{dy}{dx} = 0$:

$$-3x^2 \sin 3x + 2x \cos 3x = 0$$

$$2x \cos 3x = 3x^2 \sin 3x$$

$$2 \cos 3x = 3x \sin 3x$$

$$2 = 3x \times \frac{\sin 3x}{\cos 3x}$$

$$\frac{2}{3x} = \frac{\sin 3x}{\cos 3x}$$

$$\tan 3x = \frac{2}{3x}$$

$$3x = \tan^{-1}\left(\frac{2}{3x}\right)$$

$$x = \frac{1}{3} \tan^{-1}\left(\frac{2}{3x}\right)$$

$$a = \frac{1}{3} \tan^{-1}\left(\frac{2}{3a}\right) \quad \text{QED}$$

- (b) Use an iterative formula based on the equation in (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \frac{1}{3} \tan^{-1} \left(\frac{2}{3a_n} \right)$$

$$a_1 = \frac{\pi}{6}$$

$$a_2 = 0.3017$$

$$a_3 = 0.3820$$

$$a_4 = 0.3502$$

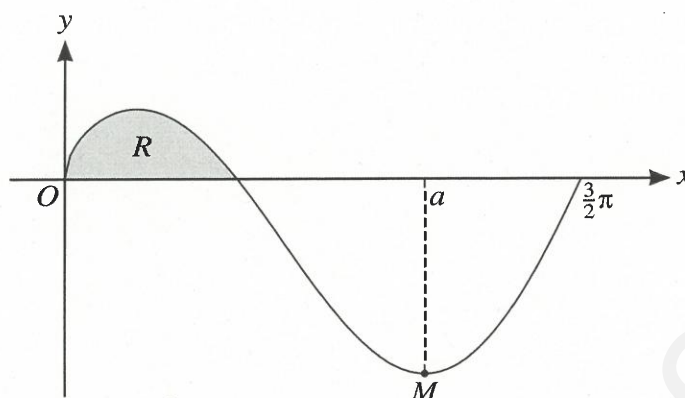
$$a_5 = 0.3624$$

$$a_6 = 0.3576$$

$$a_7 = 0.3595$$

a_5, a_6 and a_7 all round to 0.36 to 2dp,
so $a = 0.36$

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The diagram shows the curve $y = \sqrt{x} \cos x$, for $0 \leq x \leq \frac{3}{2}\pi$, and its minimum point M , where $x = a$. The shaded region between the curve and the x -axis is denoted by R .

- (a) Show that a satisfies the equation $\tan a = \frac{1}{2a}$. [3]

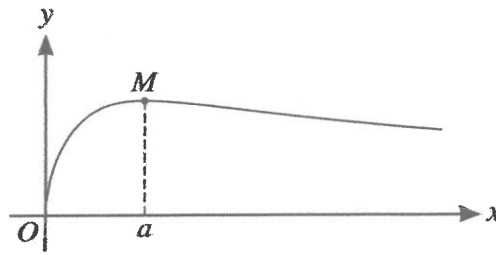
$$\begin{aligned}
 u &= x^{\frac{1}{2}} & v &= \cos x \\
 \frac{du}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} & \frac{dv}{dx} &= -\sin x \\
 \frac{dy}{dx} &= -x^{\frac{1}{2}} \sin x + \frac{1}{2}x^{-\frac{1}{2}} \cos x \\
 \frac{dy}{dx} = 0: & \quad -\sqrt{x} \sin x + \frac{1}{2} \frac{\cos x}{\sqrt{x}} = 0 \\
 & \quad -x \sin x + \frac{1}{2} \cos x = 0 \\
 & \quad x \sin x = \frac{1}{2} \cos x \quad \rightarrow \quad \tan a = \frac{1}{2a} \quad \text{QED}
 \end{aligned}$$

- (b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\frac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a .

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$\begin{aligned}
 a_1 &= 3.0000 & a_3, a_4 \text{ and } a_5 & \text{ all round} \\
 a_2 &= 3.3067 & \text{to } 3.29 \text{ to 2dp, so} \\
 a_3 &= 3.2917 & x &= 3.29 \\
 a_4 &= 3.2923 \\
 a_5 &= 3.2923
 \end{aligned}$$

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The diagram shows the curve $y = \frac{\tan^{-1}x}{\sqrt{x}}$ and its maximum point M where $x = a$.

