

- 1 A curve with equation $y = f(x)$ is such that $f'(x) = 6x^2 - \frac{8}{x^2}$. It is given that the curve passes through the point $(2, 7)$.

Find $f(x)$.

[3]

$$f(x) = \int (6x^2 - 8x^{-2}) dx$$

$$= \frac{1}{3} \times 6x^3 + 8x^{-1} + C$$

$$= 2x^3 + 8x^{-1} + C$$

sub. $(2, 7)$:

$$7 = 2(2)^3 + 8(2)^{-1} + C$$

$$7 = 2 \times 8 + 8 \times \frac{1}{2} + C$$

$$7 = 16 + 4 + C$$

$$7 = 20 + C$$

$$\underline{C = -13}$$

$$\rightarrow \underline{f(x) = 2x^3 + 8x^{-1} - 13}$$

7 The point (4, 7) lies on the curve $y = f(x)$ and it is given that $f'(x) = 6x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}$.

(b) Find the equation of the curve.

[4]

$$f(x) = \int (6x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}) dx$$

$$= 12x^{\frac{1}{2}} + 8x^{-\frac{1}{2}} + C$$

Sub. (4, 7):

$$7 = 12(4)^{\frac{1}{2}} + 8(4)^{-\frac{1}{2}} + C$$

$$7 = 12 \times 2 + 8 \times \frac{1}{2} + C$$

$$7 = 24 + 4 + C$$

$$7 = 28 + C$$

$$C = -21$$

$$\rightarrow \underline{f(x) = 12x^{\frac{1}{2}} + 8x^{-\frac{1}{2}} - 21}$$

- 2 The equation of a curve is such that $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$. It is given that the point (4, 7) lies on the curve.

Find the equation of the curve.

[4]

$$y = \int (3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}) dx$$

$$= \frac{2}{3} \times 3x^{\frac{3}{2}} - 2 \times 3x^{\frac{1}{2}} + C$$

$$= 2x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + C$$

Sub. (4, 7):

$$7 = 2(4)^{\frac{3}{2}} - 6(4)^{\frac{1}{2}} + C$$

$$7 = 2 \times 8 - 6 \times 2 + C$$

$$7 = 16 - 12 + C$$

$$7 = 4 + C$$

$$C = 3$$

$$\rightarrow \underline{y = 2x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 3}$$

- 10 At the point $(4, -1)$ on a curve, the gradient of the curve is $-\frac{3}{2}$. It is given that $\frac{dy}{dx} = x^{-\frac{1}{2}} + k$, where k is a constant.

(a) Show that $k = -2$.

[1]

$$\begin{aligned} \frac{dy}{dx} &= -\frac{3}{2} \text{ at } (4, -1): & -\frac{3}{2} &= (4)^{-\frac{1}{2}} + k \\ & & -\frac{3}{2} &= \frac{1}{2} + k \\ & & -2 &= k \end{aligned}$$

(b) Find the equation of the curve.

[4]

$$\begin{aligned} y &= \int (x^{-\frac{1}{2}} - 2) dx \\ &= 2x^{\frac{1}{2}} - 2x + C \end{aligned}$$

Sub. $(4, -1)$:

$$-1 = 2(4)^{\frac{1}{2}} - 2(4) + C$$

$$-1 = 2 \times 2 - 8 + C$$

$$-1 = -4 + C$$

$$C = 3$$

$$\rightarrow y = 2x^{\frac{1}{2}} - 2x + 3$$

2 The function f is defined by $f(x) = \frac{2}{(x+2)^2}$ for $x > -2$.

(b) The equation of a curve is such that $\frac{dy}{dx} = f(x)$. It is given that the point $(-1, -1)$ lies on the curve.

Find the equation of the curve.

[2]

$$y = \int 2(x+2)^{-2} dx$$

$$= -2(x+2)^{-1} + C$$

Sub. $(-1, -1)$:

$$-1 = -2(-1+2)^{-1} + C$$

$$-1 = -2(1)^{-1} + C$$

$$-1 = -2 + C$$

$$C = 1$$

$$\rightarrow \underline{y = -2(x+2)^{-1} + 1}$$

- 6 A curve is such that $\frac{dy}{dx} = \frac{6}{(3x-2)^3}$ and A (1, -3) lies on the curve. A point is moving along the curve and at A the y-coordinate of the point is increasing at 3 units per second.

