7 Let $f(x) = 8x^3 + 54x^2 - 17x - 21$.

(a) Show that $x + 7$ is a factor of $f(x)$. [1]

$f(-7) = 8(-7)^3 + 54(-7)^2 - 17(-7) - 21$

$= -2744 + 2646 + 119 - 21$

$= 0$

so $(x+7)$ is a factor

(b) Find the quotient when $f(x)$ is divided by $x + 7$. [2]

$$
\begin{array}{r|l}
 & 8x^2 \quad - 2x \quad - 3 \\
x + 7 & 8x^3 + 54x^2 - 17x - 21 \\
 & \underline{8x^3 + 56x^2} \\
 & \quad -2x^2 - 17x \\
 & \quad \underline{-2x^2 - 14x} \\
 & \qquad -3x - 21 \\
 & \qquad \underline{-3x - 21}
\end{array}
$$

quotient $= 8x^2 - 2x - 3$

9 (a) Find the quotient and remainder when $x^4 + 16$ is divided by $x^2 + 4$. [3]

$$
\begin{array}{r|l}
 & x^2 \quad\quad\quad - 4 \\
x^2 + 0x + 4 & x^4 + 0x^3 + 0x^2 + 0x + 16 \\
 & \underline{x^4 + 0x^3 + 4x^2} \downarrow \quad \downarrow \\
 & \quad\quad\quad -4x^2 + 0x + 16 \\
 & \quad\quad\quad \underline{-4x^2 + 0x - 16} \\
 & \quad\quad\quad\quad\quad\quad\quad 32
\end{array}
$$

$$\frac{x^4 + 16}{x^2 + 4} \equiv x^2 - 4 + \frac{32}{x^2 + 4}$$

Quotient: $\underline{x^2 - 4}$

Remainder: $\underline{32}$

1 Find the quotient and remainder when $x^4 - 3x^3 + 9x^2 - 12x + 27$ is divided by $x^2 + 5$. [3]

$$
\begin{array}{r|l}
 & x^2 \quad - 3x \quad + 4 \\
\hline
x^2 + 0x + 5 & x^4 - 3x^3 + 9x^2 - 12x + 27 \\
 & \underline{x^4 + 0x^3 + 5x^2} \downarrow \\
 & \quad - 3x^3 + 4x^2 - 12x \\
 & \quad \underline{- 3x^3 + 0x^2 - 15x} \\
 & \qquad 4x^2 + 3x + 27 \\
 & \qquad \underline{4x^2 + 0x + 20} \\
 & \qquad\qquad 3x + 7
\end{array}
$$

so $\dfrac{x^4 - 3x^3 + 9x^2 - 12x + 27}{x^2 + 5} \equiv \underbrace{x^2 - 3x + 4}_{\text{quotient}} + \dfrac{\overbrace{3x+7}^{\text{remainder}}}{x^2+5}$

3 The polynomial $2x^3 + ax^2 - 11x + b$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 12.

① ②

Find the values of a and b. [5]

①: $p\left(\frac{1}{2}\right) = 0$:

$2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 - 11\left(\frac{1}{2}\right) + b = 0$

$2 \times \frac{1}{8} + a \times \frac{1}{4} - \frac{11}{2} + b = 0$

$\frac{1}{4} + \frac{a}{4} - \frac{11}{2} + b = 0$

$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \times 4$

$1 + a - 22 + 4b = 0$

$a + 4b = 21$ Ⓐ

②: $p(-1) = 12$:

$2(-1)^3 + a(-1)^2 - 11(-1) + b = 12$

$2 \times -1 + a \times 1 + 11 + b = 12$

$-2 + a + 11 + b = 12$

$a + b = 3$ Ⓑ

Ⓐ − Ⓑ: $3b = 18$

$b = 6$

→ Ⓑ: $a + 6 = 3$

$a = -3$

2 The polynomial $ax^3 + 5x^2 - 4x + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 2.

Find the values of a and b. [5]