(a) Show that a satisfies the equation

$$a = \tan\left(\frac{2a}{1+a^2}\right). \quad [4]$$

$$y = \frac{\tan^{-1}x}{\sqrt{x}}$$

$$u = \tan^{-1}x$$

$$v = x^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{1+x^2}$$

$$\frac{dv}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}} \times \frac{1}{1+x^2} - \frac{1}{2}x^{-\frac{1}{2}} \tan^{-1}x}{(x^{\frac{1}{2}})^2}$$

$$\frac{dy}{dx} = 0: \frac{\frac{\sqrt{x}}{1+x^2} - \frac{1}{2} \times \frac{\tan^{-1}x}{\sqrt{x}}}{x} = 0$$

$$\frac{\sqrt{x}}{1+x^2} - \frac{\tan^{-1}x}{2\sqrt{x}} = 0$$

$$\frac{\tan^{-1}x}{2\sqrt{x}} = \frac{\sqrt{x}}{1+x^2}$$

$$\tan^{-1}x = \frac{2x}{1+x^2}$$

$$x = \tan\left(\frac{2x}{1+x^2}\right) \quad \text{QED}$$

- (b) Verify by calculation that
- a
- lies between 1.3 and 1.5.

[2]

$$\text{Set } = 0: \quad \tan\left(\frac{2a}{1+a^2}\right) - a = 0$$

$$a = 1.3: \quad \tan\left(\frac{2 \times 1.3}{1 + 1.3^2}\right) - 1.3 = 0.148$$

$$a = 1.5: \quad \tan\left(\frac{2 \times 1.5}{1 + 1.5^2}\right) - 1.5 = -0.178$$

Given there's a sign change (and the function is continuous) between $x=1.3$ and $x=1.5$, there is a root between these two values.

- (c) Use an iterative formula based on the equation in part (a) to determine
- a
- correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \tan\left(\frac{2a_n}{1+a_n^2}\right)$$

$$a_1 = 1.3000$$

$$a_2 = 1.4484$$

$$a_3 = 1.3552$$

$$a_4 = 1.4148$$

$$a_5 = 1.3769$$

$$a_6 = 1.4012$$

$$a_7 = 1.3857$$

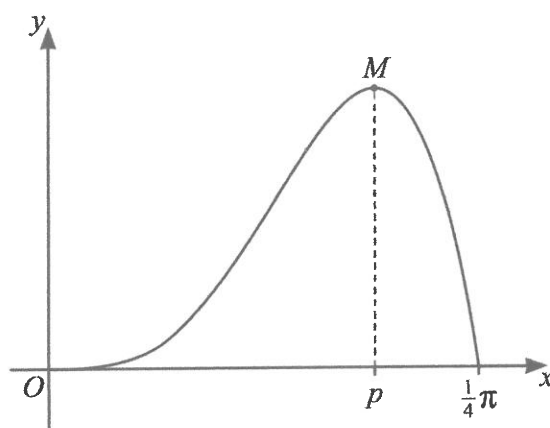
$$a_8 = 1.3956$$

$$a_9 = 1.3893$$

$$a_{10} = 1.3933$$

$$a_{11} = 1.3907$$

a_9, a_{10} and a_{11} all round to 1.39 to 2dp, so $x = \underline{\underline{1.39}}$



The diagram shows the curve $y = x^3 \cos 2x$ for $0 \leq x \leq \frac{1}{4}\pi$. The curve has a maximum point at M , where $x = p$.

- (a) Show that p satisfies the equation $p = \frac{1}{2} \tan^{-1}\left(\frac{3}{2p}\right)$. [3]

$$y = x^3 \cos 2x$$

$$u = x^3$$

$$v = \cos 2x$$

$$\frac{du}{dx} = 3x^2$$

$$\frac{dv}{dx} = -2\sin 2x$$

$$\frac{dy}{dx} = -2x^3 \sin 2x + 3x^2 \cos 2x$$

$$\text{SP, so } \frac{dy}{dx} = 0: -2x^3 \sin 2x + 3x^2 \cos 2x = 0$$

$$2x^3 \sin 2x = 3x^2 \cos 2x$$

$$\div \cos 2x \quad \div \cos 2x$$

$$2x^3 \tan 2x = 3x^2$$

$$\div 2x^3 \quad \div 2x^3$$

$$\tan 2x = \frac{3}{2x}$$

$$2x = \tan^{-1}\left(\frac{3}{2x}\right)$$

$$x = \frac{1}{2} \tan^{-1}\left(\frac{3}{2x}\right)$$

QED

- (b) Show by calculation that $0.5 < p < 0.7$.

[2]

$$\text{Set } = 0: p - \frac{1}{2} \tan^{-1} \left(\frac{3}{2p} \right) = 0$$

$$x = 0.5: 0.5 - \frac{1}{2} \tan^{-1} \left(\frac{3}{2 \times 0.5} \right) = -0.125$$

$$x = 0.7: 0.7 - \frac{1}{2} \tan^{-1} \left(\frac{3}{2 \times 0.7} \right) = 0.133$$

Sign change (and continuous function)
between 0.5 and 0.7 shows there's
a root between 0.5 and 0.7.