(b) Find the equation of the curve.

[4]

$$y = \int 6(3x-2)^{-3} dx$$

$$= -3 \times \frac{1}{3} (3x-2)^{-2} + C$$

$$= -(3x-2)^{-2} + C$$

Sub. (1, -3):

$$-3 = -(3-2)^{-2} + C$$

$$-3 = -(1)^{-2} + C$$

$$-3 = -1 + C$$

$$C = -2$$

$$\rightarrow \underline{y = -(3x-2)^{-2} - 2}$$

7 The curve $y = f(x)$ is such that $f'(x) = \frac{-3}{(x+2)^4}$.

(b) Find $f(x)$ given that the curve passes through the point $(-1, 5)$.

[3]

$$y = \int -3(x+2)^{-4} dx$$

$$= \frac{-1}{3} \times -3(x+2)^{-3} + C$$

$$= (x+2)^{-3} + C$$

sub. $(-1, 5)$:

$$5 = (-1+2)^{-3} + C$$

$$5 = 1^{-3} + C$$

$$5 = 1 + C$$

$$C = 4$$

$$\rightarrow \underline{y = (x+2)^{-3} + 4}$$

- 10 The gradient of a curve at the point (x, y) is given by $\frac{dy}{dx} = 2(x+3)^{\frac{1}{2}} - x$. The curve has a stationary point at $(a, 14)$, where a is a positive constant.

(c) Find the equation of the curve.

[4]

from (a): $a = 6$

$$y = \int (2(x+3)^{\frac{1}{2}} - x) dx$$

$$= \frac{4}{3}(x+3)^{\frac{3}{2}} - \frac{1}{2}x^2 + C$$

sub. $(6, 14)$:

$$14 = \frac{4}{3}(6+3)^{\frac{3}{2}} - \frac{1}{2}(6)^2 + C$$

$$14 = \frac{4}{3}(9)^{\frac{3}{2}} - \frac{1}{2} \times 36 + C$$

$$14 = \frac{4}{3} \times 27 - 18 + C$$

$$14 = 36 - 18 + C$$

$$14 = 18 + C$$

$$\underline{C = -4}$$

$$\rightarrow \underline{y = \frac{4}{3}(x+3)^{\frac{3}{2}} - \frac{1}{2}x^2 - 4}$$

9 A function f is such that $f'(x) = 6(2x-3)^2 - 6x$ for $x \in \mathbb{R}$.

(b) Given that $f(1) = -1$, find $f(x)$.

[4]

Integrate:

$$f(x) = \int (6(2x-3)^2 - 6x) dx$$

$$= \frac{1}{3} \times \frac{1}{2} \times 6(2x-3)^3 - 3x^2 + C$$

$$= (2x-3)^3 - 3x^2 + C$$

$f(1) = -1$:

$$-1 = (2(1)-3)^3 - 3(1)^2 + C$$

$$-1 = (-1)^3 - 3 + C$$

$$-1 = -1 - 3 + C$$

$$-1 = -4 + C$$

$$C = 3$$

$$\rightarrow \underline{f(x) = (2x-3)^3 - 3x^2 + 3}$$

- 11 The gradient of a curve is given by $\frac{dy}{dx} = 6(3x-5)^3 - kx^2$, where k is a constant. The curve has a stationary point at $(2, -3.5)$.

(b) Find the equation of the curve.

[4]

from (a): $k = \frac{3}{2}$

$$y = \int \left[6(3x-5)^3 - \frac{3}{2}x^2 \right] dx$$

$$= \frac{1}{4} \times \frac{1}{3} \times 6(3x-5)^4 - \frac{1}{3} \times \frac{3}{2}x^3 + C$$

$$= \frac{1}{2}(3x-5)^4 - \frac{1}{2}x^3 + C$$

sub. $(2, -3.5)$:

$$-3.5 = \frac{1}{2}(3 \times 2 - 5)^4 - \frac{1}{2}(2)^3 + C$$

$$-3.5 = \frac{1}{2}(1)^4 - \frac{1}{2} \times 8 + C$$

$$-3.5 = \frac{1}{2} - 4 + C$$

$$-3.5 = -3.5 + C$$

$$C = 0$$

$$\rightarrow \underline{y = \frac{1}{2}(3x-5)^4 - \frac{1}{2}x^3}$$

- 9 A curve which passes through (0, 3) has equation $y = f(x)$. It is given that $f'(x) = 1 - \frac{2}{(x-1)^3}$.

