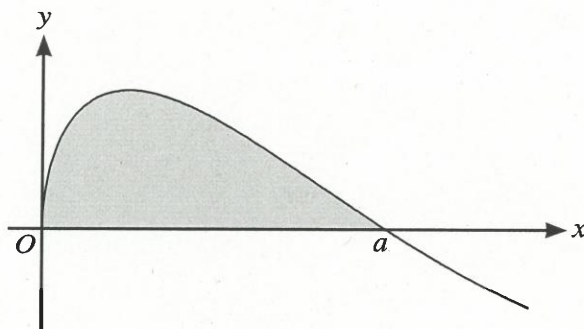




The diagram shows part of the curve $y = \sin \sqrt{x}$. This part of the curve intersects the x -axis at the point where $x = a$.

- (a) State the exact value of a .

[1]

$y=0$: $0 = \sin \sqrt{x}$ $\rightarrow \sqrt{x} = 0$ or $\sqrt{x} = \pi$
 $\sin^{-1}(0) = \sqrt{x}$ X not the point we want from diagram $x = \pi^2$

- (b) Using the substitution $u = \sqrt{x}$, find the exact area of the shaded region in the first quadrant bounded by this part of the curve and the x -axis.

[7]

$\int_0^{\pi^2} \sin \sqrt{x} \, dx$
 $\int_{x=0}^{x=\pi^2} \sin u \times 2u \, du$
 $u = \sqrt{\pi^2} = \pi$
 $u = \sqrt{0} = 0$

$\sqrt{x} = u$
 $x = u^2$
 $\frac{dx}{du} = 2u$
 $dx = 2u \, du$

$\int_0^{\pi} 2u \sin u \, du$

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$$u = 2x$$

$$\frac{du}{dx} = 2$$

$$V = -\cos u$$

$$\frac{dV}{du} = \sin u$$

$$= -2u \cos u - \int_0^{\pi} -2 \cos u \, du$$

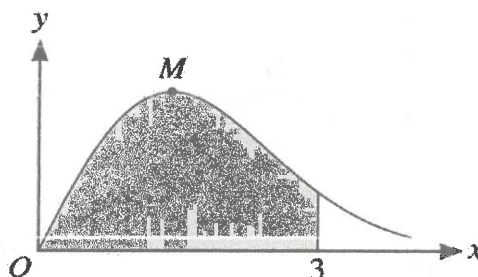
$$= -2u \cos u + \int_0^{\pi} 2 \cos u \, du$$

$$= [-2u \cos u + 2 \sin u]_0^{\pi}$$

$$= [-2\pi \cos(\pi) + 2 \sin(\pi)] - [0 + 0]$$

$$= [-2\pi \times -1 + 0] - [0]$$

$$= \underline{2\pi}$$



The diagram shows the curve $y = xe^{-\frac{1}{4}x^2}$, for $x \geq 0$, and its maximum point M .

- (b) Using the substitution $x = \sqrt{u}$, or otherwise, find by integration the exact area of the shaded region bounded by the curve, the x -axis and the line $x = 3$. [5]

$$\int_{x=0}^{x=3} x e^{-\frac{1}{4}x^2} dx$$

$$x = \sqrt{u}$$

$$x^2 = u$$

$$2x = \frac{du}{dx}$$

$$du = 2x dx$$

$$dx = \frac{du}{2x}$$

$$3 = \sqrt{u} \quad 0 = \sqrt{u}$$

$$9 = u \quad 0 = u$$

$$\int_0^9 x e^{-\frac{1}{4}u} \frac{du}{2x}$$

$$= \int_0^9 \frac{1}{2} e^{-\frac{1}{4}u} du$$

$$= \left[-4 \times \frac{1}{2} e^{-\frac{1}{4}u} \right]_0^9$$

$$= \left[-2e^{-\frac{9}{4}} \right] - \left[-2e^0 \right]$$

$$= -2e^{-\frac{9}{4}} + 2e^0$$

$$= \underline{\underline{-2e^{-\frac{9}{4}} + 2}}$$

9 Let $f(x) = \frac{1}{(9-x)\sqrt{x}}$.

(b) Using the substitution $u = \sqrt{x}$, show that $\int_0^4 f(x) dx = \frac{1}{3} \ln 5$.

