1 Solve the equation

$$\ln(1 + e^{-3x}) = 2.$$

Give the answer correct to 3 decimal places. [3]

$\ln(1 + e^{-3x}) = 2$

$1 + e^{-3x} = e^2$

$e^{-3x} = e^2 - 1$

$\ln(e^{-3x}) = \ln(e^2 - 1)$

$-3x = \ln(e^2 - 1)$

$x = \frac{-1}{3}\ln(e^2 - 1)$

$x = -0.618$

1 Solve the equation $\ln(x + 5) = 5 + \ln x$. Give your answer correct to 3 decimal places. [4]

$$\ln(x+5) - \ln x = 5$$

$$\ln\left(\frac{x+5}{x}\right) = 5$$

$$\frac{x+5}{x} = e^5$$

$$x + 5 = xe^5$$

$$x - xe^5 = -5$$

$$x(1 - e^5) = -5$$

$$x = \frac{-5}{1 - e^5}$$

$$x = 0.034$$

1 Solve the equation $\ln(x^3 - 3) = 3\ln x - \ln 3$. Give your answer correct to 3 significant figures. [3]

$$\ln(x^3-3) = 3\ln x - \ln 3$$

$$\ln(x^3-3) = \ln x^3 - \ln 3$$

$$\ln(x^3-3) = \ln\left(\frac{x^3}{3}\right)$$

$$x^3 - 3 = \frac{x^3}{3}$$

$\times 3$ $\times 3$ $\times 3$

$$3x^3 - 9 = x^3$$

$-x^3$ $-x^3$

$$2x^3 - 9 = 0$$

$$2x^3 = 9$$

$$x^3 = \frac{9}{2}$$

$$x = \sqrt[3]{\frac{9}{2}}$$

$$x = 1.65$$

1 Solve the equation $\ln(e^{2x} + 3) = 2x + \ln 3$, giving your answer correct to 3 decimal places. [4]

$$\ln(e^{2x} + 3) - \ln 3 = 2x$$

$$\ln\left(\frac{e^{2x} + 3}{3}\right) = 2x$$

$$\frac{e^{2x} + 3}{3} = e^{2x}$$

$$\underset{-e^{2x}}{e^{2x}} + 3 = \underset{-e^{2x}}{3e^{2x}}$$

$$3 = 2e^{2x}$$

$$e^{2x} = \frac{3}{2}$$

$$2x = \ln\left(\frac{3}{2}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{3}{2}\right)$$

$$x = 0.203$$

2 Solve the equation $\ln(x-5) = 7 - \ln x$. Give your answer correct to 2 decimal places. [4]

$\ln(x-5) + \ln x = 7$

$\ln[x(x-5)] = 7$

$x(x-5) = e^7$

$x^2 - 5x = e^7$

$x^2 - 5x - e^7 = 0$

$x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-e^7)}}{2(1)}$

$\underline{x = 35.71}$ or $x = -30.71$ x

Gives ln of negative when substituted into Q.

3 (a) Show that the equation $\log_3(2x+1) = 1 + 2\log_3(x-1)$ can be written as a quadratic equation in x. [3]

$$\log_3(2x+1) - 2\log_3(x-1) = 1$$

$$\log_3(2x+1) - \log_3(x-1)^2 = 1$$

$$\log_3\left(\frac{2x+1}{(x-1)^2}\right) = 1$$

$$\frac{2x+1}{(x-1)^2} = 3^1$$

$$2x+1 = 3(x-1)^2$$

$$2x+1 = 3(x^2-2x+1)$$

$$2x+1 = 3x^2-6x+3$$

$$3x^2-8x+2=0$$

QED

$\log_3(2x+1) = 1 + 2\log_3(x-1)$

(b) Hence solve the equation $\log_3(4y+1) = 1 + 2\log_3(2y-1)$, giving your answer correct to 2 decimal places. [2]

let $x = 2y$ $\quad 3x^2 - 8x + 2 = 0$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(2)}}{2(3)}$$

