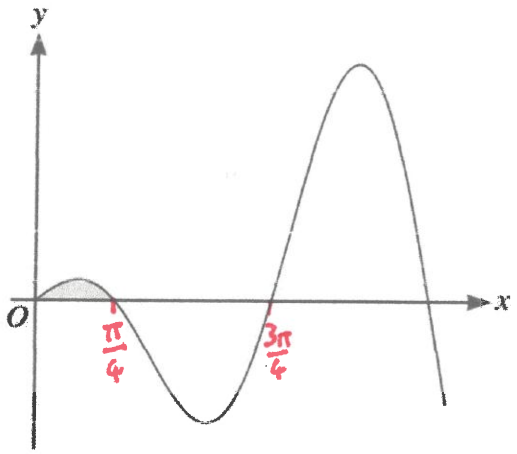
I L A T E



The diagram shows the curve $y = x\cos 2x$, for $x \geqslant 0$.

(b) Find the exact area of the shaded region shown in the diagram, bounded by the curve and the x-axis. [5]

roots: $x\cos 2x = 0$

$x = 0$ or $\cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$ ✓

$u = x \qquad v = \frac{1}{2}\sin 2x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \cos 2x$

$\rightarrow x \times \frac{1}{2}\sin 2x - \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin 2x\,dx$

$= \left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{1}{2}\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\cos\left(\frac{\pi}{2}\right)\right] - \left[0 + \frac{1}{4}\cos(0)\right]$

$= \left[\frac{\pi}{8} + 0\right] - \left[0 + \frac{1}{4}\right]$

$= \frac{\pi}{8} - \frac{1}{4}$

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4 Find the exact value of $\int_{\frac{1}{3}\pi}^{\pi} x \sin \frac{1}{2}x \, dx$. [5]

$u = x \qquad V = -2\cos\frac{1}{2}x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \sin\frac{1}{2}x$

$= -2x\cos\frac{1}{2}x - \int_{\frac{\pi}{3}}^{\pi} -2\cos\frac{1}{2}x \, dx$

$= -2x\cos\frac{1}{2}x + \int_{\frac{\pi}{3}}^{\pi} 2\cos\frac{1}{2}x \, dx$

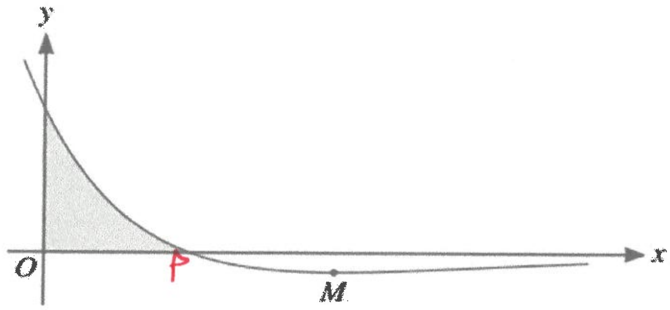
$= \left[-2x\cos\frac{1}{2}x + 4\sin\frac{1}{2}x\right]_{\frac{\pi}{3}}^{\pi}$

$= \left[-2\pi \times 0 + 4 \times 1\right] - \left[-\frac{2\pi}{3} \times \frac{\sqrt{3}}{2} + 4 \times \frac{1}{2}\right]$

$= [4] - \left[\frac{-\sqrt{3}\pi}{3} + 2\right]$

$= 4 - 2 + \frac{\sqrt{3}\pi}{3}$

$= 2 + \frac{\sqrt{3}\pi}{3}$



The diagram shows the curve $y = (2-x)e^{-\frac{1}{2}x}$, and its minimum point M.