[6]

$$\int_0^4 \frac{1}{(9-x)\sqrt{x}} dx$$

$$\sqrt{x} = u$$

$$x = u^2$$

$$\frac{dx}{du} = 2u$$

$$dx = 2u du$$

$$\int_{x=0}^{x=4} \frac{1}{(9-u^2)u} \times 2u du$$

$$u = \sqrt{4}$$

$$= 2$$

$$u = \sqrt{0}$$

$$= 0$$

$$\int_0^2 \frac{2u}{(9-u^2)u} du$$

$$= \int_0^2 \frac{2}{9-u^2} du$$

$$= 2 \int_0^2 \frac{1}{9-u^2} du$$

formula book:

$$\frac{1}{a^2-x^2} \text{ with } a=3$$

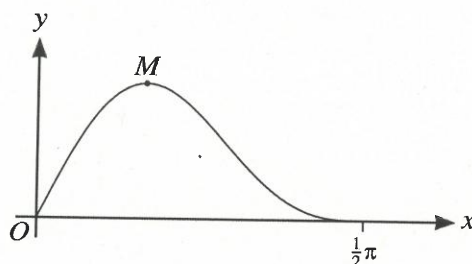
$$= 2 \left[\frac{1}{6} \ln \left| \frac{3+u}{3-u} \right| \right]_0^2$$

$$= \frac{1}{3} \left[\ln \left(\frac{5}{1} \right) \right] - \frac{1}{3} \left[\ln \left(\frac{3}{3} \right) \right]$$

$$= \frac{1}{3} \ln 5 - \frac{1}{3} \ln 1$$

$$\leftarrow = 0$$

$$= \frac{1}{3} \ln 5 \quad \text{QED}$$



The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leq x \leq \frac{1}{2}\pi$, and its maximum point M .

- (a) Using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the x -axis. [5]

$$\int_0^{\frac{\pi}{2}} \sin 2x \cos^2 x \, dx$$

$$\sin 2x \cos x \times \frac{du}{\cos x}$$

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x \, dx$$

$$dx = \frac{du}{\cos x}$$

$$u = \sin\left(\frac{\pi}{2}\right) = 1$$

$$u = \sin(0) = 0$$

$$\int_0^1 \sin 2x \cos x \, du$$

$$= \int_0^1 2 \sin x \cos x \times \cos x \, du$$

$$= \int_0^1 2 \sin x \cos^2 x \, du$$

$$= \int_0^1 2 \sin x (1 - \sin^2 x) \, du$$

$$= \int_0^1 (2 \sin x - 2 \sin^3 x) \, du$$

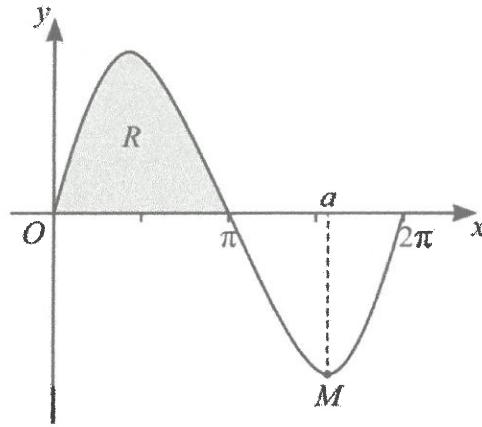
$$= \int_0^1 (2u - 2u^3) \, du$$

$$= \left[u^2 - \frac{1}{2} u^4 \right]_0^1$$

$$= \left[1^2 - \frac{1}{2} \times 1^4 \right] - \left[0^2 - \frac{1}{2} \times 0^4 \right]$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$



The diagram shows the curve $y = 2 \sin x \sqrt{2 + \cos x}$, for $0 \leq x \leq 2\pi$, and its minimum point M , where $x = a$.

- (b) Use the substitution $u = 2 + \cos x$ to find the exact area of the shaded region R . [6]

$$\begin{aligned}
 & \int_0^{\pi} 2 \sin x (2 + \cos x)^{\frac{1}{2}} dx \\
 &= \int_{x=0}^{x=\pi} 2 \sin x (2 + \cos x)^{\frac{1}{2}} \frac{du}{-\sin x} \\
 &= \int_{x=0}^{x=\pi} -2 (2 + \cos x)^{\frac{1}{2}} du \\
 & \quad u = 2 + \cos(\pi) = 2 + -1 = 1 \\
 & \quad u = 2 + \cos(0) = 2 + 1 = 3 \\
 &= \int_3^1 -2 u^{\frac{1}{2}} du \\
 &= \left[-2 \times \frac{2}{3} u^{\frac{3}{2}} \right]_3^1 \\
 &= \left[-\frac{4}{3} (1)^{\frac{3}{2}} \right] - \left[-\frac{4}{3} (3)^{\frac{3}{2}} \right] \\
 &= -\frac{4}{3} + \frac{4}{3} \times 3\sqrt{3} \\
 &= -\frac{4}{3} + 4\sqrt{3}
 \end{aligned}$$

