

- 5 The function f is defined by $f(x) = 2x^2 + 3$ for $x \geq 0$.

(a) Find and simplify an expression for $ff(x)$.

[2]

$$\begin{aligned}
 ff(x) &= f(2x^2 + 3) \\
 &= 2(2x^2 + 3)^2 + 3 \\
 &= 2(4x^4 + 12x^2 + 9) + 3 \\
 &= 8x^4 + 24x^2 + 18 + 3 \\
 &= \underline{8x^4 + 24x^2 + 21}
 \end{aligned}$$

(b) Solve the equation $ff(x) = 34x^2 + 19$.

[4]

$$\begin{aligned}
 34x^2 + 19 &= 8x^4 + 24x^2 + 21 \\
 -34x^2 \quad & \quad -34x^2
 \end{aligned}$$

$$\begin{aligned}
 19 &= 8x^4 - 10x^2 + 21 \\
 -19 \quad & \quad -19
 \end{aligned}$$

$$\begin{aligned}
 8x^4 - 10x^2 + 2 &= 0 \\
 \div 2 \quad \div 2 \quad \div 2 \quad \div 2
 \end{aligned}$$

$$4x^4 - 5x^2 + 1 = 0$$

Let $Y = x^2$:

$$4Y^2 - 5Y + 1 = 0$$

$$(4Y - 1)(Y - 1) = 0$$

$$4Y - 1 = 0 \quad \text{or} \quad Y - 1 = 0$$

$$Y = \frac{1}{4}$$

$$Y = 1$$

$$x^2 = \frac{1}{4}$$

$$x^2 = 1$$

$$x = \pm \frac{1}{2} \quad \text{or} \quad x = \pm 1$$

but $x \geq 0$, so $x = \frac{1}{2}$ or $x = 1$ only

- 5 The function f is defined for $x \in \mathbb{R}$ by

$$f: x \mapsto a - 2x,$$

where a is a constant.

- (a) Express $ff(x)$ and $f^{-1}(x)$ in terms of a and x .

[4]

$$\begin{aligned} ff(x) &= a - 2(a - 2x) \\ &= a - 2a + 4x \\ &= \underline{4x - a} \end{aligned}$$

$$\begin{aligned} y &= a - 2x \\ y + 2x &= a \\ 2x &= a - y \\ x &= \frac{a - y}{2} \end{aligned}$$

$$f^{-1}(x) = \underline{\frac{a - x}{2}}$$

- (b) Given that $ff(x) = f^{-1}(x)$, find x in terms of a .

[2]

$$4x - a = \frac{a - x}{2}$$

$$8x - 2a = a - x$$

$$9x - 2a = a$$

$$9x = 3a$$

$$x = \frac{3a}{9}$$

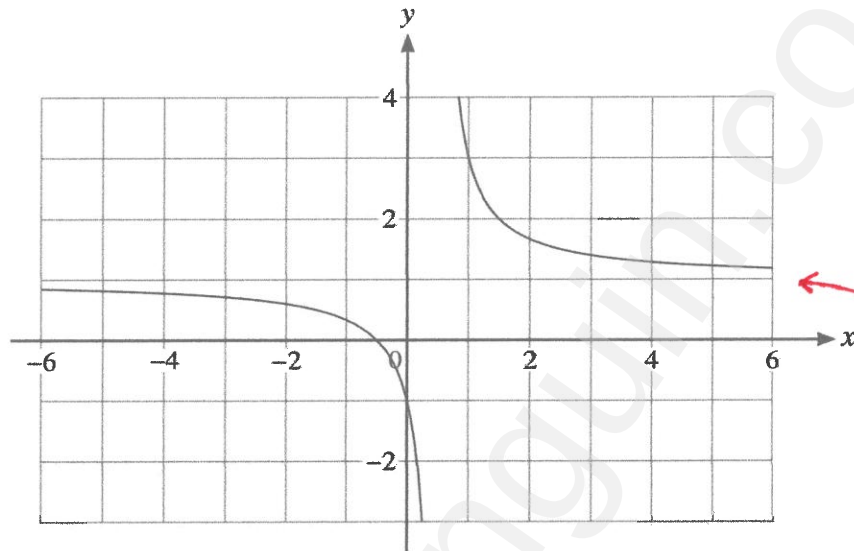
$$x = \underline{\frac{a}{3}}$$

10 Functions f and g are defined as follows:

$$f(x) = \frac{2x+1}{2x-1} \quad \text{for } x \neq \frac{1}{2},$$

$$g(x) = x^2 + 4 \quad \text{for } x \in \mathbb{R}.$$

(a)



because of asymptote at $y=1$

The diagram shows part of the graph of $y = f(x)$.

State the domain of f^{-1} .

Domain of f^{-1} = Range of f , so $x \in \mathbb{R}, x \neq 1$

[1]

(b) Find an expression for $f^{-1}(x)$.

