

- 4 The equation of a curve is $y = f(x)$, where $f(x) = (2x - 1)\sqrt{3x - 2} - 2$. The following points lie on the curve. Non-exact values have been given correct to 5 decimal places.

$A(2, 4)$, $B(2.0001, k)$, $C(2.001, 4.00625)$, $D(2.01, 4.06261)$, $E(2.1, 4.63566)$, $F(3, 11.22876)$

- (a) Find the value of k . Give your answer correct to 5 decimal places. [1]

sub. $x = 2.0001$: $(2 \times 2.0001 - 1) \cdot \sqrt{3(2.0001) - 2} - 2$
 $= 4.00063$

The table shows the gradients of the chords AB , AC , AD and AF .

Chord	AB	AC	AD	AE	AF
Gradient of chord	6.2501	6.2511	6.2608		7.2288

- (b) Find the gradient of the chord AE . Give your answer correct to 4 decimal places. [1]

$m = \frac{4.63566 - 4}{2.1 - 2}$
 $= 6.3566$

- (c) Deduce the value of $f'(2)$ using the values in the table. [1]

What is the gradient becoming as the chords are progressively getting shorter?
 AB is the shortest chord so the gradient appears to be getting closer to 6.25
 (check using calculator)

- 3 The equation of a curve is $y = (x - 3)\sqrt{x + 1} + 3$. The following points lie on the curve. Non-exact values are rounded to 4 decimal places.

A (2, k) B (2.9, 2.8025) C (2.99, 2.9800) D (2.999, 2.9980) E (3, 3)

- (a) Find k , giving your answer correct to 4 decimal places. [1]

$$\begin{aligned} \text{Sub. } x=2: \quad y &= (2-3)\sqrt{2+1} + 3 \\ &= (-1)\sqrt{3} + 3 \\ &= -\sqrt{3} + 3 \\ &= \underline{\underline{1.2679}} \end{aligned}$$

- (b) Find the gradient of AE, giving your answer correct to 4 decimal places. [1]

$$m = \frac{3 - 1.2679}{3 - 2}$$

$$m = \underline{\underline{1.7321}}$$

The gradients of BE, CE and DE, rounded to 4 decimal places, are 1.9748, 1.9975 and 1.9997 respectively.

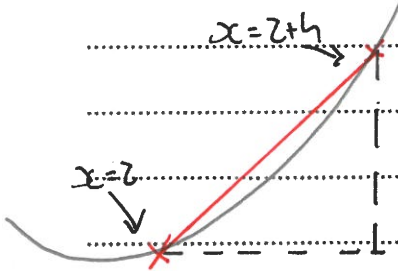
- (c) State, giving a reason for your answer, what the values of the four gradients suggest about the gradient of the curve at the point E. [2]

Gradient of curve at E = 2

The gradient of the chords is getting closer to 2 as the chords get shorter, so as chord $\rightarrow 0$ gradient $\rightarrow 2$.

- 3 The equation of a curve is $y = 2x^2 - 3$. Two points A and B with x -coordinates 2 and $(2+h)$ respectively lie on the curve.

(a) Find and simplify an expression for the gradient of the chord AB in terms of h . [3]



Find y -coords:

$x=2: y = 2(2)^2 - 3$
 $= 8 - 3 = 5 \rightarrow (2, 5)$

$x=2+h: y = 2(2+h)^2 - 3$
 $= 2(4 + 4h + h^2) - 3$
 $= 8 + 8h + 2h^2 - 3$
 $= 2h^2 + 8h + 5 \rightarrow (2+h, 2h^2 + 8h + 5)$

Gradient $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{2h^2 + 8h + 5 - 5}{2+h - 2} = \frac{2h^2 + 8h}{h} = 2h + 8$

- (b) Explain how the gradient of the curve at the point A can be deduced from the answer to part (a), and state the value of this gradient. [2]

To find the gradient at A , imagine making the chord as small as possible, so h becomes closer and closer to zero.

When $h \rightarrow 0$, gradient $= 2(0) + 8$
 $= \underline{\underline{8}}$



- 2 The curve $y = x^2 - \frac{a}{x}$ has a stationary point at $(-3, b)$.

Find the values of the constants a and b .