- 7 (a) Use the substitution $u = \cos x$ to show that

$$\int_0^{\pi} \sin 2x e^{2\cos x} dx = \int_{-1}^1 2ue^{2u} du. \quad [4]$$

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$du = -\sin x dx$$

$$dx = -\frac{du}{\sin x}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\rightarrow \int_{x=0}^{x=\pi} 2 \sin x \cos x e^{2\cos x} \times -\frac{du}{\sin x}$$

$$= \int_{x=0}^{x=\pi} -2 \cos x e^{2\cos x} du$$

$$u = \cos \pi$$

$$= -1$$

$$u = \cos(0)$$

$$= 1$$

$$\int_{-1}^1 -2ue^{2u} du$$

Swap limits and multiply by -1 :

$$\int_{-1}^1 2ue^{2u} du \quad \text{QED}$$

(b) Hence find the exact value of $\int_0^\pi \sin 2x e^{2\cos x} dx$.

[4]

$$\int_0^\pi 2u e^{2u} du$$

$$u = 2x$$

$$\frac{du}{dx} = 2$$

$$V = \frac{1}{2} e^{2u}$$

$$\frac{dV}{du} = e^{2u}$$

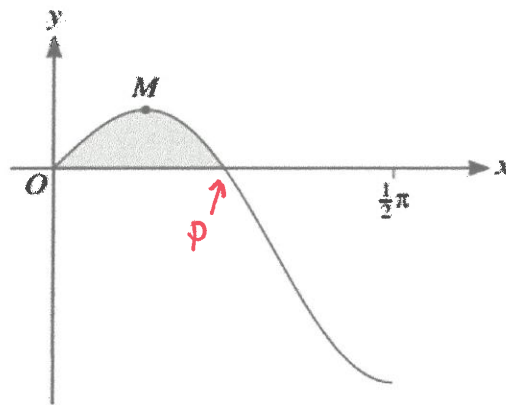
$$\rightarrow ue^{2u} - \int e^{2u} du$$

$$= \left[ue^{2u} - \frac{1}{2} e^{2u} \right]_{-1}^1$$

$$= \left[1e^2 - \frac{1}{2} e^2 \right] - \left[-e^{-2} - \frac{1}{2} e^{-2} \right]$$

$$= e^2 - \frac{1}{2} e^2 + e^{-2} + \frac{1}{2} e^{-2}$$

$$= \frac{1}{2} e^2 + \frac{3}{2} e^{-2}$$



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leq x \leq \frac{1}{2}\pi$, and its maximum point M .

- (b) Using the substitution $u = \cos x$, find the area of the shaded region enclosed by the curve and the x -axis in the first quadrant, giving your answer in a simplified exact form. [5]

Find point P: $\sin x \cos 2x = 0$

$$\sin x = 0 \quad \cos 2x = 0$$

$$x = 0, \pi \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\int_0^{\frac{\pi}{4}}$$

$$\sin x \cos 2x \, dx$$

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$dx = \frac{du}{-\sin x}$$

$$\int_0^{\frac{\pi}{4}}$$

$$\sin x \cos 2x \times \frac{du}{-\sin x}$$

$$u = \cos\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}$$

$$u = \cos(0)$$

$$= 1$$

$$= -\int_1^{\frac{1}{\sqrt{2}}} \cos 2x \, du$$

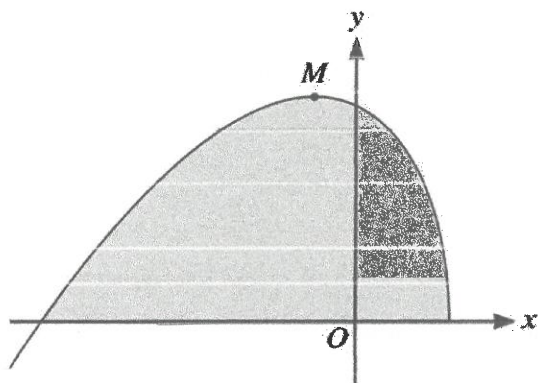
$$= -\int_1^{\frac{1}{\sqrt{2}}} (2\cos^2 x - 1) \, du$$

