7 Let $f(x) = \dfrac{\cos x}{1 + \sin x}$.

(b) Find $\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x)\,dx$. Give your answer in a simplified exact form. [4]

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x}\,dx$$

$$= \Big[\ln|1+\sin x|\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[\ln\left|1+\sin\left(\tfrac{\pi}{2}\right)\right|\right] - \left[\ln\left|1+\sin\left(\tfrac{\pi}{6}\right)\right|\right]$$

$$= \ln 2 - \ln \tfrac{3}{2}$$

$$= \ln\left(\frac{2 \times 2}{\frac{3}{2} \times 2}\right)$$

$$= \ln\left(\frac{4}{3}\right)$$

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4 (a) Prove that $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} \equiv \tan^2\theta$. [2]

LHS: $\dfrac{1-\cos 2\theta}{1+\cos 2\theta}$ ← $1-2\sin^2\theta$ ← $2\cos^2\theta - 1$

$= \dfrac{1-(1-2\sin^2\theta)}{1+2\cos^2\theta-1}$

$= \dfrac{1-1+2\sin^2\theta}{2\cos^2\theta}$

$= \dfrac{2\sin^2\theta}{2\cos^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$ QED

(b) Hence find the exact value of $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1-\cos 2\theta}{1+\cos 2\theta}\, d\theta$. [4]

$\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2\theta \, d\theta$ ← $1+\tan^2\theta = \sec^2\theta$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta$

$= \left[\tan\theta - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left[\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right] - \left[\tan\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right]$

$= \sqrt{3} - \frac{\pi}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{6}$

$= \frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{6} - \frac{\pi}{3}$

$= \frac{2\sqrt{3}}{3} - \frac{\pi}{6}$

6 (a) Using the expansions of $\sin(3x+2x)$ and $\sin(3x-2x)$, show that

$$\tfrac{1}{2}(\sin 5x + \sin x) \equiv \sin 3x \cos 2x.$$ [3]

$\sin 5x$: $\sin(3x+2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$

$\sin x$: $\sin(3x-2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$

+ + +

$\sin 5x + \sin x = 2\sin 3x \cos 2x$

$\div 2$ $\div 2$

$\frac{1}{2}(\sin 5x + \sin x) = \sin 3x \cos 2x$ QED

(b) Hence show that $\int_0^{\frac{1}{4}\pi} \sin 3x \cos 2x \, dx = \frac{1}{5}(3 - \sqrt{2})$. [3]

$$\int_0^{\frac{\pi}{4}} \sin 3x \cos 2x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(\sin 5x + \sin x) \, dx \quad \text{(from part (a))}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 5x \, dx + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin x \, dx$$

$$= \left[\frac{-1}{10}\cos 5x\right]_0^{\frac{\pi}{4}} + \left[\frac{-1}{2}\cos x\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{-1}{10}\cos 5x\right]_0^{\frac{\pi}{4}} - \left[\frac{1}{2}\cos x\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{-1}{10}\cos\left(\frac{5\pi}{4}\right) - \frac{-1}{10}\cos(0)\right) - \left(\frac{1}{2}\cos\left(\frac{\pi}{4}\right) - \frac{1}{2}\cos(0)\right)$$

$$= \left(\frac{-1}{10} \times \frac{-\sqrt{2}}{2} + \frac{1}{10}\right) - \left(\frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{20} + \frac{1}{10} - \frac{\sqrt{2}}{4} + \frac{1}{2}$$

$$= \frac{3}{5} - \frac{\sqrt{2}}{5}$$

$$= \frac{1}{5}(3 - \sqrt{2})$$

6 **(a)** Prove that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan\theta$. [3]

LHS: $\dfrac{1}{\sin 2\theta} - \dfrac{\cos 2\theta}{\sin 2\theta}$

$\rightarrow \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

$= \dfrac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$

$= \dfrac{\cancel{2}\sin^{\cancel{2}}\theta}{\cancel{2}\cancel{\sin\theta}\cos\theta}$