[4]

$$y = x^2 - ax^{-1}$$

$$\frac{dy}{dx} = 2x + ax^{-2}$$

$$\text{SP: } \frac{dy}{dx} = 0: 2x + \frac{a}{x^2} = 0$$

$$x = -3: 2(-3) + \frac{a}{(-3)^2} = 0$$

$$-6 + \frac{a}{9} = 0$$

$$\frac{a}{9} = 6$$

$$\underline{a = 54}$$

$$\text{Eqn: } y = x^2 - \frac{54}{x}$$

sub. $(-3, b)$:

$$b = (-3)^2 - \frac{54}{-3}$$

$$= 9 - -18$$

$$\underline{b = 27}$$

- 11 The gradient of a curve is given by $\frac{dy}{dx} = 6(3x - 5)^3 - kx^2$, where k is a constant. The curve has a stationary point at $(2, -3.5)$.

(a) Find the value of k .

[2]

SP: $\frac{dy}{dx} = 0$: $6(3x - 5)^3 - kx^2 = 0$

$x=2$: $6(3(2) - 5)^3 - k(2)^2 = 0$

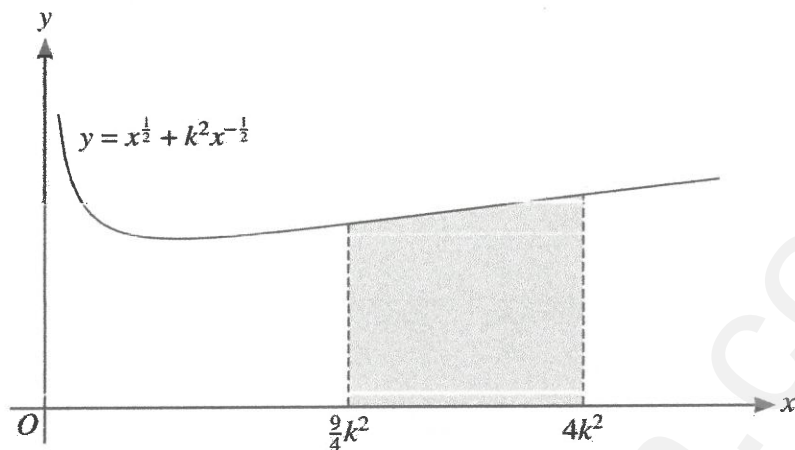
$6(1)^3 - 4k = 0$

$6 - 4k = 0$

$4k = 6$

$k = \frac{3}{2}$

11



The diagram shows part of the curve with equation $y = x^{\frac{1}{2}} + k^2 x^{-\frac{1}{2}}$, where k is a positive constant.

- (a) Find the coordinates of the minimum point of the curve, giving your answer in terms of k . [4]

$$y = x^{\frac{1}{2}} + k^2 x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}k^2 x^{-\frac{3}{2}}$$

$$\text{SP: } \frac{dy}{dx} = 0: \quad \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}k^2 x^{-\frac{3}{2}} = 0$$

$$x^{-\frac{1}{2}} - k^2 x^{-\frac{3}{2}} = 0$$

$$x^{-\frac{1}{2}} = k^2 x^{-\frac{3}{2}}$$

$$\frac{1}{x^{\frac{1}{2}}} = \frac{k^2}{x^{\frac{3}{2}}}$$

$$x^{\frac{3}{2}} = k^2 x^{\frac{1}{2}}$$

$$x^{\frac{3}{2}} - k^2 x^{\frac{1}{2}} = 0$$

$$x^{\frac{1}{2}}(x - k^2) = 0$$

$$x^{\frac{1}{2}} = 0 \quad \text{or} \quad x - k^2 = 0$$

not the point we want
- can see from graph

$$x = k^2$$

$$\text{Sub. } x = k^2: \quad y = (k^2)^{\frac{1}{2}} + k^2 (k^2)^{-\frac{1}{2}}$$

$$= k^1 + k^2 \times k^{-1}$$

$$= k + k$$

$$= 2k$$

$$\rightarrow (k^2, 2k)$$

- 10 At the point $(4, -1)$ on a curve, the gradient of the curve is $-\frac{3}{2}$. It is given that $\frac{dy}{dx} = x^{-\frac{1}{2}} + k$, where k is a constant.