(b) Find the area of the shaded region bounded by the curve and the axes. Give your answer in terms of e. [5]

Find point P: $(2-x)e^{-\frac{1}{2}x} = 0$

$2-x=0 \quad e^{-\frac{1}{2}x}=0$ ×

$x=2$ ✓

$\int_0^2 \underset{u}{(2-x)}\, \underset{dv}{e^{-\frac{1}{2}x}}\, dx$ $\qquad u = 2-x \quad v = -2e^{-\frac{1}{2}x}$

$\frac{du}{dx} = -1 \quad \frac{dv}{dx} = e^{-\frac{1}{2}x}$

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$= -2(2-x)e^{-\frac{1}{2}x} - \int_0^2 2e^{-\frac{1}{2}x}\, dx$

$= \left[2(x-2)e^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}\right]_0^2$

$= \left[2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}\right]_0^2$

$= \left[2xe^{-\frac{1}{2}x}\right]_0^2$

$= \left(4e^{-\frac{1}{2}\times 2}\right) - (0)$

$= 4e^{-1}$

$= \frac{4}{e}$

3 Find the exact value of $\int_0^{\frac{1}{4}\pi} x \sec^2 x \, dx$. [5]

$u = x \qquad v = \tan x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \sec^2 x$

$= x \tan x - \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= x \tan x - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$

$= \left[x \tan x + \ln |\cos x| \right]_0^{\frac{\pi}{4}}$

$= \left[\frac{\pi}{4} \times 1 + \ln\left(\frac{\sqrt{2}}{2}\right) \right] - \left[0 + \ln(1) \right]$

$= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right)$

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2 Find the exact value of $\int_1^3 x^2 \ln 3x \, dx$. Give your answer in the form $a \ln b + c$, where a and c are rational and b is an integer. [5]

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$u = \ln 3x \qquad v = \frac{1}{3}x^3$

$\frac{du}{dx} = \frac{3}{3x} = \frac{1}{x} \qquad \frac{dv}{dx} = x^2$

$= \left[\frac{1}{3}x^3 \ln|3x| - \int \frac{1}{3}x^3 \times \frac{1}{x} \, dx\right]_1^3$

$= \left[\frac{1}{3}x^3 \ln|3x| - \int \frac{1}{3}x^2 \, dx\right]_1^3$

$= \left[\frac{1}{3}x^3 \ln|3x| - \frac{1}{9}x^3\right]_1^3$

$= \left[\frac{1}{3} \times 3^3 \times \ln 9 - \frac{1}{9} \times 3^3\right] - \left[\frac{1}{3} \times 1^3 \times \ln 3 - \frac{1}{9} \times 1^3\right]$

$= \left[9 \ln 9 - 3\right] - \left[\frac{1}{3}\ln 3 - \frac{1}{9}\right]$

$= 9 \ln 9 - 3 - \frac{1}{3}\ln 3 + \frac{1}{9}$

$= 9 \ln 9 - \frac{1}{3}\ln 3 - \frac{26}{9}$

$= 9 \ln 3^2 - \frac{1}{3}\ln 3 - \frac{26}{9}$

$= 18 \ln 3 - \frac{1}{3}\ln 3 - \frac{26}{9}$

$= \frac{53}{3}\ln 3 - \frac{26}{9}$

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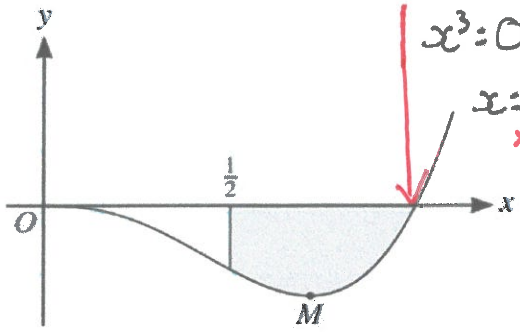

$0 = x^3 \ln x$

$x^3 = 0$ or $\ln x = 0$

$x = 0$ or $x = e^0$

$x = 1$

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The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M.

(b) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = \frac{1}{2}$. [5]

$u = \ln x \qquad v = \frac{1}{4}x^4$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^3$

$= \frac{1}{4}x^4 \ln x - \int_{\frac{1}{2}}^{1} \frac{1}{4}x^4 \times \frac{1}{x}\, dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int_{\frac{1}{2}}^{1} x^3\, dx$

$= \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_{\frac{1}{2}}^{1}$

$= \left[\frac{1}{4}(1)^4 \ln(1) - \frac{1}{16}(1)^4\right] - \left[\frac{1}{4}\left(\frac{1}{2}\right)^4 \ln\left(\frac{1}{2}\right) - \frac{1}{16}\left(\frac{1}{2}\right)^4\right]$

$= \left[0 - \frac{1}{16}\right] - \left[\frac{1}{64}\ln\left(\frac{1}{2}\right) - \frac{1}{256}\right]$