- (c) Use an iterative formula based on the equation in part (a) to calculate p correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

[3]

$$p_{n+1} = \frac{1}{2} \tan^{-1} \left(\frac{3}{2p_n} \right)$$

$$p_1 = 0.50000$$

$$p_2 = 0.62452$$

$$p_3 = 0.58814$$

$$p_4 = 0.59856$$

$$p_5 = 0.59556$$

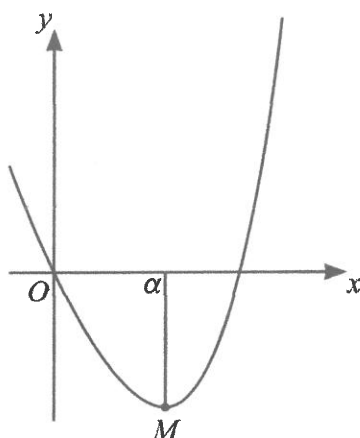
$$p_6 = 0.59642$$

$$p_7 = 0.59617$$

p_5, p_6 and p_7 all round to 0.596 to 3dp



7



The diagram shows the curve $y = xe^{2x} - 5x$ and its minimum point M , where $x = \alpha$.

- (a) Show that α satisfies the equation $\alpha = \frac{1}{2} \ln\left(\frac{5}{1+2\alpha}\right)$.

[3]

$$y = \overset{u}{x} \overset{v}{e^{2x}} - 5x$$

$$u = x \quad v = e^{2x}$$

$$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = 2e^{2x}$$

$$\rightarrow 2xe^{2x} + e^{2x}$$

$$\frac{dy}{dx} = 2xe^{2x} + e^{2x} - 5$$

SP, so $\frac{dy}{dx} = 0$:

$$2xe^{2x} + e^{2x} - 5 = 0$$

$$e^{2x}(2x + 1) - 5 = 0$$

$$e^{2x}(2x + 1) = 5$$

$$e^{2x} = \frac{5}{1+2x}$$

$$2x = \ln\left(\frac{5}{1+2x}\right)$$

$$x = \frac{1}{2} \ln\left(\frac{5}{1+2x}\right)$$

$$\alpha = \frac{1}{2} \ln\left(\frac{5}{1+2\alpha}\right)$$

- (b) Verify by calculation that α lies between 0.4 and 0.5 .

[2]

$$\text{Set } = 0: \frac{1}{2} \ln\left(\frac{5}{1+2\alpha}\right) - \alpha = 0$$

$$\alpha = 0.4: \frac{1}{2} \ln\left(\frac{5}{1+0.8}\right) - 0.4 = 0.111$$

$$\alpha = 0.5: \frac{1}{2} \ln\left(\frac{5}{1+1}\right) - 0.5 = -0.042$$

Given there's a sign change (and the function is continuous) between $x=0.4$ and $x=0.5$, there is a root between 0.4 and 0.5.

- (c) Use an iterative formula based on the equation in part (a) to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$\alpha_{n+1} = \frac{1}{2} \ln\left(\frac{5}{1+2\alpha_n}\right)$$

$$\alpha_1 = 0.4000$$

$$\alpha_2 = 0.5108$$

$$\alpha_3 = 0.4528$$

$$\alpha_4 = 0.4823$$

$$\alpha_5 = 0.4671$$

$$\alpha_6 = 0.4749$$

$$\alpha_7 = 0.4709$$

α_5, α_6 and α_7 all round to 0.47 to 2dp, so $\alpha = \underline{\underline{0.47}}$

- 8 The curve with equation $y = \frac{x^3}{e^x - 1}$ has a stationary point at $x = p$, where $p > 0$.

(a) Show that $p = 3(1 - e^{-p})$.

[3]

$$y = \frac{x^3}{e^x - 1}$$

$$u = x^3$$

$$v = e^x - 1$$

$$\frac{du}{dx} = 3x^2$$

$$\frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{3x^2(e^x - 1) - x^3 e^x}{(e^x - 1)^2}$$

$$\frac{dy}{dx} = 0 \text{ when } x = p: \frac{3p^2(e^p - 1) - p^3 e^p}{(e^p - 1)^2} = 0$$

$$3p^2(e^p - 1) - p^3 e^p = 0$$

$$3p^2(e^p - 1) = p^3 e^p$$

$$3(e^p - 1) = p e^p$$

$$p e^p = 3(e^p - 1)$$

$$p = \frac{3e^p - 3}{e^p}$$

$$p = \frac{3e^p}{e^p} - \frac{3}{e^p}$$

$$p = 3 - 3e^{-p}$$

$$p = 3(1 - e^{-p})$$

QED

- (b) Verify by calculation that p lies between 2.5 and 3. [2]

$$\text{set } = 0: \quad p - 3(1 - e^{-p}) = 0$$

$$p = 2.5: \quad 2.5 - 3(1 - e^{-2.5}) = -0.254$$

$$p = 3: \quad 3 - 3(1 - e^{-3}) = 0.149$$

Sign change (and continuous function) between 2.5 and 3 shows there's a root between $x=2.5$ and $x=3$.