$$= -\int_1^{\frac{1}{\sqrt{2}}} (2u^2 - 1) \, du = -\left[\frac{2}{3}u^3 - u\right]_1^{\frac{1}{\sqrt{2}}}$$

$$= -\left[\frac{2}{3} \times \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - \left[\frac{2}{3} - 1\right]$$

$$= \frac{-1}{3\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{2}{3} - 1$$

$$= \frac{2}{3\sqrt{2}} - \frac{1}{3}$$



The diagram shows the curve $y = (x+5)\sqrt{3-2x}$ and its maximum point M .

- (b) Using the substitution $u = 3 - 2x$, find by integration the area of the shaded region bounded by the curve and the x -axis. Give your answer in the form $a\sqrt{13}$, where a is a rational number. [5]

roots: $(x+5)\sqrt{3-2x} = 0$

$x+5=0$ or $\sqrt{3-2x}=0$

$x = -5$

$3-2x=0$

$x = \frac{3}{2}$

$u = 3 - 2\left(\frac{3}{2}\right)$
 $= 0$

$u = 3 - 2(-5)$

$u = 13$

$u = 3 - 2x$

$\frac{du}{dx} = -2$

$du = -2 dx$

$dx = \frac{du}{-2}$

$\int_{13}^0 (x+5) u^{\frac{1}{2}} \times \frac{du}{-2}$

$u = 3 - 2x$

$2x = 3 - u$

$x = \frac{3-u}{2}$

$x+5 = \frac{3-u}{2} + 5$

$= \frac{3-u}{2} + \frac{10}{2}$

$= \frac{13-u}{2}$

$\int_{13}^0 \left(\frac{13-u}{2}\right) u^{\frac{1}{2}} \times \frac{du}{-2}$

$= -\frac{1}{4} \int_{13}^0 (13-u) u^{\frac{1}{2}} du = -\frac{1}{4} \int_{13}^0 (13u^{\frac{1}{2}} - u^{\frac{3}{2}}) du$

$= -\frac{1}{4} \left[\frac{26}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right]_{13}^0 = -\frac{1}{4} [0 - 0] - \frac{1}{4} \left[\frac{26}{3} (13)^{\frac{3}{2}} - \frac{2}{5} (13)^{\frac{5}{2}} \right]$

$= \frac{1}{4} \left[\frac{26}{3} \times 13\sqrt{13} - \frac{2}{5} \times 13^2\sqrt{13} \right] = \frac{1}{4} \left[\frac{338}{3}\sqrt{13} - \frac{338}{5}\sqrt{13} \right]$

$= \frac{1}{4} \times \frac{676}{15} \sqrt{13}$

$= \frac{169}{15} \sqrt{13}$

- 8 Use the substitution $u = 1 - \sin x$ to find the exact value of

$$\int_{\pi}^{\frac{3\pi}{2}} \frac{\sin 2x}{\sqrt{1 - \sin x}} dx.$$

Give your answer in the form $a + b\sqrt{2}$ where a and b are rational numbers to be determined. [7]

$$\int_{\pi}^{\frac{3\pi}{2}} \frac{\sin 2x}{\sqrt{1 - \sin x}} dx$$

$$u = 1 - \sin x$$

$$\frac{du}{dx} = -\cos x$$

$$du = -\cos x dx$$

$$dx = \frac{du}{-\cos x}$$

$$u = 1 - \sin\left(\frac{3\pi}{2}\right) = 2$$

$$u = 1 - \sin(\pi) = 1$$

$$\int_1^2 \frac{\sin 2x}{\sqrt{1 - \sin x}} \times \frac{du}{-\cos x}$$

$$= \int_1^2 \frac{2 \sin x \cos x}{\sqrt{1 - \sin x}} \times \frac{du}{-\cos x}$$

$$= \int_1^2 \frac{-2 \sin x}{\sqrt{1 - \sin x}} du$$

$$= \int_1^2 \frac{-2(1 - u)}{\sqrt{u}} du$$

$$= \int_1^2 \frac{2u - 2}{u^{\frac{1}{2}}} du$$

$$= \int_1^2 (2u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}) du$$

$$= \left[\frac{2}{3} \times 2u^{\frac{3}{2}} - 2 \times 2u^{\frac{1}{2}} \right]_1^2$$