[3]

$$\begin{aligned} y &= \frac{2x+1}{2x-1} \\ y(2x-1) &= 2x+1 \\ 2xy - y &= 2x+1 \\ 2xy - 2x &= y+1 \\ x(2y-2) &= y+1 \\ x &= \frac{y+1}{2y-2} \\ f^{-1}(x) &= \frac{x+1}{2x-2} \end{aligned}$$

(c) Find $gf^{-1}(3)$.

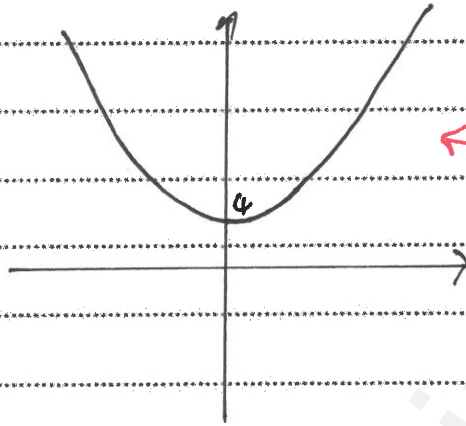
[2]

$$\begin{aligned} f^{-1}(3) &= \frac{3+1}{2(3)-2} \\ &= \frac{4}{4} = 1 \\ gf^{-1}(3) &= g(1) \\ &= (1)^2 + 4 \\ &= 5 \end{aligned}$$

(d) Explain why $g^{-1}(x)$ cannot be found.

[1]

$$g(x) = x^2 + 4:$$



← Many-to-One function

The inverse of a many-to-one does not exist (it would become a "one-to-many")

- 8 Functions f and g are defined by

$$f(x) = (x+a)^2 - a \text{ for } x \leq -a,$$

$$g(x) = 2x - 1 \text{ for } x \in \mathbb{R},$$

where a is a positive constant.

- (a) Find an expression for $f^{-1}(x)$.

[3]

$$y = (x+a)^2 - a$$

$$y+a = (x+a)^2$$

$$\pm \sqrt{y+a} = x+a$$

$x \leq -a$, so $x+a$ is ≤ 0 for all x ,

so we want negative square root.

$$-\sqrt{y+a} = x+a$$

$$x = -a - \sqrt{y+a}$$

$$\underline{f^{-1}(x) = -a - \sqrt{x+a}}$$

- (b) (i) State the domain of the function f^{-1} . = range of $f(x)$

[1]

Minimum point of $f(x)$ is $(-a, -a)$ so range is $f(x) \geq -a \rightarrow$ domain of f^{-1} is $\underline{x \geq -a}$

- (ii) State the range of the function f^{-1} .

[1]

Range of f^{-1} is domain of f : $\underline{f^{-1}(x) \leq -a}$

(c) Given that $a = \frac{7}{2}$, solve the equation $gf(x) = 0$.

[3]

$$f(x) = \left(x + \frac{7}{2}\right)^2 - \frac{7}{2}$$

$$gf(x) = 2\left[\left(x + \frac{7}{2}\right)^2 - \frac{7}{2}\right] - 1$$

$$= 2\left(x + \frac{7}{2}\right)^2 - 7 - 1$$

$$= 2\left(x + \frac{7}{2}\right)^2 - 8$$

$$gf(x) = 0:$$

$$2\left(x + \frac{7}{2}\right)^2 - 8 = 0$$

$$2\left(x + \frac{7}{2}\right)^2 = 8$$

$$\left(x + \frac{7}{2}\right)^2 = 4$$

$$x + \frac{7}{2} = \pm 2$$

$$x = -\frac{7}{2} \pm 2$$

$$x = -\frac{7}{2} + 2 \quad \text{or} \quad x = -\frac{7}{2} - 2$$

$$x = \frac{-3}{2} \quad \text{or} \quad x = \frac{-11}{2}$$

But domain is $x \leq -\frac{7}{2}$ so $\frac{-3}{2}$ is outside domain:

$$\underline{\underline{x = \frac{-11}{2}}}$$

2 The function f is defined by $f(x) = -2x^2 - 8x - 13$ for $x < -3$.

(a) Express $f(x)$ in the form $-2(x+a)^2 + b$, where a and b are integers. [2]

$$\begin{aligned} f(x) &= -2[x^2 + 4x] - 13 \\ &= -2[(x+2)^2 - 4] - 13 \\ &= -2(x+2)^2 + 8 - 13 \\ &= \underline{-2(x+2)^2 - 5} \end{aligned}$$

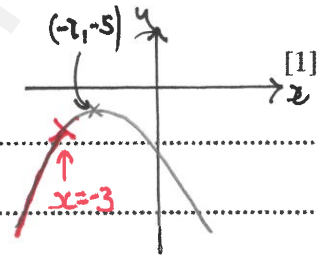
(b) Find the range of f . [1]