$p(-2) = 0$

$a \times (-2)^3 + 5(-2)^2 - 4(-2) + b = 0$

$-8a + 20 + 8 + b = 0$

$-8a + b = -28$ ①

$p(-1) = 2$

$a(-1)^3 + 5(-1)^2 - 4(-1) + b = 2$

$-a + 5 + 4 + b = 2$

$-a + b = -7$ ②

② − ①: $-a + b = -7$

$-8a + b = -28$

$7a = 21$

$\underline{a = 3}$

sub → ②: $-3 + b = -7$

$\underline{b = -4}$

3 The polynomial $ax^3 + x^2 + bx + 3$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 2)$ the remainder is 5.

Find the values of a and b. [5]

$p\left(\frac{1}{2}\right) = 0: \quad a\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 3 = 0$

$a \times \frac{1}{8} + \frac{1}{4} + \frac{b}{2} + 3 = 0$

$\times 8 \quad \times 8 \quad \times 8 \quad \times 8 \quad \times 8$

$a + 2 + 4b + 24 = 0$

$a + 4b + 26 = 0$

$a + 4b = -26$ ①

$p(-2) = 5: \quad a(-2)^3 + (-2)^2 + b(-2) + 3 = 5$

$-8a + 4 - 2b + 3 = 5$

$-8a - 2b + 7 = 5$

$-8a - 2b = -2$ ②

② $\times 2$: $-16a - 4b = -4$ ③

① + ③: $a + 4b = -26$

$-16a - 4b = -4$

$-15a = -30$

$\underline{a = 2}$

$\rightarrow$ ①: $2 + 4b = -26$

$4b = -28$

$\underline{b = -7}$

3 The polynomial $2x^3 + ax^2 + bx + 6$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $(x+2)$ the remainder is -38 and when $p(x)$ is divided by $(2x-1)$ the remainder is $\frac{19}{2}$.

① ②

Find the values of a and b. [5]

①: $p(-2) = -38$:

$2(-2)^3 + a(-2)^2 + b(-2) + 6 = -38$

$2\times -8 + a\times 4 - 2b + 6 = -38$

$-16 + 4a - 2b + 6 = -38$

$4a - 2b - 10 = -38$

$4a - 2b = -28$ Ⓐ

②: $p\left(\frac{1}{2}\right) = \frac{19}{2}$:

$2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 6 = \frac{19}{2}$

$2\times\frac{1}{8} + a\times\frac{1}{4} + \frac{b}{2} + 6 = 19/2$

$\frac{1}{4} + \frac{a}{4} + b/2 + 6 = 19/2$

$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \times 4$

$1 + a + 2b + 24 = 38$

$a + 2b + 25 = 38$

$a + 2b = 13$ Ⓑ

Ⓐ + Ⓑ: $5a = -15$

$\underline{a = -3}$

→ Ⓑ: $-3 + 2b = 13$

$2b = 16$

$\underline{b = 8}$

5 The polynomial $ax^3 - 10x^2 + bx + 8$, where a and b are constants, is denoted by $p(x)$. It is given that $(x-2)$ is a factor of both $p(x)$ and $p'(x)$.

(a) Find the values of a and b. [5]

$p(2) = 0: \quad a(2)^3 - 10(2)^2 + b(2) + 8 = 0$

$8a - 40 + 2b + 8 = 0$

$8a + 2b - 32 = 0$

$8a + 2b = 32$ ①

$p'(x) = 3ax^2 - 20x + b$

$p'(2) = 0: \quad 3a(2)^2 - 20(2) + b = 0$

$12a - 40 + b = 0$

$12a + b = 40$ ②

②×2: $24a + 2b = 80$ ③

③ − ①: $16a = 48$

$\underline{a = 3}$

→ ②: $12(3) + b = 40$

$36 + b = 40$

$\underline{b = 4}$

(b) When a and b have these values, factorise $\mathrm{p}(x)$ completely. [3]