(a) Find the equation of the curve.

[4]

$$f(x) = \int (1 - 2(x-1)^{-3}) dx$$

$$= x + (x-1)^{-2} + C$$

sub. (0,3):

$$3 = 0 + (-1)^{-2} + C$$

$$3 = 1 + C$$

$$\underline{C = 2}$$

$$\rightarrow \underline{f(x) = x + (x-1)^{-2} + 2}$$

The tangent to the curve at $(0, 3)$ intersects the curve again at one other point, P .

- (b) Show that the x -coordinate of P satisfies the equation $(2x + 1)(x - 1)^2 - 1 = 0$.

[4]

gradient at $(0, 3)$:

$$f'(0) = 1 - \frac{2}{(-1)^3}$$

$$= 1 - \frac{2}{-1}$$

$$= 3$$

eqn of tangent:

$$y - 3 = 3(x - 0)$$

$$y - 3 = 3x$$

$$y = 3x + 3$$

solve simultaneously with $f(x)$:

$$3x + 3 = x + (x - 1)^2 + 2$$

$$2x + 1 = (x - 1)^2$$

$$2x + 1 = \frac{1}{(x - 1)^2}$$

$$(2x + 1)(x - 1)^2 = 1$$

$$(2x + 1)(x - 1)^2 - 1 = 0 \quad \text{QED}$$

- (c) Verify that $x = \frac{3}{2}$ satisfies this equation and hence find the y -coordinate of P .

[2]

Sub. $x = \frac{3}{2}$: $(2(\frac{3}{2}) + 1)(\frac{3}{2} - 1)^2 - 1$

$$= (3 + 1)(\frac{1}{2})^2 - 1$$

$$= 4 \times \frac{1}{4} - 1$$

$$= 1 - 1$$

$$= 0 \quad \text{QED}$$

Sub $x = \frac{3}{2}$ into $y = 3x + 3$:

$$y = 3(\frac{3}{2}) + 3$$

$$= \frac{9}{2} + 3$$

$$= \frac{15}{2}$$

- 9 A curve is such that $\frac{d^2y}{dx^2} = \frac{6}{x^4} - \frac{5}{x^3}$. It is given that the curve has a stationary point at $(\frac{1}{2}, 9)$.

- (a) Use the expression for $\frac{d^2y}{dx^2}$ to determine whether the stationary point is a maximum or a minimum point. [2]

Sub. $x = \frac{1}{2}$ into $\frac{d^2y}{dx^2} = \frac{6}{(\frac{1}{2})^4} - \frac{5}{(\frac{1}{2})^3}$

$$= 96 - 40$$

$$= 56$$

$$\frac{d^2y}{dx^2} > 0, \text{ so } \underline{\text{minimum}}$$

- (b) Find the equation of the curve. [7]

$$\frac{dy}{dx} = \int (6x^{-4} - 5x^{-3}) dx$$

$$= -2x^{-3} + \frac{5}{2}x^{-2} + C$$

$\frac{dy}{dx} = 0$ at $x = \frac{1}{2}$:

$$0 = -2\left(\frac{1}{2}\right)^{-3} + \frac{5}{2}\left(\frac{1}{2}\right)^{-2} + C$$

$$0 = -16 + 10 + C$$

$$C = 6$$

$$\rightarrow \frac{dy}{dx} = -2x^{-3} + \frac{5}{2}x^{-2} + 6$$

$$y = \int (-2x^{-3} + \frac{5}{2}x^{-2} + 6) dx$$

$$= x^{-2} - \frac{5}{2}x^{-1} + 6x + C$$

Sub. $(\frac{1}{2}, 9)$:

$$9 = \left(\frac{1}{2}\right)^{-2} - \frac{5}{2}\left(\frac{1}{2}\right)^{-1} + 6\left(\frac{1}{2}\right) + C$$

$$9 = 4 - 5 + 3 + C$$

$$C = 7$$

$$\rightarrow \underline{y = x^{-2} - \frac{5}{2}x^{-1} + 6x + 7}$$