- (c) Use an iterative formula based on the equation in part (a) to determine p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$p_{n+1} = 3(1 - e^{-p_n})$$

$$p_1 = 2.5000$$

$$p_2 = 2.7537$$

$$p_3 = 2.8089$$

$$p_4 = 2.8192$$

$$p_5 = 2.8210$$

$$p_6 = 2.8214$$

p_4, p_5 and p_6 all round to 2.82 to 2dp
so $p = 2.82$

- 9 Let $f(x) = \frac{e^{2x} + 1}{e^{2x} - 1}$, for $x > 0$.

(a) The equation $x = f(x)$ has one root, denoted by a .

Verify by calculation that a lies between 1 and 1.5.

[2]

$$x = f(x): \quad x = \frac{e^{2x} + 1}{e^{2x} - 1}$$

$$\text{Set } = 0: \quad x(e^{2x} - 1) = e^{2x} + 1$$

$$x(e^{2x} - 1) - e^{2x} - 1 = 0$$

$$x = 1: \quad 1(e^{2(1)} - 1) - e^{2(1)} - 1 = -2$$

$$x = 1.5: \quad 1.5(e^{2(1.5)} - 1) - e^{2(1.5)} - 1 = 7.543$$

Sign change (and continuous function) between 1 and 1.5 shows there's a root between $x = 1$ and $x = 1.5$.

- (b) Use an iterative formula based on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \frac{e^{2a_n} + 1}{e^{2a_n} - 1}$$

$$a_1 = 1.0000$$

$$a_2 = 1.3130$$

$$a_3 = 1.1560$$

$$a_4 = 1.2199$$

$$a_5 = 1.1910$$

$$a_6 = 1.2035$$

$$a_7 = 1.1980$$

$$a_8 = 1.2004$$

a_6, a_7 and a_8 all round

to 1.20 to 2dp, so

$$\underline{a = 1.20}$$

(c) Find $f'(x)$. Hence find the exact value of x for which $f'(x) = -8$.

[6]

$$f(x) = \frac{e^{2x} + 1}{e^{2x} - 1}$$

$$u = e^{2x} + 1$$

$$\frac{du}{dx} = 2e^{2x}$$

$$v = e^{2x} - 1$$

$$\frac{dv}{dx} = 2e^{2x}$$

$$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1)}{(e^{2x} - 1)^2}$$

$$f'(x) = -8: \quad \frac{2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1)}{(e^{2x} - 1)^2} = -8$$

$$2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1) = -8(e^{2x} - 1)^2$$

$$2e^{4x} - 2e^{2x} - 2e^{4x} - 2e^{2x} = -8(e^{4x} - 2e^{2x} + 1)$$

$$-4e^{2x} = -8e^{4x} + 16e^{2x} - 8$$

$$8e^{4x} - 20e^{2x} + 8 = 0$$

$$2e^{4x} - 5e^{2x} + 2 = 0$$

$$\text{let } y = e^{2x}: \quad 2y^2 - 5y + 2 = 0$$

$$(2y - 1)(y - 2) = 0$$

$$2y - 1 = 0$$

or

$$y - 2 = 0$$

$$y = \frac{1}{2}$$

or

$$y = 2$$

$$e^{2x} = \frac{1}{2}$$

or

$$e^{2x} = 2$$

$$2x = \ln\left(\frac{1}{2}\right)$$

or

$$2x = \ln 2$$

$$x = \frac{1}{2} \ln\left(\frac{1}{2}\right)$$

$$x = \frac{1}{2} \ln 2$$

↑
negative, but $x > 0$
so reject solution.

8 The constant a is such that $\int_1^a \frac{\ln x}{\sqrt{x}} dx = 6$.

(a) Show that $a = \exp\left(\frac{1}{\sqrt{a}} + 2\right)$.

[5]

[$\exp(x)$ is an alternative notation for e^x .]

$$\int_1^a x^{-\frac{1}{2}} \ln x dx = 6$$

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = 2x^{\frac{1}{2}}$$

$$\frac{dv}{dx} = x^{-\frac{1}{2}}$$

$$= 2x^{\frac{1}{2}} \ln x - \int_1^a 2x^{\frac{1}{2}} \times \frac{1}{x} dx$$

$$= 2x^{\frac{1}{2}} \ln x - 2 \int_1^a x^{-\frac{1}{2}} dx$$

$$= \left[2x^{\frac{1}{2}} \ln x - 4x^{\frac{1}{2}} \right]_1^a$$

$$= \left[2a^{\frac{1}{2}} \ln a - 4a^{\frac{1}{2}} \right] - [0 - 4]$$

$$= 2\sqrt{a} \ln a - 4\sqrt{a} + 4 = 6$$

$$2\sqrt{a} \ln a - 4\sqrt{a} = 2$$

$$2\sqrt{a} \ln a = 2 + 4\sqrt{a}$$

$$\sqrt{a} \ln a = 1 + 2\sqrt{a}$$

$$\ln a = \frac{1}{\sqrt{a}} + 2$$

$$a = \exp\left(\frac{1}{\sqrt{a}} + 2\right) \quad \text{QED}$$

- (b) Verify by calculation that a lies between 9 and 11.