$$= \left[\frac{4}{3} u^{3/2} - 4u^{1/2} \right]_1^{13}$$

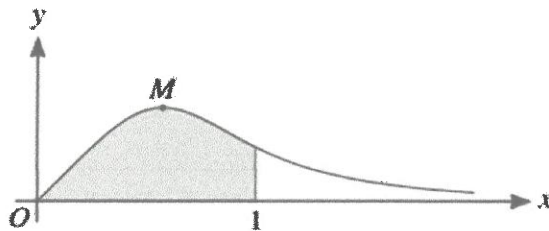
$$= \left[\frac{4}{3} (2)^{3/2} - 4(2)^{1/2} \right] - \left[\frac{4}{3} (1)^{3/2} - 4(1)^{1/2} \right]$$

$$= \left[\frac{4}{3} \times 2\sqrt{2} - 4\sqrt{2} \right] - \left[\frac{4}{3} - 4 \right]$$

$$= \left[\frac{8}{3}\sqrt{2} - 4\sqrt{2} \right] - \left[-\frac{8}{3} \right]$$

$$= \left[-\frac{4}{3}\sqrt{2} \right] - \left[-\frac{8}{3} \right]$$

$$= \underline{\underline{\frac{8}{3} - \frac{4}{3}\sqrt{2}}}$$



The diagram shows the curve $y = \frac{x}{1+3x^4}$, for $x \geq 0$, and its maximum point M .

- (b) Using the substitution $u = \sqrt{3}x^2$, find by integration the exact area of the shaded region bounded by the curve, the x -axis and the line $x = 1$. [5]

$$\int_0^1 \frac{x}{1+3x^4} dx$$

$$u = \sqrt{3}x^2$$

$$\frac{du}{dx} = 2\sqrt{3}x$$

$$dx = \frac{du}{2\sqrt{3}x}$$

$$\int_{x=0}^{x=1} \frac{x}{1+3x^4} \cdot \frac{du}{2\sqrt{3}x}$$

$$= \int_{x=0}^{x=1} \frac{1}{2\sqrt{3}} \cdot \frac{1}{1+u^2} du$$

$$u = \sqrt{3}x^2$$

$$= \sqrt{3}$$

$$u = \sqrt{3} \times 0^2$$

$$= 0$$

$$= \frac{1}{2\sqrt{3}} \int_0^{\sqrt{3}} \frac{1}{1+u^2} du$$

$$= \frac{1}{2\sqrt{3}} \left[\tan^{-1} u \right]_0^{\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}} \left[\tan^{-1} \sqrt{3} \right] - \frac{1}{2\sqrt{3}} \left[\tan^{-1}(0) \right]$$

$$= \frac{1}{2\sqrt{3}} \left(\frac{\pi}{3} \right)$$

$$= \frac{\pi}{6\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{18}$$

6 Let $I = \int_0^3 \frac{27}{(9+x^2)^2} dx$.

(a) Using the substitution $x = 3 \tan \theta$, show that $I = \int_0^{\frac{1}{4}\pi} \cos^2 \theta d\theta$.

[4]

$$\int_0^3 \frac{27}{(9+x^2)^2} dx$$

$$x = 3 \tan \theta$$

$$\frac{dx}{d\theta} = 3 \sec^2 \theta$$

$$dx = 3 \sec^2 \theta d\theta$$

$$3 = 3 \tan \theta \quad 0 = 3 \tan \theta$$

$$\tan \theta = 1 \quad \tan \theta = 0$$

$$\theta = \frac{\pi}{4} \quad \theta = 0$$

$$\int_0^{\frac{\pi}{4}} \frac{27 \times 3}{(9 + (3 \tan \theta)^2)^2} \times \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{(9 + 9 \tan^2 \theta)^2} \times \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{[9(1 + \tan^2 \theta)]^2} \times \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{(9 \sec^2 \theta)^2} \times \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{81 \sec^4 \theta} \times \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\sec^4 \theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} \cos^2 \theta d\theta$$

QED

(b) Hence find the exact value of I .