$= -\frac{1}{16} - \frac{1}{64}\ln\left(\frac{1}{2}\right) + \frac{1}{256}$

$= \frac{-15}{256} - \frac{1}{64}\ln 2^{-1}$

$= \underbrace{\frac{-15}{256} + \frac{1}{64}\ln 2}$

this is negative because it is below the x-axis. We want the area, which is positive, so multiply everything by -1:

$\frac{15}{256} - \frac{1}{64}\ln 2$

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9 The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

(b) Show that $\int_1^8 y \, dx = 18 \ln 2 - 9$. [5]

$$\int_1^8 \underbrace{x^{-2/3}}_{dv} \underbrace{\ln x}_{u} \, dx \qquad u = \ln x \qquad v = 3x^{\frac{1}{3}}$$

$$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^{-2/3}$$

$$= 3x^{\frac{1}{3}} \ln x - \int_1^8 3x^{\frac{1}{3}} \times \frac{1}{x} \, dx$$

$$= 3x^{\frac{1}{3}} \ln x - \int_1^8 3x^{-2/3} \, dx$$

$$= \left[3x^{\frac{1}{3}} \ln x - 9x^{\frac{1}{3}} \right]_1^8$$

$$= \left[3(8)^{\frac{1}{3}} \ln(8) - 9(8)^{\frac{1}{3}} \right] - \left[3(1)^{\frac{1}{3}} \ln(1) - 9(1)^{\frac{1}{3}} \right]$$

$$= [6 \ln 8 - 18] - [0 - 9]$$

$$= 6 \ln 8 - 18 + 9$$

$$= 6 \ln 8 - 9$$

$$= 6 \ln 2^3 - 9$$

$$= \underline{18 \ln 2 - 9} \quad \text{QED}$$

I L A T E

2 Find the exact value of $\int_0^1 (2-x)e^{-2x}\,dx$. [5]

$u = 2 - x$ $\quad v = -\frac{1}{2}e^{-2x}$

$\frac{du}{dx} = -1$ $\quad \frac{dv}{dx} = e^{-2x}$

$= -\frac{1}{2}e^{-2x}(2-x) - \int_0^1 \frac{1}{2}e^{-2x}\,dx$

$= \left[-\frac{1}{2}e^{-2x}(2-x) + \frac{1}{4}e^{-2x}\right]_0^1$

$= \left[-\frac{1}{2}e^{-2}(2-1) + \frac{1}{4}e^{-2}\right] - \left[\frac{-1}{2}e^{0}(2-0) + \frac{1}{4}e^{0}\right]$

$= \left[-\frac{1}{2}e^{-2} + \frac{1}{4}e^{-2}\right] - \left[\frac{-1}{2}\times 2 + \frac{1}{4}\right]$

$= \left[\frac{-1}{4}e^{-2}\right] - \left[-1 + \frac{1}{4}\right]$

$= \frac{-1}{4}e^{-2} + 1 - \frac{1}{4}$

$= \underline{\frac{-1}{4}e^{-2} + \frac{3}{4}}$

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4 Find $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} x \sec^2 x \, dx$. Give your answer in a simplified exact form. [7]

$u = x \qquad v = \tan x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \sec^2 x$

$= x\tan x - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x \, dx$

$= x\tan x - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx$

$= \Big[x\tan x + \ln|\cos x|\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left[\frac{\pi}{3} \times \sqrt{3} + \ln\left(\frac{1}{2}\right)\right] - \left[\frac{\pi}{6} \times \frac{\sqrt{3}}{3} + \ln\left(\frac{\sqrt{3}}{2}\right)\right]$

$= \frac{\sqrt{3}\pi}{3} + \ln\left(\frac{1}{2}\right) - \frac{\sqrt{3}\pi}{18} - \ln\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{6\sqrt{3}\pi}{18} - \frac{\sqrt{3}\pi}{18} + \ln\left(\frac{1}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{5\sqrt{3}\pi}{18} + \ln\left(\frac{\frac{1}{2} \times 2}{\frac{\sqrt{3}}{2} \times 2}\right)$

$= \frac{5\sqrt{3}\pi}{18} + \ln\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{5\sqrt{3}\pi}{18} + \ln 3^{-\frac{1}{2}}$