Maximum point of $f(x)$ is at $(-2, -5)$:

Find y -value when $x = -3$:

$$\begin{aligned} y &= -2(-3)^2 - 8(-3) - 13 \\ &= -18 + 24 - 13 \\ &= -7 \end{aligned}$$

so $\underline{f(x) < -7}$



(c) Find an expression for $f^{-1}(x)$. [3]

$$\begin{aligned} y &= -2(x+2)^2 - 5 \\ y + 5 &= -2(x+2)^2 \\ \frac{y+5}{-2} &= (x+2)^2 \\ \frac{-y-5}{2} &= (x+2)^2 \\ \pm \sqrt{\frac{-y-5}{2}} &= x+2 \end{aligned}$$

domain of f is $x < -3$,
so $x+2$ is negative for all
values of x , so we want the
negative root.

$$\begin{aligned} x+2 &= -\sqrt{\frac{-y-5}{2}} \\ x &= -2 - \sqrt{\frac{-y-5}{2}} \\ f^{-1}(x) &= \underline{-2 - \sqrt{\frac{-x-5}{2}}} \end{aligned}$$

- 6 The function f is defined by $f(x) = \frac{2x}{3x-1}$ for $x > \frac{1}{3}$.

(a) Find an expression for $f^{-1}(x)$.

[3]

$$y = \frac{2x}{3x-1}$$

$$y(3x-1) = 2x$$

$$3xy - y = 2x$$

$$3xy - 2x = y$$

$$x(3y-2) = y$$

$$x = \frac{y}{3y-2}$$

$$f^{-1}(x) = \frac{x}{3x-2}$$

- (b) Show that $\frac{2}{3} + \frac{2}{3(3x-1)}$ can be expressed as $\frac{2x}{3x-1}$.

[2]

$$\frac{2}{3} + \frac{2}{3(3x-1)}$$

$$\frac{2(3x-1)}{3(3x-1)} + \frac{2}{3(3x-1)}$$

$$\frac{6x-2+2}{3(3x-1)}$$

$$\frac{6x}{3(3x-1)}$$

$$\frac{2x}{3x-1} \quad \text{QED}$$

- (c) State the range of f .

[1]

sub. something a little bigger than $\frac{1}{3}$: $f(0.3333334) = \infty$
 sub. a very big number: $f(9999999) = \frac{2}{3}$
 so range: $f(x) > \frac{2}{3}$ between $\frac{2}{3}$ and ∞

8 Functions f and g are defined as follows:

$$f: x \mapsto x^2 - 1 \text{ for } x < 0,$$

$$g: x \mapsto \frac{1}{2x+1} \text{ for } x < -\frac{1}{2}.$$

(a) Solve the equation $fg(x) = 3$.

[4]

$$fg(x) = \left(\frac{1}{2x+1}\right)^2 - 1$$

$$fg(x) = 3:$$

$$\left(\frac{1}{2x+1}\right)^2 - 1 = 3$$

$$\left(\frac{1}{2x+1}\right)^2 = 4$$

$$\frac{1}{2x+1} = \pm 2$$

$$\frac{1}{2x+1} = -2$$

$$\frac{1}{2x+1} = 2$$

$$1 = -2(2x+1)$$

$$1 = 2(2x+1)$$

$$1 = -4x - 2$$

$$1 = 4x + 2$$

$$3 = -4x$$

$$-1 = 4x$$

$$x = \frac{-3}{4}$$

$$x = \frac{-1}{4}$$

But domain of g is $x < -\frac{1}{2}$, so $x = \frac{-3}{4}$ only

(b) Find an expression for $(fg)^{-1}(x)$.

[3]

$$f(g(x)) = \left(\frac{1}{2x+1}\right)^2 - 1$$

$$y = \frac{1}{(2x+1)^2} - 1$$

$$y+1 = \frac{1}{(2x+1)^2}$$

$$(2x+1)^2 (y+1) = 1$$

$$(2x+1)^2 = \frac{1}{y+1}$$

$$2x+1 = \pm \sqrt{\frac{1}{y+1}}$$

domain for $f(g(x))$ is $x < -\frac{1}{2}$. When $x < -\frac{1}{2}$,

$2x+1$ is negative, so we want the negative root.

$$\rightarrow 2x+1 = -\sqrt{\frac{1}{y+1}}$$

$$2x+1 = \frac{-1}{\sqrt{y+1}}$$

$$2x = \frac{-1}{\sqrt{y+1}} - 1$$

$$x = \frac{-1}{2\sqrt{y+1}} - \frac{1}{2}$$

$$f^{-1}(x) = \frac{-1}{2\sqrt{x+1}} - \frac{1}{2}$$

- 4 The function f is defined as follows:

$$f(x) = \sqrt{x} - 1 \text{ for } x > 1.$$

- (a) Find an expression for $f^{-1}(x)$.