[4]

$$\int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta$$

$$2\cos^2 \theta - 1 = \cos 2\theta$$

$$2\cos^2 \theta = 1 + \cos 2\theta$$

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta$$

$$\int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{8} + \frac{1}{4} \sin \left(\frac{\pi}{2} \right) \right] - [0 + 0]$$

$$= \underline{\underline{\frac{\pi}{8} + \frac{1}{4}}}$$

- 4 Using the substitution $u = \sqrt{x}$, find the exact value of

$$\int_3^{\infty} \frac{1}{(x+1)\sqrt{x}} dx.$$

[6]

$$\int_3^{\infty} \frac{1}{(x+1)\sqrt{x}} dx$$

$$\int_3^{\infty} \frac{1}{(x+1)\sqrt{x}} \times 2u du$$

$$\sqrt{x} = u$$

$$x = u^2$$

$$\frac{dx}{du} = 2u$$

$$dx = 2u du$$

$$u = \sqrt{\infty}$$

$$u = \sqrt{3}$$

$$= \infty$$

$$\int_{\sqrt{3}}^{\infty} \frac{1}{(u^2+1)u} \times 2u du$$

$$= \int_{\sqrt{3}}^{\infty} \frac{2u}{(u^2+1)u} du$$

$$= \int_{\sqrt{3}}^{\infty} \frac{2}{1+u^2} du$$

$$= 2 \int_{\sqrt{3}}^{\infty} \frac{1}{1+u^2} du$$

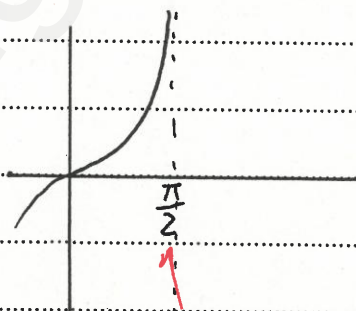
$$= 2 \left[\tan^{-1}(u) \right]_{\sqrt{3}}^{\infty}$$

$$= 2 \left[\tan^{-1}(\infty) \right] - 2 \left[\tan^{-1}(\sqrt{3}) \right]$$

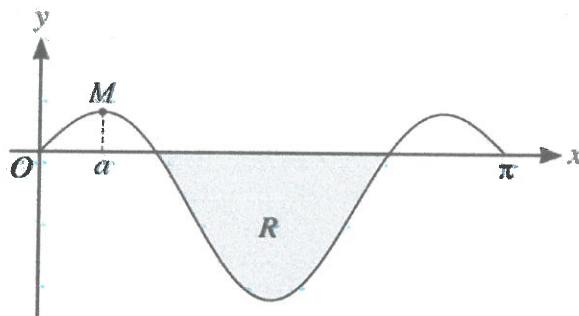
$$= 2 \left(\frac{\pi}{2} \right) - 2 \left(\frac{\pi}{3} \right)$$

$$= \pi - \frac{2\pi}{3}$$

$$= \frac{\pi}{3}$$



$\tan\left(\frac{\pi}{2}\right) = \infty$
 so $\frac{\pi}{2} = \tan^{-1}(\infty)$



The diagram shows the curve $y = \sin x \cos 2x$, for $0 \leq x \leq \pi$, and a maximum point M , where $x = a$. The shaded region between the curve and the x -axis is denoted by R .

(b) Find the exact area of the region R , giving your answer in simplified form.

[4]

roots: $\sin x \cos 2x = 0$

$\sin x = 0$ or $\cos 2x = 0$

$x = 0, \pi$ $2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\int \sin x \cos 2x \, dx$

$\uparrow (2\cos^2 x - 1)$

$= \int \sin x (2\cos^2 x - 1) \, dx$

$= \int 2\cos^2 x \sin x \, dx - \int \sin x \, dx$

integrate normally

use substitution $u = \cos x$

$= \int_{x=\frac{\pi}{4}}^{x=\frac{3\pi}{4}} 2\cos^2 x \sin x \, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

$u = \cos\left(\frac{3\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2}$

$u = \cos\left(\frac{\pi}{4}\right)$

$= \frac{\sqrt{2}}{2}$

$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 2u^2 \times \sin x \times \frac{du}{-\sin x} - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \, dx$

$= \int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} -2u^2 \, du - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \, dx$

$= \left[-\frac{2}{3} u^3 \right]_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} - \left[-\cos x \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$= \left[-\frac{2}{3} \left(\frac{\sqrt{2}}{2} \right)^3 \right] - \left[-\frac{2}{3} \left(\frac{\sqrt{2}}{2} \right)^3 \right] - \left[-\cos\left(\frac{3\pi}{4}\right) \right] + \left[-\cos\left(\frac{\pi}{4}\right) \right]$

$= \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

$= -\frac{2\sqrt{2}}{3}$

Area is positive,

[Turn over]

so $= \frac{2\sqrt{2}}{3}$

- 11 Use the substitution $2x = \tan \theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{12}{(1+4x^2)^2} dx.$$

Give your answer in the form $a + b\pi$, where a and b are rational numbers.