$= \frac{5\sqrt{3}\pi}{18} - \frac{1}{2}\ln 3$

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(b) Find the exact value of $\int_0^{\frac{2}{a}} xe^{-ax}\,dx$. [5]

$u = x \qquad v = \frac{-1}{a}e^{-ax}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = e^{-ax}$

$\rightarrow x \times \frac{-1}{a}e^{-ax} - \int_0^{2/a} \frac{-1}{a}e^{-ax}\,dx$

$= \frac{-x}{a}e^{-ax} + \int_0^{2/a} \frac{1}{a}e^{-ax}\,dx$

$= \left[\frac{-x}{a}e^{-ax} - \frac{1}{a^2}e^{-ax}\right]_0^{2/a}$

$= \left[\frac{-\frac{2}{a}}{a}e^{-a\times\frac{2}{a}} - \frac{1}{a^2}e^{-a\times\frac{2}{a}}\right] - \left[0 - \frac{1}{a^2}e^0\right]$

$= \left[\frac{-2}{a^2}e^{-2} - \frac{1}{a^2}e^{-2}\right] - \left[-\frac{1}{a^2}\right]$

$= \frac{-3}{a^2}e^{-2} + \frac{1}{a^2}$

$= \frac{1}{a^2}(1 - 3e^{-2})$

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(b) By using integration by parts, show that for all $a > 1$, $\int_1^a \frac{\ln x}{x^4}\,dx < \frac{1}{9}$. [6]

$$\int x^{-4}\ln x\,dx \qquad u = \ln x \qquad V = \frac{-1}{3}x^{-3}$$

$$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^{-4}$$

$$= \frac{-1}{3}x^{-3} \times \ln x - \int_1^a \frac{-1}{3}x^{-3} \times \frac{1}{x}\,dx$$

$$= \frac{-1}{3}x^{-3}\ln x + \int_1^a \frac{1}{3}x^{-4}\,dx$$

$$= \left[\frac{-1}{3}x^{-3}\ln x - \frac{1}{9}x^{-3}\right]_1^a$$

$$= \left[\frac{-1}{3}a^{-3}\ln a - \frac{1}{9}a^{-3}\right] - \left[\frac{-1}{3} \times \ln 1 - \frac{1}{9}\right]$$

$$= \left[\frac{-1}{3a^3}\ln a - \frac{1}{9a^3}\right] - \left[-\frac{1}{9}\right]$$

$$= \frac{-1}{3a^3}\ln a - \frac{1}{9a^3} + \frac{1}{9}$$

$$= \frac{1}{9} - \frac{1}{3a^3}\ln a - \frac{1}{9a^3}$$

if $a > 1$, both of these terms are negative,
so get subtracted from $\frac{1}{9}$,
therefore for all $a > 1$, the integral $< \frac{1}{9}$.

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3 Find the exact value of

$$\int_1^4 x^{\frac{3}{2}} \ln x \, dx.$$

[5]

$u = \ln x \qquad v = \frac{2}{5}x^{\frac{5}{2}}$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^{\frac{3}{2}}$

$= \frac{2}{5}x^{\frac{5}{2}}\ln x - \int_1^4 \frac{2}{5}x^{\frac{5}{2}} \times \frac{1}{x}\,dx$

$= \frac{2}{5}x^{\frac{5}{2}}\ln x - \frac{2}{5}\int_1^4 x^{\frac{3}{2}}\,dx$

$= \left[\frac{2}{5}x^{\frac{5}{2}}\ln x - \frac{2}{5}\times\frac{2}{5}x^{\frac{5}{2}}\right]_1^4$

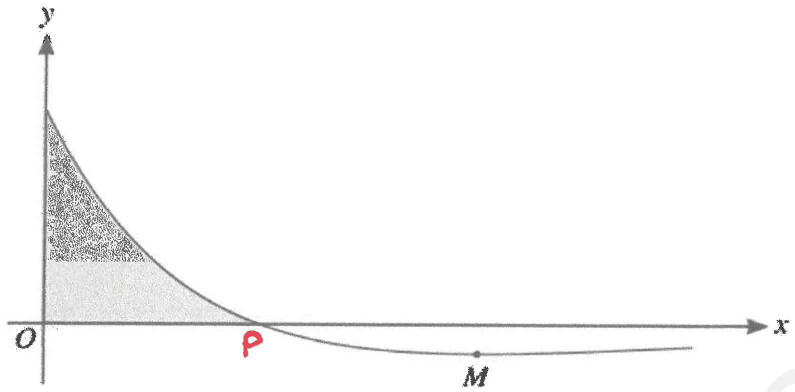
$= \left[\frac{2}{5}x^{\frac{5}{2}}\ln x - \frac{4}{25}x^{\frac{5}{2}}\right]_1^4$