[2]

$$\text{Set } = 0: \quad a - \exp\left(\frac{1}{\sqrt{a}} + 2\right) = 0$$

$$a = 9: \quad 9 - \exp\left(\frac{1}{\sqrt{9}} + 2\right) = -1.312$$

$$a = 11: \quad 11 - \exp\left(\frac{1}{\sqrt{11}} + 2\right) = 1.011$$

Sign change (and continuous function) between 9 and 11 shows there's a root between $x=9$ and $x=11$.

- (c) Use an iterative formula based on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \exp\left(\frac{1}{\sqrt{a_n}} + 2\right)$$

$$a_1 = 9.0000$$

$$a_2 = 10.3123$$

$$a_3 = 10.0886$$

$$a_4 = 10.1233$$

$$a_5 = 10.1178$$

$$a_6 = 10.1187$$

a_4, a_5 and a_6 all round to 10.12 to 2dp, therefore $x = 10.12$

11 The equation of a curve is $y = \sqrt{\tan x}$, for $0 \leq x < \frac{1}{2}\pi$.

(a) Express $\frac{dy}{dx}$ in terms of $\tan x$, and verify that $\frac{dy}{dx} = 1$ when $x = \frac{1}{4}\pi$. [4]

$$y = (\tan x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(\tan x)^{-\frac{1}{2}} \times \sec^2 x \quad \leftarrow \sec^2 x = 1 + \tan^2 x$$

$$= \frac{1}{2}(\tan x)^{-\frac{1}{2}} \times (1 + \tan^2 x)$$

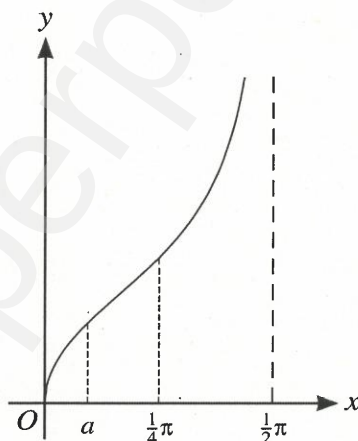
$$x = \frac{\pi}{4} : \quad \frac{dy}{dx} = \frac{1}{2}(\tan(\frac{\pi}{4}))^{-\frac{1}{2}} \times (1 + \tan^2(\frac{\pi}{4}))$$

$$= \frac{1}{2}(1)^{-\frac{1}{2}} \times (1 + 1^2)$$

$$= \frac{1}{2} \times 1 \times 2$$

$$= \underline{1}$$

The value of $\frac{dy}{dx}$ is also 1 at another point on the curve where $x = a$, as shown in the diagram.



(b) Show that $t^3 + t^2 + 3t - 1 = 0$, where $t = \tan a$. [4]

$$\frac{1}{2}(\tan a)^{-\frac{1}{2}} \times (1 + \tan^2 a) = 1$$

$$\frac{1}{2\sqrt{\tan a}} \times (1 + \tan^2 a) = 1$$

$$\frac{1}{2\sqrt{t}} (1 + t^2) = 1$$

$$1 + t^2 = 2\sqrt{t}$$

$$(1 + t^2)^2 = (2\sqrt{t})^2$$

$$1 + 2t^2 + t^4 = 4t$$

$$t^4 + 2t^2 - 4t + 1 = 0$$

from part (a): $\tan(\frac{\pi}{4})$ is a root:

when $\alpha = \frac{\pi}{4}$ $t = \tan(\frac{\pi}{4})$

$t = 1$ so $(t-1)$ is a factor:

$$t-1 \overline{) \begin{array}{r} t^4 + 0t^3 + 2t^2 - 4t + 1 \\ t^4 - t^3 \\ \hline t^3 + 2t^2 - 4t + 1 \end{array}}$$

$$\begin{array}{r} t^3 + 2t^2 - 4t + 1 \\ t^3 - t^2 \\ \hline 3t^2 - 4t + 1 \end{array}$$

$$\begin{array}{r} 3t^2 - 4t + 1 \\ 3t^2 - 3t \\ \hline -t + 1 \end{array}$$

$$(t-1)(t^3 + t^2 + 3t - 1) = 0$$

$$\text{so } \underline{t^3 + t^2 + 3t - 1 = 0}$$

QED

(c) Use the iterative formula

$$a_{n+1} = \tan^{-1} \left(\frac{1}{3} (1 - \tan^2 a_n - \tan^3 a_n) \right)$$

to determine a correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

from the graph in part (b), a is between 0 and $\frac{\pi}{4}$.