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$ QED

(b) Hence show that $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\operatorname{cosec} 2\theta - \cot 2\theta)\,d\theta = \tfrac{1}{2}\ln 2$. [4]

$\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan\theta\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta}\, d\theta$

$= \Big[-\ln|\cos\theta|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \Big[\ln|\cos\theta|\Big]_{\frac{\pi}{3}}^{\frac{\pi}{4}}$ Switching limits and changing sign of $\ln|\cos\theta|$

$= \left[\ln|\cos(\tfrac{\pi}{4})|\right] - \left[\ln|\cos(\tfrac{\pi}{3})|\right]$

$= \ln\left(\frac{\sqrt{2}}{2}\right) - \ln\left(\frac{1}{2}\right)$

$= \ln\left(\dfrac{\frac{\sqrt{2}}{2}\times 2}{\frac{1}{2}\times 2}\right)$

$= \ln\sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2$

7 (a) Show that $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$. [3]

Change LHS to either just $\cos\theta$ or $\sin\theta$:

LHS: $\cos^4\theta - (\sin^2\theta)^2$

$1-\cos^2\theta$

$= \cos^4\theta - (1-\cos^2\theta)^2$

$= \cos^4\theta - (1 - 2\cos^2\theta + \cos^4\theta)$

$= \cancel{\cos^4\theta} - 1 + 2\cos^2\theta - \cancel{\cos^4\theta}$

$= 2\cos^2\theta - 1$

$= \underline{\underline{\cos 2\theta}}$ QED

(b) Hence find the exact value of $\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} (\underbrace{\cos^4\theta - \sin^4\theta}_{\cos 2\theta} + 4\sin^2\theta\cos^2\theta)\, d\theta$. [6]

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (\cos 2\theta + 4\sin^2\theta\cos^2\theta)\, d\theta$$

$2\sin\theta\cos\theta = \sin 2\theta$

$(2\sin\theta\cos\theta)^2 = \sin^2 2\theta$

$4\sin^2\theta\cos^2\theta = \sin^2 2\theta$

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (\cos 2\theta + \sin^2 2\theta)\, d\theta$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 4\theta = 1 - 2\sin^2 2\theta$

$\sin^2 2\theta = \frac{1}{2} - \frac{1}{2}\cos 4\theta$

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \left(\cos 2\theta + \frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \left[\frac{1}{2}\sin 2\theta + \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right]_{-\frac{\pi}{8}}^{\frac{\pi}{8}}$$

$$= \left[\frac{1}{2}\sin\left(\frac{\pi}{4}\right) + \frac{1}{2}\left(\frac{\pi}{8}\right) - \frac{1}{8}\sin\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{2}\sin\left(\frac{-\pi}{4}\right) + \frac{1}{2}\left(\frac{-\pi}{8}\right) - \frac{1}{8}\sin\left(\frac{-\pi}{2}\right)\right]$$

$$= \left[\frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\pi}{16} - \frac{1}{8}\right] - \left[\frac{1}{2} \times \frac{-\sqrt{2}}{2} - \frac{\pi}{16} + \frac{1}{8}\right]$$

$$= \frac{\sqrt{2}}{4} + \frac{\pi}{16} - \frac{1}{8} + \frac{\sqrt{2}}{4} + \frac{\pi}{16} - \frac{1}{8}$$

$$= \frac{\sqrt{2}}{2} + \frac{\pi}{8} - \frac{1}{4}$$

8 **(a)** Express $3\cos 2x - \sqrt{3}\sin 2x$ in the form $R\cos(2x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and α. [3]