$= \left[\frac{2}{5}(4)^{\frac{5}{2}}\ln 4 - \frac{4}{25}(4)^{\frac{5}{2}}\right] - \left[\frac{2}{5}\ln(1) - \frac{4}{25}(1)\right]$

$= \left[\frac{2}{5}\times 32\ln 4 - \frac{4}{25}\times 32\right] - \left[0 - \frac{4}{25}\right]$

$= \frac{64}{5}\ln 4 - \frac{128}{25} + \frac{4}{25}$

$= \frac{64}{5}\ln 4 - \frac{124}{25}$



The diagram shows part of the curve $y = (3-x)e^{-\frac{1}{3}x}$ for $x \geqslant 0$, and its minimum point M.

(b) Find the area of the shaded region bounded by the curve and the axes, giving your answer in terms of e. [5]

Find point P: $(3-x)e^{-\frac{1}{3}x} = 0$

$3-x=0$ or $e^{-\frac{1}{3}x}=0$ ×

$x=3$ ✓

$\int_0^3 (3-x)e^{-\frac{1}{3}x}\,dx$ u dv

$u = 3-x$ $\quad V = -3e^{-\frac{1}{3}x}$

$\frac{du}{dx} = -1$ $\quad \frac{dv}{dx} = e^{-\frac{1}{3}x}$

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$= -3(3-x)e^{-\frac{1}{3}x} - \int_0^3 3e^{-\frac{1}{3}x}\,dx$

$= 3(x-3)e^{-\frac{1}{3}x} - \int_0^3 3e^{-\frac{1}{3}x}\,dx$

$= \left[3(x-3)e^{-\frac{1}{3}x} + 9e^{-\frac{1}{3}x}\right]_0^3$

$= \left[3xe^{-\frac{1}{3}x} - 9e^{-\frac{1}{3}x} + 9e^{-\frac{1}{3}x}\right]_0^3$

$= \left[3xe^{-\frac{1}{3}x}\right]_0^3$

$= \left[9e^{-\frac{1}{3}\times 3}\right] - [0]$

$= 9e^{-1} = \frac{9}{e}$

11 Find the exact value of $\int_0^{\pi} \overset{u}{x^2} \overset{dv}{\cos\tfrac{1}{3}x}\,dx$. [6]

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$u = x^2 \qquad v = 3\sin\left(\tfrac{1}{3}x\right)$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \cos\left(\tfrac{1}{3}x\right)$

$\rightarrow 3x^2\sin\left(\tfrac{1}{3}x\right) - \int \overset{u}{6x}\, \overset{dv}{\sin\left(\tfrac{1}{3}x\right)}\,dx$

$u = 6x \qquad v = -3\cos\left(\tfrac{1}{3}x\right)$

$\frac{du}{dx} = 6 \qquad \frac{dv}{dx} = \sin\left(\tfrac{1}{3}x\right)$

$\rightarrow 3x^2\sin\left(\tfrac{1}{3}x\right) - \left[-18x\cos\left(\tfrac{1}{3}x\right) + \int 18\cos\left(\tfrac{1}{3}x\right)dx\right]$

$= \left[3x^2\sin\left(\tfrac{1}{3}x\right) + 18x\cos\left(\tfrac{1}{3}x\right) - 54\sin\left(\tfrac{1}{3}x\right)\right]_0^{\pi}$

$= \left[3\pi^2\sin\left(\tfrac{\pi}{3}\right) + 18\pi\cos\left(\tfrac{\pi}{3}\right) - 54\sin\left(\tfrac{\pi}{3}\right)\right] - [0]$

$= 3\pi^2 \times \frac{\sqrt{3}}{2} + 18\pi \times \frac{1}{2} - 54 \times \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}\pi^2 + 9\pi - 27\sqrt{3}$