[9]

$$\int_0^{\frac{1}{2}} \frac{12}{(1+(2x)^2)^2} dx$$

$$2x = \tan \theta$$

$$x = \frac{1}{2} \tan \theta$$

$$\frac{dx}{d\theta} = \frac{1}{2} \sec^2 \theta$$

$$dx = \frac{1}{2} \sec^2 \theta d\theta$$

$$\int_0^{\frac{1}{2}} \frac{12}{(1+(2x)^2)^2} \times \frac{1}{2} \sec^2 \theta d\theta$$

$$2\left(\frac{1}{2}\right) = \tan \theta$$

$$1 = \tan \theta$$

$$\theta = \frac{\pi}{4}$$

$$2(0) = \tan \theta$$

$$0 = \tan \theta$$

$$\theta = 0$$

$$= \int_0^{\frac{\pi}{4}} \frac{12}{(1+\tan^2 \theta)^2} \times \frac{1}{2} \sec^2 \theta d\theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{12}{(\sec^2 \theta)^2} \times \frac{1}{2} \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{6 \sec^2 \theta}{\sec^4 \theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{6}{\sec^2 \theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} 6 \cos^2 \theta d\theta$$

$$\cos 2\theta = 2\cos^2 \theta - 1$$

$$\cos 2\theta + 1 = 2\cos^2 \theta$$

$$6 \cos^2 \theta = 3 \cos 2\theta + 3$$

$$= \int_0^{\frac{\pi}{4}} (3\cos 2\theta + 3) d\theta$$

$$= \left[\frac{3}{2} \sin 2\theta + 3\theta \right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2} \sin\left(\frac{\pi}{2}\right) + \frac{3\pi}{4} \right] - [0 + 0]$$

$$= \frac{3}{2} \times 1 + \frac{3\pi}{4}$$

$$= \underline{\underline{\frac{3}{2} + \frac{3\pi}{4}}}$$

11 Let $f(x) = \frac{2e^{2x}}{e^{2x} - 3e^x + 2}$.

(b) Use the substitution $u = e^x$ and partial fractions to find the exact value of $\int_{\ln 3}^{\ln 5} f(x) dx$.

Give your answer in the form $\ln a$, where a is a rational number in its simplest form.

[9]

$$\int_{\ln 3}^{\ln 5} \frac{2e^{2x}}{e^{2x} - 3e^x + 2} dx$$

$$u = e^x$$

$$\frac{du}{dx} = e^x$$

$$du = e^x dx$$

$$dx = \frac{du}{e^x}$$

$$\int_{x=\ln 3}^{x=\ln 5} \frac{2e^{2x}}{e^{2x} - 3e^x + 2} \cdot \frac{du}{e^x}$$

$$u = e^{\ln 5} = 5$$

$$u = e^{\ln 3} = 3$$

$$= \int_3^5 \frac{2e^x}{e^{2x} - 3e^x + 2} du$$

$$u = e^x$$

$$u^2 = (e^x)^2 = e^{2x}$$

$$= \int_3^5 \frac{2u}{u^2 - 3u + 2} du$$

Partial Fractions:

$$\frac{2u}{(u-2)(u-1)} = \frac{A}{u-2} + \frac{B}{u-1}$$

$$2u = A(u-1) + B(u-2)$$

$$u=1: 2 = 0 + -B$$

$$B = -2$$

$$u=2: 4 = A$$

$$[4 \ln |u-2| - 2 \ln |u-1|]_3^5$$

$$\rightarrow \int_3^5 \left(\frac{4}{u-2} + \frac{-2}{u-1} \right) du = [4 \ln 3 - 2 \ln 4] - [4 \ln 1 - 2 \ln 2]$$

$$= 4 \ln 3 - 2 \ln 4 + 2 \ln 2$$

$$= \ln 3^4 - \ln 4^2 + \ln 2^2$$

$$= \ln 81 - \ln 16 + \ln 4$$

$$= \ln \left(\frac{81 \times 4}{16} \right) = \ln \frac{81}{4}$$

9709/32/O/N/24