[1]

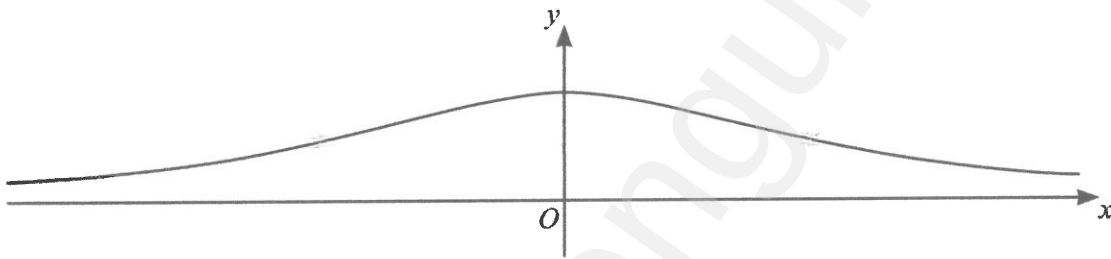
$$y = \sqrt{x} - 1$$

$$y + 1 = \sqrt{x}$$

$$(y + 1)^2 = x$$

$$x = (y + 1)^2$$

$$\underline{f^{-1}(x) = (x + 1)^2}$$



The diagram shows the graph of $y = g(x)$ where $g(x) = \frac{1}{x^2 + 2}$ for $x \in \mathbb{R}$.

- (b) State the range of g and explain whether g^{-1} exists.

[2]

Max. value of y is when $x = 0$:

$$y = \frac{1}{0+2} = \frac{1}{2}$$

Min. value of y is when $x \rightarrow \pm \infty$:

$$y = \frac{1}{\infty + 2} = 0 \text{ (but never actually reaches 0)}$$

$$\text{so } \underline{0 < g(x) \leq \frac{1}{2}}$$

$g(x)$ is a many-to-one function. Only one-to-one functions have an inverse, so g^{-1} does not exist.

The function h is defined by $h(x) = \frac{1}{x^2 + 2}$ for $x \geq 0$.

- (c) Solve the equation $hf(x) = f\left(\frac{25}{16}\right)$. Give your answer in the form $a + b\sqrt{c}$, where a , b and c are integers. [4]

$$f\left(\frac{25}{16}\right) = \sqrt{\frac{25}{16}} - 1$$

$$= \frac{5}{4} - 1$$

$$= \frac{1}{4}$$

$$hf(x) = \frac{1}{(f(x))^2 + 2} = \frac{1}{(\sqrt{x} - 1)^2 + 2}$$

$$= \frac{1}{x - 2\sqrt{x} + 1 + 2}$$

$$= \frac{1}{x - 2\sqrt{x} + 3}$$

$$hf(x) = \frac{1}{4}:$$

$$\frac{1}{x - 2\sqrt{x} + 3} = \frac{1}{4}$$

$$4 = x - 2\sqrt{x} + 3$$

$$x - 2\sqrt{x} - 1 = 0$$

$$x - 2x^{\frac{1}{2}} - 1 = 0$$

$$\text{let } Y = x^{\frac{1}{2}}:$$

$$Y^2 - 2Y - 1 = 0$$

$$Y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$Y = 1 + \sqrt{2} \quad \text{or} \quad Y = 1 - \sqrt{2}$$

$$\sqrt{x} = 1 + \sqrt{2} \quad \text{or} \quad \sqrt{x} = 1 - \sqrt{2}$$

$$x = (1 + \sqrt{2})^2 \quad x = (1 - \sqrt{2})^2$$

$$= 1 + 2\sqrt{2} + 2$$

$$= 1 - 2\sqrt{2} + 2$$

$$= 3 + 2\sqrt{2}$$

$$= 3 - 2\sqrt{2}$$

X $\leftarrow$ can't be this because domain of $f(x)$ is $x \geq 1$



5 The function f is defined by $f(x) = \frac{2x+1}{2x-1}$ for $x < \frac{1}{2}$.

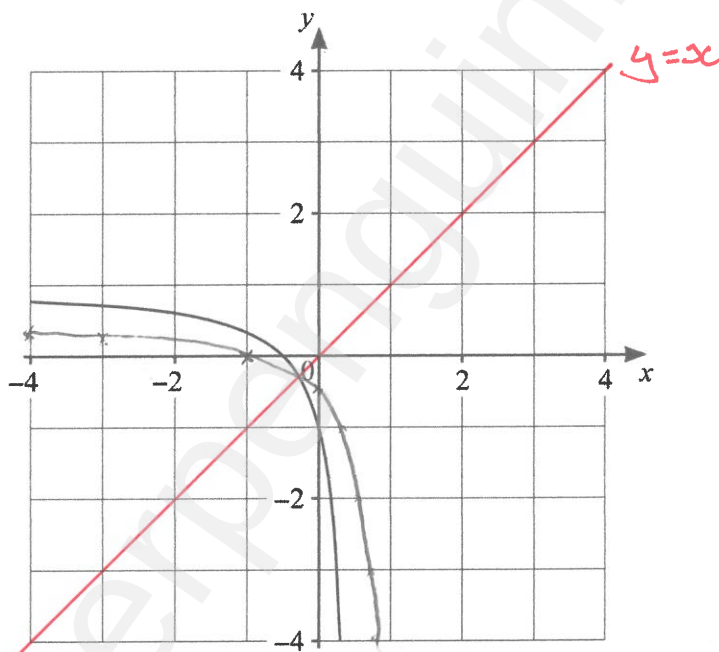
(a) (i) State the value of $f(-1)$.