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4 Using integration by parts, find the exact value of $\int_0^2 \tan^{-1}(\frac{1}{2}x)\,dx$. [5]

$u = \tan^{-1}(\frac{1}{2}x)$ $\quad v = x$

$\frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+(\frac{1}{2}x)^2}$ $\quad \frac{dv}{dx} = 1$

$\rightarrow x\tan^{-1}(\frac{1}{2}x) - \frac{1}{2}\int_0^2 \frac{x}{1+(\frac{1}{2}x)^2}\,dx$

$x\tan^{-1}(\frac{1}{2}x) - \frac{1}{2}\int_0^2 \frac{x}{1+\frac{x^2}{4}}\,dx$ $\times 4$ $\times 4$

$x\tan^{-1}(\frac{1}{2}x) - \frac{1}{2}\int_0^2 \frac{4x}{4+x^2}\,dx$

$x\tan^{-1}(\frac{1}{2}x) - \int_0^2 \frac{2x}{4+x^2}\,dx$

$= \left[x\tan^{-1}(\frac{1}{2}x) - \ln|4+x^2|\right]_0^2$

$= \left[2\tan^{-1}(\frac{1}{2}\times 2) - \ln|4+4|\right] - \left[0 - \ln|4|\right]$

$= \left[2\tan^{-1}(1) - \ln 8\right] - \left[-\ln 4\right]$

$= 2\times\frac{\pi}{4} - \ln 8 + \ln 4$

$= \frac{\pi}{2} - \ln 8 + \ln 4$

$= \frac{\pi}{2} - \ln 8 + \ln 4$

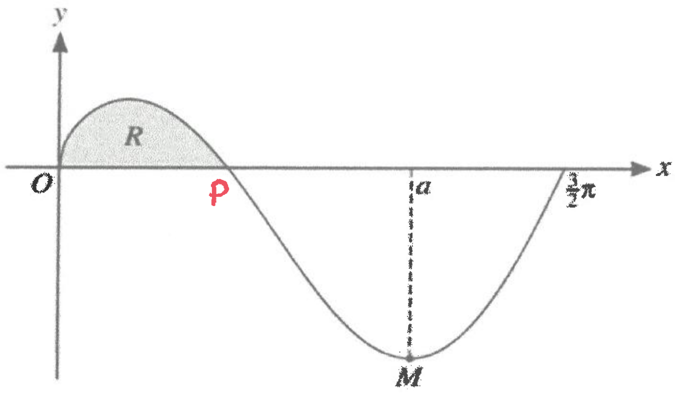
$= \frac{\pi}{2} + \ln 4 - \ln 8$

$\rightarrow \frac{\pi}{2} + \ln(\frac{4}{8})$

$= \frac{\pi}{2} + \ln(\frac{1}{2})$

$= \frac{\pi}{2} + \ln 2^{-1}$

$= \frac{\pi}{2} - \ln 2$



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]

Volume $= \pi\int y^2\,dx$

Point P: $\sqrt{x}\cos x = 0$

$\sqrt{x} = 0 \qquad \cos x = 0$

$x = 0 \qquad x = \frac{\pi}{2}$

$= \pi\int_0^{\frac{\pi}{2}} (\sqrt{x}\cos x)^2\,dx$

$= \pi\int_0^{\frac{\pi}{2}} x\cos^2 x\,dx$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{1}{2}\cos 2x + \frac{1}{2} = \cos^2 x$

$= \pi\int_0^{\frac{\pi}{2}} x\left(\frac{1}{2}\cos 2x + \frac{1}{2}\right)dx$

$= \pi\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}x\right)dx + \pi\int_0^{\frac{\pi}{2}} \frac{1}{2}x\cos 2x\,dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} x\,dx + \frac{\pi}{2}\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$

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$u = x \qquad v = \frac{1}{2}\sin 2x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \cos 2x$

$= \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + \frac{\pi}{2}\left[\frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right]$

$= \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + \frac{\pi}{2}\left[\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left(\frac{\pi^2}{8}\right) - \frac{\pi}{2}(0) + \frac{\pi}{2}\left(0 - \frac{1}{4}\right) - \frac{\pi}{2}\left(0 + \frac{1}{4}\right)$