$$a_1 = 0.0000$$

$$a_2 = 0.3218$$

$$a_3 = 0.2767$$

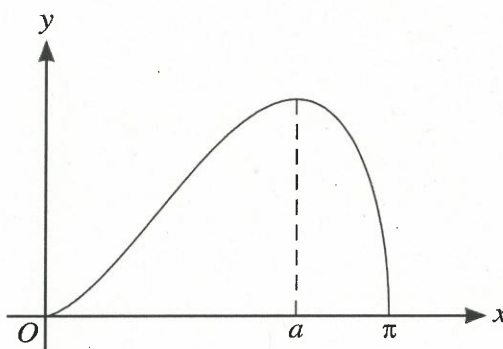
$$a_4 = 0.2904$$

$$a_5 = 0.2866$$

$$a_6 = 0.2877$$

a_4, a_5 and a_6 all round to 0.29 to 2dp, so

$$\underline{a = 0.29}$$



The curve $y = x\sqrt{\sin x}$ has one stationary point in the interval $0 < x < \pi$, where $x = a$ (see diagram).

(a) Show that $\tan a = -\frac{1}{2}a$.

[4]

$$y = x^u \sqrt{\sin x}^v$$

$$u = x$$

$$\frac{du}{dx} = 1$$

$$v = (\sin x)^{\frac{1}{2}}$$

$$\frac{dv}{dx} = \frac{1}{2}(\sin x)^{-\frac{1}{2}} \times \cos x$$

$$\frac{dy}{dx} = \frac{1}{2}x(\sin x)^{-\frac{1}{2}}\cos x + (\sin x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 0: \quad \frac{1}{2}x \frac{\cos x}{\sqrt{\sin x}} + \frac{\sqrt{\sin x}}{x \sqrt{\sin x}} = 0$$

$$\frac{1}{2}x \cos x + \sin x = 0$$

$$\sin x = -\frac{1}{2}x \cos x$$

$$\frac{\sin x}{\cos x} = -\frac{1}{2}x$$

$$\tan x = -\frac{1}{2}x$$

$$\text{or } \tan a = -\frac{1}{2}a \quad \text{QED}$$

- (b) Verify by calculation that a lies between 2 and 2.5. [2]

Set $= 0$: $\tan x + \frac{1}{2}x = 0$

$x = 2$: $\tan(2) + \frac{1}{2}(2) = -1.185$

$x = 2.5$: $\tan(2.5) + \frac{1}{2}(2.5) = 0.503$

Given there's a sign change (and the function is continuous) between $x=2$ and $x=2.5$, there's a root between $x=2$ and $x=2.5$.

- (c) Show that if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula $x_{n+1} = \pi - \tan^{-1}(\frac{1}{2}x_n)$ converges, then it converges to a , the root of the equation in part (a). [2]

$$x = \pi - \tan^{-1}\left(\frac{1}{2}x\right)$$

$$\tan^{-1}\left(\frac{1}{2}x\right) = \pi - x$$

$$\frac{1}{2}x = \tan(\pi - x)$$

$$\frac{1}{2}x = \frac{\tan \pi - \tan x}{1 + \tan \pi \tan x}$$

$$\frac{1}{2}x = \frac{0 - \tan x}{1 + 0}$$

$$\frac{1}{2}x = -\tan x$$

$$\tan x = -\frac{1}{2}x \quad \text{QED}$$

- (d) Use the iterative formula given in part (c) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$x_1 = 2.0000$$

$$x_2 = 2.3562$$

$$x_3 = 2.2746$$

$$x_4 = 2.2920$$

$$x_5 = 2.2883$$

$$x_6 = 2.2891$$

x_4, x_5 and x_6 all round

to 2.29 to 2dp

so $x = 2.29$

9 The constant a is such that $\int_0^a x e^{-2x} dx = \frac{1}{8}$.

(a) Show that $a = \frac{1}{2} \ln(4a + 2)$.