[1]

$$f(-1) = \frac{2(-1)+1}{2(-1)-1}$$

$$= \frac{-1}{-3} = \underline{\underline{\frac{1}{3}}}$$

(ii)



The diagram shows the graph of $y = f(x)$. Sketch the graph of $y = f^{-1}(x)$ on this diagram. Show any relevant mirror line. [2]

(iii) Find an expression for $f^{-1}(x)$ and state the domain of the function f^{-1} . [4]

$$y = \frac{2x+1}{2x-1}$$

$$y(2x-1) = 2x+1$$

$$2xy - y = 2x+1$$

$$2xy - 2x = y+1$$

$$x(2y-2) = y+1$$

$$x = \frac{y+1}{2y-2}$$

$$f^{-1}(x) = \underline{\underline{\frac{x+1}{2x-2}}}$$

Domain of $f^{-1}(x)$ = Range of $f(x)$:

$$f(x) = \frac{2x+1}{2x-1} \quad x < \frac{1}{2}$$

Sub. something a tiny bit smaller than $\frac{1}{2}$, e.g. $x = 0.49$:

$$= \frac{2(0.49)+1}{2(0.49)-1} = \frac{1.98}{-0.02} = -99 \leftarrow \text{implies } f(x) \rightarrow -\infty$$

Sub. something at other end of domain, e.g. $x = -1000$:

$$= \frac{2(-1000)+1}{2(-1000)-1} = \frac{-1999}{-2001} = 0.999 \leftarrow \text{implies } f(x) \rightarrow 1$$

So domain of $f^{-1}(x)$ is $x < 1$

The function g is defined by $g(x) = 3x + 2$ for $x \in \mathbb{R}$.

(b) Solve the equation $f(x) = gf\left(\frac{1}{4}\right)$.

[3]

$$\begin{aligned} f\left(\frac{1}{4}\right) &= \frac{2\left(\frac{1}{4}\right)+1}{2\left(\frac{1}{4}\right)-1} = \frac{1.5}{-0.5} \\ &= \underline{\underline{-3}} \end{aligned}$$

$$gf\left(\frac{1}{4}\right) = g(-3)$$

$$= 3(-3) + 2$$

$$= \underline{\underline{-7}}$$

$$f(x) = gf\left(\frac{1}{4}\right):$$

$$\frac{2x+1}{2x-1} = -7$$

$$2x+1 = -14x+7$$

$$16x = 6$$

$$x = \underline{\underline{\frac{3}{8}}}$$

- 8 The functions f and g are defined as follows, where a and b are constants.

$$f(x) = 1 + \frac{2a}{x-a} \text{ for } x > a$$

$$g(x) = bx - 2 \text{ for } x \in \mathbb{R}$$

- (a) Given that $f(7) = \frac{5}{2}$ and $gf(5) = 4$, find the values of a and b .

[4]

$$f(7) = \frac{5}{2}:$$

$$1 + \frac{2a}{7-a} = \frac{5}{2}$$

$$\frac{2a}{7-a} = \frac{3}{2}$$

$$4a = 3(7-a)$$

$$4a = 21 - 3a$$

$$7a = 21$$

$$\underline{a = 3}$$

→

$$f(x) = 1 + \frac{6}{x-3}$$

$$gf(5) = 4: \quad f(5) = 1 + \frac{6}{5-3}$$

$$= 1 + 3$$

$$= 4$$

$$gf(5) = g(4)$$

$$= 4b - 2$$

$$\rightarrow 4b - 2 = 4$$

$$4b = 6$$

$$b = \frac{3}{2}$$

$$\underline{b = \frac{3}{2}}$$

→

$$g(x) = \frac{3}{2}x - 2$$

For the rest of this question, you should use the value of a which you found in (a).

(b) Find the domain of f^{-1} .

[1]

domain of $f^{-1} = \text{range of } f$

$$f(x) = 1 + \frac{6}{x-3} \text{ for } x > 3$$

when $x \rightarrow 3$, $f(x) \rightarrow \infty$
 and when $x \rightarrow \infty$, $f(x) \rightarrow 1$
 so range of $f(x)$ is $f(x) > 1$

domain of f^{-1} :
 $x > 1$

(c) Find an expression for $f^{-1}(x)$.