$= \frac{\pi^3}{16} - \frac{\pi}{8} - \frac{\pi}{8}$

$= \frac{\pi^3}{16} - \frac{\pi}{4}$

10 **(a)** Given that $2x = \tan y$, show that $\frac{dy}{dx} = \frac{2}{1+4x^2}$. [3]

differentiate:

$$2 = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2}{\sec^2 y} \leftarrow \sec^2 y = 1 + \tan^2 y$$

$$\frac{dy}{dx} = \frac{2}{1 + \tan^2 y} \leftarrow \tan y = 2x \text{ (from Q.)}$$

$$\frac{dy}{dx} = \frac{2}{1 + (2x)^2}$$

$$\frac{dy}{dx} = \frac{2}{1 + 4x^2} \quad \text{QED}$$

(b) Hence find the exact value of $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x \tan^{-1}(2x)\, dx$. [7]

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \underset{dv}{x} \, \underset{u}{\tan^{-1}(2x)}\, dx$$

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$u = \tan^{-1}(2x)$ $\quad\quad$ $v = \frac{1}{2}x^2$

or: $\tan(u) = 2x$ $\quad\quad$ $\frac{dv}{dx} = x$

$$\frac{du}{dx} = \frac{2}{1 + 4x^2} \leftarrow \text{from part (a)}$$

$$\text{Integral} = \frac{1}{2}x^2 \times \tan^{-1}(2x) - \int \frac{1}{\cancel{2}} x^2 \times \frac{\cancel{2}}{1 + 4x^2}\, dx$$

$$= \frac{1}{2}x^2 \tan^{-1}(2x) - \int \frac{x^2}{1 + 4x^2}\, dx$$

$\frac{x^2}{1+4x^2}$ ← improper fraction, so use algebraic division:

$$\begin{array}{r|l} & \frac{1}{4} \\ 4x^2+0x+1 & x^2+0x+0 \\ & \underline{x^2+0x+\frac{1}{4}} \\ & \qquad\quad -\frac{1}{4} \end{array}$$

$$\rightarrow \frac{x^2}{1+4x^2} = \frac{1}{4} + \frac{-\frac{1}{4}}{1+4x^2}$$

Back to whole integral:

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \int\left(\frac{1}{4} + \frac{-\frac{1}{4} \div 4}{1+4x^2 \div 4}\right)dx$$

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \int\left(\frac{1}{4} + \frac{-\frac{1}{16}}{\frac{1}{4}+x^2}\right)dx$$

$\int \frac{1}{x^2+a^2}$ with $a^2=\frac{1}{4}$ so $a=\frac{1}{2}$

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \int\left(\frac{1}{4} - \frac{1}{16}\times\frac{1}{x^2+\frac{1}{4}}\right)dx$$

$$= \left[\frac{1}{2}x^2\tan^{-1}(2x)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} - \left[\frac{1}{4}x - \frac{1}{16}\times\frac{1}{\left(\frac{1}{2}\right)}\tan^{-1}\left(\frac{x}{\left(\frac{1}{2}\right)}\right)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \left[\frac{1}{2}x^2\tan^{-1}(2x) - \frac{1}{4}x + \frac{1}{8}\tan^{-1}(2x)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \left[\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2\tan^{-1}(\sqrt{3}) - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{8}\tan^{-1}(\sqrt{3})\right] - \left[\frac{1}{2}\left(\frac{1}{2}\right)^2\tan^{-1}(1) - \frac{1}{4}\left(\frac{1}{2}\right) + \frac{1}{8}\tan^{-1}(1)\right]$$

$$= \left[\frac{3}{8}\times\frac{\pi}{3} - \frac{\sqrt{3}}{8} + \frac{1}{8}\times\frac{\pi}{3}\right] - \left[\frac{1}{8}\times\frac{\pi}{4} - \frac{1}{8} + \frac{1}{8}\times\frac{\pi}{4}\right]$$

$$= \left[\frac{4\pi}{24} - \frac{\sqrt{3}}{8}\right] - \left[\frac{2\pi}{32} - \frac{1}{8}\right]$$

$$= \underline{\frac{5\pi}{48} - \frac{\sqrt{3}}{8} + \frac{1}{8}}$$