$p(x) = 3x^3 - 10x^2 + 4x + 8$

$$
\begin{array}{r|l}
 & 3x^2 - 4x - 4 \\
x - 2 & 3x^3 - 10x^2 + 4x + 8 \\
 & 3x^3 - 6x^2 \\
 & \quad -4x^2 + 4x \\
 & \quad -4x^2 + 8x \\
 & \qquad -4x + 8 \\
 & \qquad -4x + 8
\end{array}
$$

$\rightarrow \quad (x-2)(\underbrace{3x^2 - 4x - 4}_{\text{factorise}})$

$= \quad (x-2)(3x+2)(x-2)$

1 Find the quotient and remainder when $6x^4 + x^3 - x^2 + 5x - 6$ is divided by $2x^2 - x + 1$. [3]

$$\begin{array}{r|l} & 3x^2 + 2x - 1 \\ 2x^2 - x + 1 & 6x^4 + x^3 - x^2 + 5x - 6 \\ & \underline{6x^4 - 3x^3 + 3x^2} \\ & \quad 4x^3 - 4x^2 + 5x \\ & \quad \underline{4x^3 - 2x^2 + 2x} \\ & \qquad -2x^2 + 3x - 6 \\ & \qquad \underline{-2x^2 + x - 1} \\ & \qquad\qquad 2x - 5 \end{array}$$

so $\dfrac{6x^4+x^3-x^2+5x-6}{2x^2-x+1} = 3x^2+2x-1 + \dfrac{2x-5}{2x^2-x+1}$

remainder

quotient

1 Find the quotient and remainder when $2x^4 + 1$ is divided by $x^2 - x + 2$. [3]

$$
\begin{array}{r|l}
 & 2x^2 + 2x - 2 \\
x^2 - x + 2 & 2x^4 + 0x^3 + 0x^2 + 0x + 1 \\
 & \underline{2x^4 - 2x^3 + 4x^2} \\
 & 2x^3 - 4x^2 + 0x \\
 & \underline{2x^3 - 2x^2 + 4x} \\
 & -2x^2 - 4x + 1 \\
 & \underline{-2x^2 + 2x - 4} \\
 & -6x + 5
\end{array}
$$

quotient remainder

so $\dfrac{2x^4+1}{x^2-x+2} \equiv 2x^2 + 2x - 2 + \dfrac{-6x+5}{x^2-x+2}$

2 Find the quotient and remainder when $2x^4 - 27$ is divided by $x^2 + x + 3$. [3]

$$
\begin{array}{r|l}
 & 2x^2 - 2x - 4 \\
\hline
x^2 + x + 3 & 2x^4 + 0x^3 + 0x^2 + 0x - 27 \\
 & \underline{2x^4 + 2x^3 + 6x^2} \\
 & \quad -2x^3 - 6x^2 + 0x \\
 & \quad \underline{-2x^3 - 2x^2 - 6x} \\
 & \qquad -4x^2 + 6x - 27 \\
 & \qquad \underline{-4x^2 - 4x - 12} \\
 & \qquad\qquad 10x - 15
\end{array}
$$

remainder

$$\frac{2x^4 - 27}{x^2 + x + 3} \equiv 2x^2 - 2x - 4 + \frac{10x - 15}{x^2 + x + 3}$$

quotient

1 The polynomial $4x^3+ax^2+5x+b$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x+1)$ is a factor of $p(x)$. When $p(x)$ is divided by $(x-4)$ the remainder is equal to 3 times the remainder when $p(x)$ is divided by $(x-2)$. 3R

R

Find the values of a and b. [5]

$(2x+1)$ is a factor, so $p\left(-\frac{1}{2}\right)=0$:

$$4\left(-\frac{1}{2}\right)^3+a\left(-\frac{1}{2}\right)^2+5\left(-\frac{1}{2}\right)+b=0$$

$$4\times-\frac{1}{8}+a\times\frac{1}{4}-\frac{5}{2}+b=0$$

$$-\frac{1}{2}+\frac{a}{4}-\frac{5}{2}+b=0$$

$$\frac{a}{4}+b=3$$

$$\times 4 \quad \times 4 \quad \times 4$$

$$a+4b=12 \quad (1)$$

When $p(x)$ is divided by $(x-4)$.... so $p(4)=3R$: ← remainder

$$4(4)^3+a(4)^2+5(4)+b=3R$$

$$256+16a+20+b=3R$$

$$16a+b=3R-276 \quad (2)$$

When $p(x)$ is divided by $(x-2)$.... so $p(2)=R$:

$$4(2)^3+a(2)^2+5(2)+b=R$$

$$32+4a+10+b=R$$

$$4a+b=R-42 \quad \times 3$$

$$12a+3b=3R-126 \quad (3)$$

(2) − (3): $4a-2b=-150 \quad \times 2$

$$8a-4b=-300 \quad (4)$$

(1) + (4): $9a=-288$

$$a=-32$$

→ (1): $a+4b=12$

$$-32+4b=12$$

$$4b=44$$

$$b=11$$

2 The polynomial $2x^3 - x^2 + a$, where a is a constant, is denoted by $p(x)$. It is given that $(2x + 3)$ is a factor of $p(x)$.

(a) Find the value of a. [2]

$p\left(\frac{-3}{2}\right) = 0: \quad 2\left(\frac{-3}{2}\right)^3 - \left(\frac{-3}{2}\right)^2 + a = 0$

$2\left(\frac{-27}{8}\right) - \left(\frac{9}{4}\right) + a = 0$

$\frac{-27}{4} - \frac{9}{4} + a = 0$

$\frac{-36}{4} + a = 0$

$-9 + a = 0$

$a = 9$

(b) When a has this value, solve the inequality $p(x) < 0$. [4]

$p(x) < 0: \quad 2x^3 - x^2 + 9 < 0$

factorise:

$$\begin{array}{r|l} & x^2 - 2x + 3 \\ 2x+3 & 2x^3 - x^2 + 0x + 9 \\ & \underline{2x^3 + 3x^2} \\ & -4x^2 + 0x \\ & \underline{-4x^2 - 6x} \\ & 6x + 9 \\ & \underline{6x + 9} \end{array}$$

$\rightarrow (2x+3)(x^2 - 2x + 3) < 0$

$b^2 - 4ac: \quad (-2)^2 - 4(1)(3)$

$= 4 - 12$

$= -8$

$b^2 - 4ac < 0$ so no real roots

required

critical value: $x = \frac{-3}{2}$:

try: $x < \frac{-3}{2}$ (eg -2) | $x > \frac{-3}{2}$ (eg 0)

$2(-2)^3 - (-2)^2 + 9 = -11$ | $2(0)^3 - (0)^2 + 9 = 9$

$-11 < 0$ ✓ | $9 \nless 0$ ✗

so $x < \frac{-3}{2}$

9 The polynomial $6x^3 + ax^2 + bx + 9$ is denoted by $p(x)$, where a and b are constants. It is given that $(x-3)$ is a factor of $p(x)$, and when the first derivative $p'(x)$ is divided by $(x-3)$ the remainder is 72.

(a) Find the values of a and b. [5]

$(x-3)$ is a factor : $p(3) = 0$

$6(3)^3 + a(3)^2 + b(3) + 9 = 0$

$162 + 9a + 3b + 9 = 0$

$9a + 3b = -171$ ①

differentiate $p(x)$:

$p'(x) = 18x^2 + 2ax + b$

when $\div(x-3)$, remainder $= 72$: $p'(3) = 72$:

$18(3)^2 + 2a(3) + b = 72$

$162 + 6a + b = 72$

$6a + b = -90$ $\times 3$

$18a + 3b = -270$ ②

② − ① : $9a = -99$

$\underline{a = -11}$

Sub. into ①: $9(-11) + 3b = -171$

$-99 + 3b = -171$

$3b = -72$

$\underline{b = -24}$

(b) When a and b have the values found in part **(a)**, factorise $\mathrm{p}(x)$ completely. [3]