[3]

$$y = 1 + \frac{6}{x-3}$$

$$y - 1 = \frac{6}{x-3}$$

$$(y-1)(x-3) = 6$$

$$xy - 3y - x + 3 = 6$$

$$xy - x = 3y + 3$$

$$x(y-1) = 3y + 3$$

$$x = \frac{3y+3}{y-1}$$

$$f^{-1}(x) = \frac{3x+3}{x-1}$$

7 Functions f and g are defined as follows:

$$f : x \mapsto x^2 + 2x + 3 \text{ for } x \leq -1,$$

$$g : x \mapsto 2x + 1 \text{ for } x \geq -1.$$

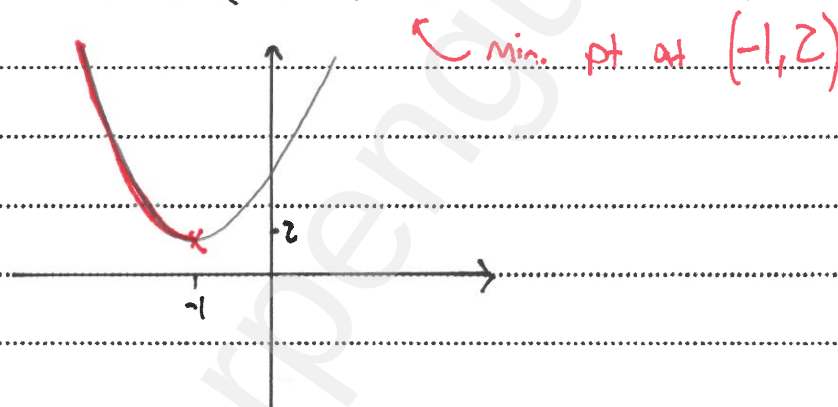
(a) Express $f(x)$ in the form $(x+a)^2 + b$ and state the range of f .

[3]

$$f(x) = x^2 + 2x + 3$$

$$= (x+1)^2 - 1 + 3$$

$$= (x+1)^2 + 2$$



$$\text{range: } f(x) \geq -1$$

(b) Find an expression for $f^{-1}(x)$.

[2]

$$y = (x+1)^2 + 2$$

$$y-2 = (x+1)^2$$

$$\pm\sqrt{y-2} = x+1$$

domain of f is $x \leq -1$, so $x+1$ is ≤ 0
(negative), meaning we want the negative
square root only:

$$-\sqrt{y-2} = x+1$$

$$-1 - \sqrt{y-2} = x$$

$$x = -1 - \sqrt{y-2}$$

$$\underline{f^{-1}(x) = -1 - \sqrt{x-2}}$$

(c) Solve the equation $gf(x) = 13$.

[3]

$$gf(x) = 2(x^2 + 2x + 3) + 1$$

$$= 2x^2 + 4x + 6 + 1$$

$$= 2x^2 + 4x + 7$$

$$gf(x) = 13:$$

$$2x^2 + 4x + 7 = 13$$

$$2x^2 + 4x - 6 = 0$$

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$$x = -3 \text{ or } x = 1$$

But domain of f is $x \leq -1$, so $x = -3$ only

from -2 to ∞

- 7 The function f is defined by $f(x) = 2 - \frac{5}{x+2}$ for $x > -2$.

(a) State the range of f .

[1]

sub. something a little bigger than -2: $f(-1.999...) = -\infty$
 sub. a very big number: $f(999999) = 2$
 so range: $f(x) < 2$ } from $-\infty$ to 2

(b) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1} .

[4]

domain of $f^{-1}(x)$ is range of $f(x)$, so: $x < 2$

$$y = 2 - \frac{5}{x+2}$$

$$y - 2 = -\frac{5}{x+2}$$

$$2 - y = \frac{5}{x+2}$$

$$(2-y)(x+2) = 5$$

$$2x + 4 - xy - 2y = 5$$

$$2x - xy = 1 + 2y$$

$$x(2-y) = 1 + 2y$$

$$x = \frac{1 + 2y}{2-y}$$

$$f^{-1}(x) = \frac{1 + 2x}{2-x}$$

The function g is defined by $g(x) = x + 3$ for $x > 0$.

- (c) Obtain an expression for $fg(x)$ giving your answer in the form $\frac{ax+b}{cx+d}$, where a , b , c and d are integers. [3]

$$fg(x) = 2 - \frac{5}{x+3+2}$$

$$= 2 - \frac{5}{x+5}$$

$$= \frac{2(x+5)}{x+5} - \frac{5}{x+5}$$

$$= \frac{2x+10-5}{x+5}$$

$$= \frac{2x+5}{x+5}$$

- 9 Functions f and g are defined as follows:

$$f(x) = (x-2)^2 - 4 \text{ for } x \geq 2,$$

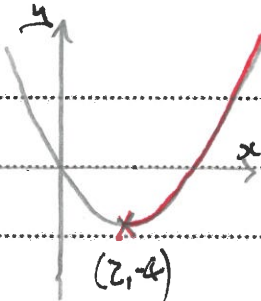
$$g(x) = ax + 2 \text{ for } x \in \mathbb{R},$$

where a is a constant.