[5]

$$u = x$$

$$\frac{du}{dx} = 1$$

$$v = -\frac{1}{2} e^{-2x}$$

$$\frac{dv}{dx} = e^{-2x}$$

$$\rightarrow -\frac{1}{2} x e^{-2x} + \int_0^a \frac{1}{2} e^{-2x} dx$$

$$= \left[-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^a$$

$$= \left[-\frac{1}{2} (a) e^{-2a} - \frac{1}{4} e^{-2a} \right] - \left[0 - \frac{1}{4} e^0 \right]$$

$$= -\frac{a}{2} e^{-2a} - \frac{1}{4} e^{-2a} + \frac{1}{4}$$

integral = $\frac{1}{8}$:

$$\frac{-a}{2} e^{-2a} - \frac{1}{4} e^{-2a} + \frac{1}{4} = \frac{1}{8}$$

$$-4ae^{-2a} - 2e^{-2a} + 2 = 1$$

$$-4ae^{-2a} - 2e^{-2a} + 1 = 0$$

$$4ae^{-2a} + 2e^{-2a} = 1$$

$$e^{-2a} (4a + 2) = 1$$

$$e^{-2a} = \frac{1}{4a + 2}$$

$$e^{2a} = 4a + 2$$

$$2a = \ln(4a + 2)$$

$$a = \frac{1}{2} \ln(4a + 2) \quad \text{QED}$$

- (b) Verify by calculation that a lies between 0.5 and 1.

[2]

Set = 0:

$$a - \frac{1}{2} \ln(4a + 2) = 0$$

$a = 0.5$:

$$0.5 - \frac{1}{2} \ln(2 + 2) = -0.193$$

$a = 1$:

$$1 - \frac{1}{2} \ln(4 + 2) = 0.104$$

Sign change (and continuous function) between $a = 0.5$ and $a = 1$ shows there's a root between 0.5 and 1.

- (c) Use an iterative formula based on the equation in (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \frac{1}{2} \ln(4a_n + 2)$$

$$a_1 = 0.5000$$

$$a_2 = 0.6931$$

$$a_3 = 0.7814$$

$$a_4 = 0.8171$$

$$a_5 = 0.8309$$

$$a_6 = 0.8361$$

$$a_7 = 0.8380$$

$$a_8 = 0.8388$$

a_6, a_7 and a_8 all

round to 0.84

to 2dp, so $a = 0.84$

10 The constant a is such that $\int_1^a x^2 \ln x \, dx = 4$.

(a) Show that $a = \left(\frac{35}{3 \ln a - 1} \right)^{\frac{1}{3}}$.

[5]

$$\int_1^a x^2 \ln x \, dx = 4$$

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{1}{3}x^3$$

$$\frac{dv}{dx} = x^2$$

$$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \times \frac{1}{x} \, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 \, dx$$

$$= \left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 \right]_1^a$$

$$= \left[\frac{1}{3}a^3 \ln a - \frac{1}{9}a^3 \right] - \left[\frac{1}{3} \times 1^3 \times \ln(1) - \frac{1}{9} \times 1^3 \right]$$

$$= \frac{1}{3}a^3 \ln a - \frac{1}{9}a^3 + \frac{1}{9} = 4$$

$$3a^3 \ln a - a^3 + 1 = 36$$

$$3a^3 \ln a - a^3 = 35$$

$$a^3(3 \ln a - 1) = 35$$

$$a^3 = \frac{35}{3 \ln a - 1}$$

$$a = \left(\frac{35}{3 \ln a - 1} \right)^{\frac{1}{3}}$$

QED

- (b) Verify by calculation that a lies between 2.4 and 2.8.

[2]

$$\text{Set } = 0: a - \left(\frac{35}{3 \ln a - 1} \right)^{\frac{1}{3}} = 0$$

$$a = 2.4: 2.4 - \left(\frac{35}{3 \ln(2.4) - 1} \right)^{\frac{1}{3}} = -0.382$$

$$a = 2.8: 2.8 - \left(\frac{35}{3 \ln(2.8) - 1} \right)^{\frac{1}{3}} = 0.241$$

Sign change (and continuous function) between $x = 2.4$ and 2.8 indicates root between $x = 2.4$ and $x = 2.8$.

- (c) Use an iterative formula based on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \left(\frac{35}{3 \ln a_n - 1} \right)^{\frac{1}{3}}$$

$$a_1 = 2.4000$$

$$a_0 = 2.6377$$

$$a_2 = 2.7815$$

$$a_3 = 2.5671$$

a_5, a_9 and a_{10} all round to

$$a_4 = 2.6751$$

2.64 to 2dp so $x = 2.64$.

$$a_5 = 2.6174$$

$$a_6 = 2.6473$$

$$a_7 = 2.6315$$

$$a_8 = 2.6398$$

$$a_9 = 2.6354$$

- 7 The equation of a curve is $y = \frac{x}{\cos^2 x}$, for $0 \leq x < \frac{1}{2}\pi$. At the point where $x = a$, the tangent to the curve has gradient equal to 12.