$p(x) = 6x^3 - 11x^2 - 24x + 9$

$$\begin{array}{r|l} & 6x^2 + 7x - 3 \\ x-3 & 6x^3 - 11x^2 - 24x + 9 \\ & \underline{6x^3 - 18x^2} \\ & \quad 7x^2 - 24x \\ & \quad \underline{7x^2 - 21x} \\ & \qquad -3x + 9 \\ & \qquad \underline{-3x + 9} \end{array}$$

$\rightarrow (x-3)(\overset{a}{6x^2} + \overset{b}{7x} - \overset{c}{3})$

$\rightarrow ac = -18$, two numbers: $9, -2$

$6x^2 + 9x - 2x - 3$

$3x(2x+3) - 1(2x+3)$

$(3x-1)(2x+3)$

$\rightarrow \underline{(x-3)(3x-1)(2x+3)}$

(c) Hence solve the inequality $\mathrm{p}(x) < 0$. [2]

Critical values: $(x-3)(3x-1)(2x+3) = 0$

$x = 3 \quad x = \frac{1}{3} \quad x = -\frac{3}{2}$

$x < -\frac{3}{2}$	$-\frac{3}{2} < x < \frac{1}{3}$	$\frac{1}{3} < x < 3$	$x > 3$
e.g. -2	e.g. 0	e.g. 2	e.g. 4
$p(-2) = -35$	$p(0) = 9$	$p(2) = -35$	$p(4) = 121$
$-35 < 0$	$9 \not< 0$	$-35 < 0$	$121 \not< 0$
✓	✗	✓	✗

$\underline{x < \frac{-3}{2}}$ or $\underline{\frac{1}{3} < x < 3}$

3 The polynomial $2x^4 + ax^3 + bx - 1$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $x^2 - x + 1$ the remainder is $3x + 2$.

Find the values of a and b. [5]

$$
\begin{array}{r|lllll}
 & 2x^2 & +(a+2)x & +a & & \\
\hline
x^2-x+1 & 2x^4 & +ax^3 & +0x^2 & +bx & -1 \\
 & 2x^4 & -2x^3 & +2x^2 & & \\
\hline
 & & (a+2)x^3 & -2x^2 & +bx & \\
 & & (a+2)x^3 & -(a+2)x^2 & +(a+2)x & \\
\hline
 & & & ax^2 & +(b-a-2)x & -1 \\
 & & & ax^2 & -ax & +a \\
\hline
 & & & & (b-2)x & -1-a
\end{array}
$$

$\rightarrow (b-2)x - 1 - a = 3x + 2$

Compare coeffs: $b - 2 = 3$ $\qquad -1 - a = 2$

$\underline{b = 5}$ $\qquad -1 = 2 + a$

$\underline{a = -3}$

Alternative Method:

$$\frac{2x^4 + ax^3 + bx - 1}{x^2 - x + 1} \equiv \alpha x^2 + \beta x + \gamma + \frac{3x + 2}{x^2 - x + 1}$$

$$2x^4 + ax^3 + bx - 1 \equiv (\alpha x^2 + \beta x + \gamma)(x^2 - x + 1) + 3x + 2$$

$$2x^4 + ax^3 + bx - 1 \equiv \alpha x^4 - \alpha x^3 + \alpha x^2 + \beta x^3 - \beta x^2 + \beta x + \gamma x^2 - \gamma x + \gamma + 3x + 2$$

$$\equiv \alpha x^4 + (\beta - \alpha)x^3 + (\alpha - \beta + \gamma)x^2 + (3 + \beta - \gamma)x + \gamma + 2$$

Compare coeffs:

x^4: $\underline{2 = \alpha}$ $\qquad$ const: $-1 = \gamma + 2$ $\qquad$ x^2: $0 = \alpha - \beta + \gamma$

$\underline{\gamma = -3}$ $\qquad 0 = 2 - \beta - 3$

$\underline{\beta = -1}$

x^3: $a = \beta - \alpha$ $\qquad$ x: $b = 3 + \beta - \gamma$

$= -1 - 2$ $\qquad = 3 + -1 - -3$

$\underline{a = -3}$ $\qquad = 3 - 1 + 3$

$\rightarrow \underline{b = 5}$