- (a) State the range of f .

[1]

min. pt. at $(2, -4)$:



so range is $f(x) \geq -4$

- (b) Find $f^{-1}(x)$.

[2]

$$y = (x-2)^2 - 4$$

$$y + 4 = (x-2)^2$$

$$\pm \sqrt{y+4} = x-2$$

domain of $f(x)$ is $x \geq 2$

so $x-2$ is always positive

meaning we want the positive root:

$$x = 2 + \sqrt{y+4}$$

$$f^{-1}(x) = 2 + \sqrt{x+4}$$

- (c) Given that $a = -\frac{5}{3}$, solve the equation $f(x) = g(x)$.

[3]

$$(x-2)^2 - 4 = -\frac{5}{3}x + 2$$

$$x^2 - 4x + 4 - 4 = -\frac{5}{3}x + 2$$

$$x^2 - 4x = -\frac{5}{3}x + 2$$

$$x^2 - 4x - 2 = -\frac{5}{3}x$$

$$\begin{matrix} \times 3 \\ \times 3 \\ \times 3 \\ \times 3 \end{matrix}$$

$$3x^2 - 12x - 6 = -5x$$

$$\begin{matrix} +5x \\ +5x \end{matrix}$$

$$3x^2 - 7x - 6 = 0$$

$$ac = -18 \rightarrow -9, 2$$

$$3x^2 - 9x + 2x - 6 = 0$$

$$3x(x-3) + 2(x-3) = 0$$

$$(3x+2)(x-3) = 0$$

$$3x+2=0 \quad x-3=0$$

$$x = -\frac{2}{3} \quad \text{or} \quad x = 3$$

but domain for f is $x \geq 2$,

so $x = 3$ only

(d) Given instead that $ggf^{-1}(12) = 62$, find the possible values of a .

[5]

$$\begin{aligned} f^{-1}(12) &= 2 + \sqrt{12+4} \\ &= 2 + \sqrt{16} \\ &= 2 + 4 \\ &= 6 \end{aligned}$$

$$\begin{aligned} gf^{-1}(12) &= g(6) \\ &= 6a + 2 \end{aligned}$$

$$\begin{aligned} ggf^{-1}(12) &= g(6a + 2) \\ &= a(6a + 2) + 2 \\ &= 6a^2 + 2a + 2 \end{aligned}$$

$$ggf^{-1}(12) = 62$$

$$\rightarrow 6a^2 + 2a + 2 = 62$$

$$6a^2 + 2a - 60 = 0$$

$$a = \frac{-2 \pm \sqrt{2^2 - 4(6)(-60)}}{2(6)}$$

$$\underline{a = 3} \quad \text{or} \quad \underline{a = -\frac{10}{3}}$$

- 9 The functions f and g are defined for all real values of x by

$$f(x) = (3x-2)^2 + k \quad \text{and} \quad g(x) = 5x-1,$$

where k is a constant.

- (a) Given that the range of the function gf is $gf(x) \geq 39$, find the value of k .

[4]

If the range is $gf(x) \geq 39$, then the minimum point must be at $y = 39$. We need to find $gf(x)$ and the minimum point:

$$\begin{aligned} gf(x) &= 5[f(x)] - 1 \\ &= 5[(3x-2)^2 + k] - 1 \\ &= 5(3x-2)^2 + 5k - 1 \end{aligned}$$

Minimum point is when $3x-2 = 0$:

$$\begin{aligned} \text{min pt: } gf(x) &= 5(0)^2 + 5k - 1 \\ &= 5k - 1 \end{aligned}$$

$$\text{min pt} = 39: \quad 5k - 1 = 39$$

$$5k = 40$$

$$\underline{\underline{k = 8}}$$

- (b) For this value of k , determine the range of the function fg .

[2]

$$\begin{aligned} fg(x) &= [3(5x-1)-2]^2 + 8 \\ &= (15x-3-2)^2 + 8 \\ &= (15x-5)^2 + 8 \end{aligned}$$

Minimum point is when this $= 0$, at which point $fg(x) = 8$

$$\text{so } \underline{\underline{fg(x) \geq 8}}$$

- (c) The function h is defined for all real values of x and is such that $gh(x) = 35x + 19$.