$x = \frac{4+\sqrt{10}}{3}$ or $x = \frac{4-\sqrt{10}}{3}$

$2y = \frac{4+\sqrt{10}}{3}$ or $2y = \frac{4-\sqrt{10}}{3}$

$y = \frac{4+\sqrt{10}}{6}$ or $y = \frac{4-\sqrt{10}}{6}$

$y = 1.19$ $\quad$ $y = 0.14$ ✗

gives a negative here and logs of negatives don't exist

1 Solve the equation $2(3^{2x-1}) = 4^{x+1}$, giving your answer correct to 2 decimal places. [4]

$$2(3^{2x-1}) = 4^{x+1}$$

$$\ln[2(3^{2x-1})] = \ln 4^{x+1}$$

$$\ln 2 + \ln 3^{2x-1} = (x+1)\ln 4$$

$$\ln 2 + (2x-1)\ln 3 = (x+1)\ln 4$$

$$\ln 2 + 2x\ln 3 - \ln 3 = x\ln 4 + \ln 4$$

$$2x\ln 3 - x\ln 4 = \ln 4 + \ln 3 - \ln 2$$

$$x(2\ln 3 - \ln 4) = \ln 12 - \ln 2$$

$$x(\ln 3^2 - \ln 4) = \ln 6$$

$$x(\ln 9 - \ln 4) = \ln 6$$

$$x = \frac{\ln 6}{\ln 9 - \ln 4}$$

$$= \frac{\ln 6}{\ln\left(\frac{9}{4}\right)}$$

$$= \underline{2.21}$$

1 Solve the equation $2^{3x-1} = 5(3^{1-x})$. Give your answer in the form $\frac{\ln a}{\ln b}$ where a and b are integers. [4]

$$2^{3x-1} = 5(3^{1-x})$$
$$\ln 2^{3x-1} = \ln[5(3^{1-x})]$$
$$(3x-1)\ln 2 = \ln 5 + \ln 3^{1-x}$$
$$3x\ln 2 - \ln 2 = \ln 5 + (1-x)\ln 3$$
$$3x\ln 2 - \ln 2 = \ln 5 + \ln 3 - x\ln 3$$
$$3x\ln 2 + x\ln 3 = \ln 5 + \ln 3 + \ln 2$$
$$x(3\ln 2 + \ln 3) = \ln 15 + \ln 2$$
$$x(\ln 2^3 + \ln 3) = \ln 30$$
$$x(\ln 8 + \ln 3) = \ln 30$$
$$x\ln 24 = \ln 30$$
$$x = \frac{\ln 30}{\ln 24}$$

DO NOT WRITE IN THIS MARGIN

4 Solve the equation $5^x = 5^{x+2} - 10$. Give your answer correct to 3 decimal places. [3]

$5^x = 5^x \times 5^2 - 10$

$5^x = 5^x \times 25 - 10$

Let $Y = 5^x$:

$Y = 25Y - 10$

$24Y = 10$

$Y = \frac{5}{12}$

$5^x = \frac{5}{12}$

$\log_5 5^x = \log_5\left(\frac{5}{12}\right)$

$x = \log_5\left(\frac{5}{12}\right)$

$= \underline{-0.544}$

2 Solve the equation $4^x = 3 + 4^{-x}$. Give your answer correct to 3 decimal places. [5]

$$4^x = 3 + \frac{1}{4^x}$$

$\times 4^x \quad \times 4^x \quad \times 4^x$

$$4^{2x} = 3 \times 4^x + 1$$

$$4^{2x} - 3 \times 4^x - 1 = 0$$

let $y = 4^x$:

$$y^2 - 3y - 1 = 0$$

$$y = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)}$$

$$y = \frac{3+\sqrt{13}}{2} \quad \text{or} \quad y = \frac{3-\sqrt{13}}{2}$$

$\rightarrow \quad 4^x = \frac{3+\sqrt{13}}{2} \quad$ or $\quad 4^x = \frac{3-\sqrt{13}}{2}$