(a) Show that $a = \cos^{-1}\left(\sqrt[3]{\frac{\cos a + 2a \sin a}{12}}\right)$. [3]

$$y = \frac{x}{\cos^2 x}$$

$$u = x$$

$$\frac{du}{dx} = 1$$

$$v = \cos^2 x$$

$$\begin{aligned}\frac{dv}{dx} &= 2\cos x \times -\sin x \\ &= -2\sin x \cos x\end{aligned}$$

$$\frac{dy}{dx} = \frac{1 \times \cos^2 x + 2x \sin x \cos x}{\cos^4 x}$$

at $x=a$, $\frac{dy}{dx} = 12$: $\frac{\cos^2 a + 2a \sin a \cos a}{\cos^4 a} = 12$

$$\cos^2 a + 2a \sin a \cos a = 12 \cos^4 a$$

$$\cos a + 2a \sin a = 12 \cos^3 a$$

$$\frac{\cos a + 2a \sin a}{12} = \cos^3 a$$

$$\cos a = \sqrt[3]{\frac{\cos a + 2a \sin a}{12}}$$

$$a = \cos^{-1}\left(\sqrt[3]{\frac{\cos a + 2a \sin a}{12}}\right)$$

QED

- (b) Verify by calculation that a lies between 0.9 and 1.

[2]

Set = 0: $a - \cos^{-1}\left(\sqrt[3]{\frac{\cos a + 2\sin a}{12}}\right) = 0$

$a = 0.9$: $0.9 - \cos^{-1}\left(\sqrt[3]{\frac{\cos(0.9) + 2(0.9)\sin(0.9)}{12}}\right) = -0.0846$

$a = 1$: $1 - \cos^{-1}\left(\sqrt[3]{\frac{\cos(1) + 2(1)\sin(1)}{12}}\right) = 0.0358$

sign change (and continuous function) between 0.9 and 1 shows there's a root between $x=0.9$ and $x=1$.

- (c) Use an iterative formula based on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[3]

$$a_{n+1} = \cos^{-1}\left(\sqrt[3]{\frac{\cos(a_n) + 2a_n\sin(a_n)}{12}}\right)$$

$$a_1 = 0.9000$$

$$a_2 = 0.9846$$

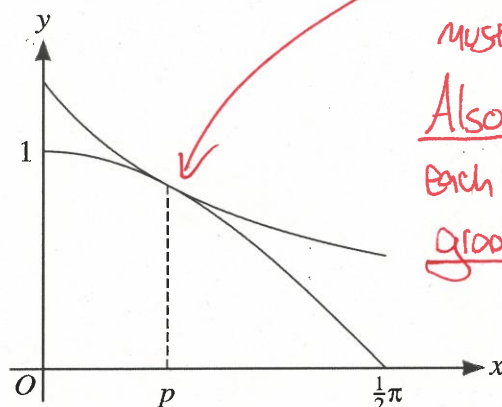
$$a_3 = 0.9673$$

$$a_4 = 0.9708$$

$$a_5 = 0.9701$$

a_3, a_4 and a_5 all round to 0.97 to 2dp

so $a = 0.97$



Lines intersect, so equations must be equal.

Also, lines are tangential to each other, meaning their gradients must be equal.

The diagram shows the curves $y = \cos x$ and $y = \frac{k}{1+x}$, where k is a constant, for $0 \leq x \leq \frac{1}{2}\pi$. The curves touch at the point where $x = p$.

(a) Show that p satisfies the equation $\tan p = \frac{1}{1+p}$. [5]

equations equal: $\cos x = \frac{k}{1+x}$ (A)

gradients: (1): $\frac{dy}{dx} = -\sin x$ (2): $y = k(1+x)^{-1}$
 $\frac{dy}{dx} = -k(1+x)^{-2}$

gradients equal: $-\sin x = \frac{-k}{(1+x)^2}$

$\sin x = \frac{k}{(1+x)^2}$ (B)

(B) $\div$ (A): $\frac{\sin x}{\cos x} = \frac{\frac{k}{(1+x)^2}}{\frac{k}{1+x}}$ $\frac{x(1+x)^2}{x(1+x)^2}$

$\tan x = \frac{k}{k(1+x)}$

$\tan x = \frac{1}{1+x}$

$\tan p = \frac{1}{1+p}$ QED

- (b) Use the iterative formula $p_{n+1} = \tan^{-1}\left(\frac{1}{1+p_n}\right)$ to determine the value of p correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$$p_1 = 0.00000$$

$$p_2 = 0.78540$$

$$p_3 = 0.51056$$

$$p_4 = 0.58477$$

$$p_5 = 0.56291$$

$$p_6 = 0.56919$$

$$p_7 = 0.56737$$

$$p_8 = 0.56790$$

$$p_9 = 0.56775$$

$$p_{10} = 0.56779$$

p_6, p_9 and p_{10} all round to 0.568 to 3dp
so $x = 0.568$

- (c) Hence find the value of k correct to 2 decimal places. [2]

Sub into (A): $\cos x = \frac{k}{1+x}$

$$\cos(0.568) = \frac{k}{1+0.568}$$

$$\cos(0.568) = \frac{k}{1.568}$$

$$k = 1.568 \cos(0.568)$$

$$= \underline{1.32}$$