Find an expression for $g^{-1}(x)$ and hence, or otherwise, find an expression for $h(x)$. [3]

$$g^{-1}(x): y = 5x - 1$$

$$y + 1 = 5x$$

$$x = \frac{y + 1}{5}$$

$$g^{-1}(x) = \frac{x + 1}{5}$$

$$gh(x) = 35x + 19$$

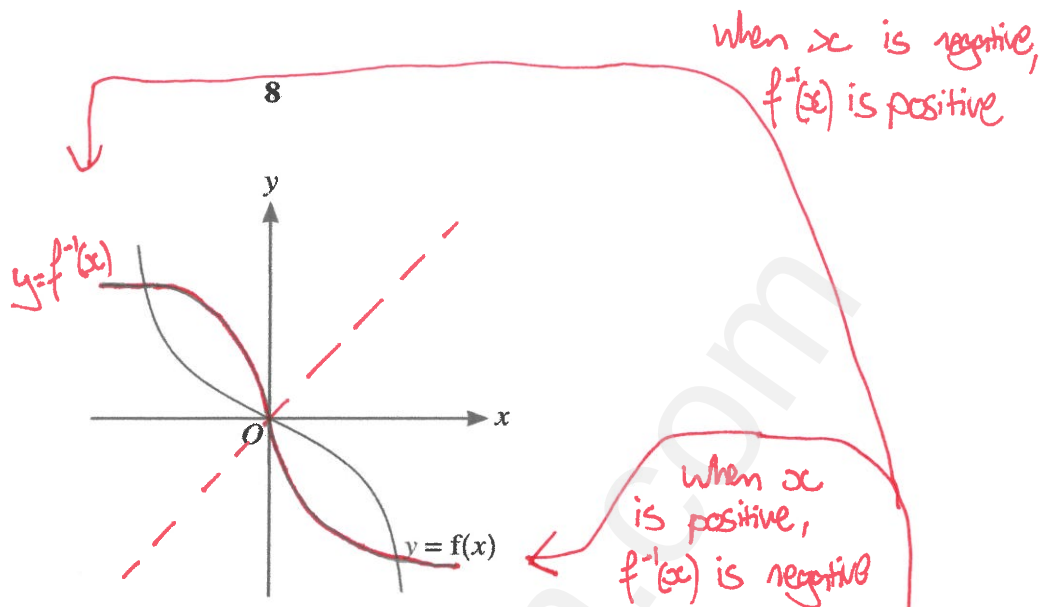
To get rid of g , we can't divide by g , we have to do g^{-1} on both sides:

$$g^{-1}(gh(x)) = g^{-1}(35x + 19)$$

$$\text{cancel out } \cancel{g^{-1}g} h(x) = \frac{(35x + 19) + 1}{5}$$

$$h(x) = \frac{35x + 20}{5}$$

$$\underline{h(x) = 7x + 4}$$



The diagram shows the graph of $y = f(x)$.

- (a) On this diagram sketch the graph of $y = f^{-1}(x)$. - reflection in $y = x$ [1]

It is now given that $f(x) = -\frac{x}{\sqrt{4-x^2}}$ where $-2 < x < 2$.

- (b) Find an expression for $f^{-1}(x)$. [4]

$$y = \frac{-x}{\sqrt{4-x^2}}$$

Square both sides

$$y^2 = \frac{x^2}{4-x^2}$$

$$y^2(4-x^2) = x^2$$

$$4y^2 - x^2y^2 = x^2$$

$$4y^2 = x^2 + x^2y^2$$

$$4y^2 = x^2(1+y^2)$$

$$x^2 = \frac{4y^2}{1+y^2}$$

$$x = \pm \sqrt{\frac{4y^2}{1+y^2}}$$

$$x = \pm \frac{2y}{\sqrt{1+y^2}}$$

$$f^{-1}(x) = \pm \frac{2x}{\sqrt{1+x^2}}$$

To decide whether it is the positive or negative square root we need to use the graph (!)

$$f^{-1}(x) = \pm \frac{2x}{\sqrt{1+x^2}}$$

this bit is always positive

So for $f^{-1}(x)$ to be negative when x is positive, we want the negative root:

$$f^{-1}(x) = -\frac{2x}{\sqrt{1+x^2}}$$

difficult!

The function g is defined by $g(x) = 2x$ for $-a < x < a$, where a is a constant.

- (c) State the maximum possible value of a for which fg can be formed. [1]

The outputs of g become the inputs of f . The domain for f is $-2 < x < 2$, so this should be the range of g . Given $g(x)$ multiplies by 2, this means the domain must be $-1 < x < 1$ ($\rightarrow$ range: $-2 < g(x) < 2$)

- (d) Assuming that fg can be formed, find and simplify an expression for $fg(x)$. [2]

$$\begin{aligned} fg(x) &= -\frac{2x}{\sqrt{4 - (2x)^2}} \\ &= -\frac{2x}{\sqrt{4 - 4x^2}} \\ &= -\frac{2x}{2\sqrt{1 - x^2}} \\ &= -\frac{x}{\sqrt{1 - x^2}} \end{aligned}$$