$3\cos 2x - \sqrt{3}\sin 2x = R\cos 2x\cos\alpha - R\sin 2x\sin\alpha$

$R\cos\alpha = 3$ ① $\quad R\sin\alpha = \sqrt{3}$ ②

② ÷ ①: $\tan\alpha = \frac{\sqrt{3}}{3}$

$\alpha = \frac{\pi}{6}$

①^2 + ②^2: $R^2 = (\sqrt{3})^2 + 3^2$

$= 3 + 9$

$= 12$

$R = \sqrt{12}$

$= 2\sqrt{3}$

→ $2\sqrt{3}\cos\left(2x + \frac{\pi}{6}\right)$

(b) Hence find the exact value of $\int_0^{\frac{1}{12}\pi} \frac{3}{(3\cos 2x - \sqrt{3}\sin 2x)^2}\,dx$, simplifying your answer. [5]

$$\int_0^{\frac{\pi}{12}} \frac{3}{\left(2\sqrt{3}\cos\left(2x+\frac{\pi}{6}\right)\right)^2}\,dx \leftarrow \text{from part (a)}$$

$$= \int_0^{\frac{\pi}{12}} \frac{3}{12\cos^2\left(2x+\frac{\pi}{6}\right)}\,dx$$

$$= \int_0^{\frac{\pi}{12}} \frac{1}{4}\sec^2\left(2x+\frac{\pi}{6}\right)dx$$

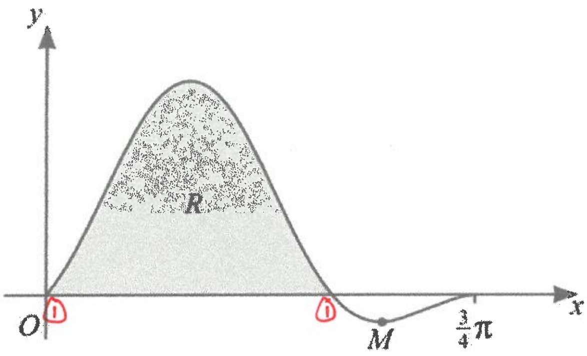
$$= \left[\frac{1}{8}\tan\left(2x+\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{12}}$$

$$= \left[\frac{1}{8}\tan\left(\frac{\pi}{6}+\frac{\pi}{6}\right)\right] - \left[\frac{1}{8}\tan\left(\frac{\pi}{6}\right)\right]$$

$$= \left[\frac{1}{8}\times\sqrt{3}\right] - \left[\frac{1}{8}\times\frac{\sqrt{3}}{3}\right]$$

$$= \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{24}$$

$$= \frac{\sqrt{3}}{12}$$



The diagram shows the curve $y = \sin 2x(1 + \sin 2x)$, for $0 \leqslant x \leqslant \frac{3}{4}\pi$, and its minimum point M. The shaded region bounded by the curve that lies above the x-axis and the x-axis itself is denoted by R.

$0 \leq 2x \leq \frac{3\pi}{2}$

(b) Find the exact area of the region R. [4]

① Find points of intersection with x-axis:

$\sin 2x(1 + \sin 2x) = 0$

$\sin 2x = 0$ or $1 + \sin 2x = 0$

$2x = 0$ or π $\quad \sin 2x = -1$

$x = 0$ or $\frac{\pi}{2}$

$2x = \frac{-\pi}{2}$ or $\frac{3\pi}{2}$

$x = \frac{-\pi}{4}$ or $\frac{3\pi}{4}$

Not in range

Not the point we want

$$\int_0^{\frac{\pi}{2}} (\sin 2x + \sin^2 2x)\, dx$$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 4x = 1 - 2\sin^2 2x$

$\sin^2 2x = \frac{1}{2} - \frac{1}{2}\cos 4x$

$$\int_0^{\frac{\pi}{2}} \left(\sin 2x + \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \left[-\frac{1}{2}\cos 2x + \frac{1}{2}x - \frac{1}{8}\sin 4x\right]_0^{\pi/2}$$

$$= \left[-\frac{1}{2}\cos(\pi) + \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{8}\sin(2\pi)\right] - \left[-\frac{1}{2}\cos(0) + 0 - \frac{1}{8}\sin(0)\right]$$

$$= \left[-\frac{1}{2}(-1) + \frac{\pi}{4} - 0\right] - \left[-\frac{1}{2}(1) + 0 - 0\right]$$

$$= \left[\frac{1}{2} + \frac{\pi}{4}\right] - \left[-\frac{1}{2}\right]$$

$$= 1 + \frac{\pi}{4}$$