$\log_4 4^x = \log_4\left(\frac{3+\sqrt{13}}{2}\right) \qquad \log_4 4^x = \log_4\left(\frac{3-\sqrt{13}}{2}\right)$

$x = \log_4\left(\frac{3+\sqrt{13}}{2}\right) \qquad$ no sol^s

$\underline{x = 0.862}$

2 Find the real root of the equation $\frac{2e^x + e^{-x}}{2 + e^x} = 3$, giving your answer correct to 3 decimal places. Your working should show clearly that the equation has only one real root. [5]

$$\frac{2e^x + e^{-x}}{2 + e^x} = 3$$

$$2e^x + e^{-x} = 3(2 + e^x)$$

$$2e^x + e^{-x} = 6 + 3e^x$$

$$-2e^x \qquad -2e^x$$

$$0 = e^x - e^{-x} + 6$$

$$e^x - \frac{1}{e^x} + 6 = 0$$

$$\times e^x \quad \times e^x \quad \times e^x$$

$$e^{2x} - 1 + 6e^x = 0$$

$$e^{2x} + 6e^x - 1 = 0$$

let $y = e^x$: $y^2 + 6y - 1 = 0$

$$y = \frac{-6 \pm \sqrt{6^2 - 4(1)(-1)}}{2(1)}$$

$y = -3 + \sqrt{10}$ or $y = -3 - \sqrt{10}$

$e^x = -3 + \sqrt{10}$ or $e^x = -3 - \sqrt{10}$

$x = \ln(-3 + \sqrt{10})$ or $x = \ln(-3 - \sqrt{10})$

$x = -1.818$

no sol^n - logs of negatives do not exist.

1 Solve the equation

$$3e^{2x} - 4e^{-2x} = 5.$$

Give the answer correct to 3 decimal places. [3]

$$3e^{2x} - \frac{4}{e^{2x}} = 5$$

$\times e^{2x}$ $\times e^{2x}$ $\times e^{2x}$

$$3e^{4x} - 4 = 5e^{2x}$$

$$3e^{4x} - 5e^{2x} - 4 = 0$$

Let $Y = e^{2x}$:

$$3Y^2 - 5Y - 4 = 0$$

$$Y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-4)}}{2(3)}$$

$$Y = \frac{5+\sqrt{73}}{6} \quad \text{or} \quad Y = \frac{5-\sqrt{73}}{6}$$

$$e^{2x} = \frac{5+\sqrt{73}}{6} \qquad e^{2x} = \frac{5-\sqrt{73}}{6}$$

✗ no solutions

$$2x = \ln\left(\frac{5+\sqrt{73}}{6}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{5+\sqrt{73}}{6}\right)$$

$$= \underline{0.407}$$

4 The positive numbers p and q are such that

$$\ln\left(\frac{p}{q}\right) = a \quad \text{and} \quad \ln\left(q^2 p\right) = b.$$

1 2

Express $\ln\left(p^7 q\right)$ in terms of a and b. [4]

$\ln(p^7 q) = \ln p^7 + \ln q$

$= 7\ln p + \ln q$

Now find $\ln p$:

1: $\ln\left(\frac{p}{q}\right) = a$

$\ln p - \ln q = a$

$\ln p = a + \ln q$ A

2: $\ln(q^2 p) = b$

$\ln q^2 + \ln p = b$

$2\ln q + \ln p = b$ B

Sub. A into B:

$2\ln q + a + \ln q = b$

$3\ln q + a = b$

$3\ln q = b - a$

$\ln q = \frac{b-a}{3}$

Sub into A: $\ln p = a + \frac{b-a}{3}$

$= \frac{3a}{3} + \frac{b-a}{3}$

$= \frac{2a+b}{3}$

$7\ln p + \ln q$

$= 7\left(\frac{2a+b}{3}\right) + \frac{b-a}{3}$

$= \frac{14a+7b}{3} + \frac{b-a}{3}$

$= \frac{13a+8b}{3}$