(c) Find the coordinates of the stationary point. [3]

$$\frac{dy}{dx} = -\frac{3}{2} \text{ when } x=4: -\frac{3}{2} = 4^{-\frac{1}{2}} + k$$

$$-\frac{3}{2} = \frac{1}{2} + k$$

$$k = -2$$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} - 2$$

$$\text{sp: } \frac{dy}{dx} = 0: x^{-\frac{1}{2}} - 2 = 0$$

$$x^{-\frac{1}{2}} = 2$$

$$\frac{1}{\sqrt{x}} = 2$$

$$\frac{1}{x} = 4$$

$$x = \frac{1}{4}$$

from (b), equation of curve: $y = 2x^{\frac{1}{2}} - 2x + 3$

$$\text{sub. } x = \frac{1}{4}: y = 2\left(\frac{1}{4}\right)^{\frac{1}{2}} - 2\left(\frac{1}{4}\right) + 3$$

$$y = \frac{7}{2}$$

$$\left(\frac{1}{4}, \frac{7}{2}\right)$$

(d) Determine the nature of the stationary point. [2]

$$\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$\text{sub. } x = \frac{1}{4}: \frac{d^2y}{dx^2} = -\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{3}{2}}$$

$$= -4$$

$$\frac{d^2y}{dx^2} < 0, \text{ so } \underline{\text{maximum point}}$$

- 9 The equation of a curve is $y = (3 - 2x)^3 + 24x$.

(a) Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

[4]

$$\frac{dy}{dx} = 3 \times -2(3-2x)^2 + 24$$

$$= -6(3-2x)^2 + 24$$

$$\frac{d^2y}{dx^2} = -12 \times -2(3-2x)^1$$

$$= 24(3-2x)$$

(b) Find the coordinates of each of the stationary points on the curve.

[3]

$$\text{SP: } \frac{dy}{dx} = 0: -6(3-2x)^2 + 24 = 0$$

$$6(3-2x)^2 = 24$$

$$(3-2x)^2 = 4$$

$$3-2x = \pm\sqrt{4}$$

$$3-2x = \pm 2$$

$$-2x = -3 \pm 2$$

$$2x = 3 \pm 2$$

$$x = \frac{3 \pm 2}{2}$$

$$x = \frac{5}{2} \quad \text{or} \quad x = \frac{1}{2}$$

$$\text{sub. } x = \frac{5}{2}:$$

$$y = (3-2 \times \frac{5}{2})^3 + 24(\frac{5}{2})$$

$$= (-2)^3 + 60$$

$$= -8 + 60$$

$$= 52 \rightarrow \underline{\underline{(\frac{5}{2}, 52)}}$$

$$\text{sub. } x = \frac{1}{2}:$$

$$y = (3-2 \times \frac{1}{2})^3 + 24(\frac{1}{2})$$

$$= (2)^3 + 12$$

$$= 8 + 12$$

$$= 20 \rightarrow \underline{\underline{(\frac{1}{2}, 20)}}$$

(c) Determine the nature of each stationary point.

[2]

$$\frac{d^2y}{dx^2} = 24(3-2x)$$

$$\text{sub. } x = \frac{5}{2}:$$

$$24(3-2 \times \frac{5}{2}) = -48$$

$$\frac{d^2y}{dx^2} < 0, \text{ so } \underline{\underline{(\frac{5}{2}, 52) \text{ is a maximum}}}$$

$$\text{sub. } x = \frac{1}{2}:$$

$$24(3-2 \times \frac{1}{2}) = 48$$

$$\frac{d^2y}{dx^2} > 0, \text{ so } \underline{\underline{(\frac{1}{2}, 20) \text{ is a minimum}}}$$

- 9 The equation of a curve is $y = 3x + 1 - 4(3x + 1)^{\frac{1}{2}}$ for $x > -\frac{1}{3}$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

[3]

$$y = 3x + 1 - 4(3x + 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 3 - 2 \times 4(3x + 1)^{-\frac{1}{2}}$$

$$= 3 - 8(3x + 1)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 3 \times 4(3x + 1)^{-\frac{3}{2}}$$

$$= 12(3x + 1)^{-\frac{3}{2}}$$

(b) Find the coordinates of the stationary point of the curve and determine its nature. [4]

SP: $\frac{dy}{dx} = 0$: $3 - 6(3x+1)^{-\frac{1}{2}} = 0$

$$6(3x+1)^{-\frac{1}{2}} = 3$$

$$2(3x+1)^{-\frac{1}{2}} = 1$$

$$(3x+1)^{-\frac{1}{2}} = \frac{1}{2}$$

$$\frac{1}{\sqrt{3x+1}} = \frac{1}{2}$$

$$2 = \sqrt{3x+1}$$

$$4 = 3x+1$$

$$3 = 3x$$

$$1 = x$$

$$x = 1$$

Sub. $x=1$ into y :

$$y = 3(1) + 1 - 4(3(1)+1)^{\frac{1}{2}}$$

$$= 3 + 1 - 4(4)^{\frac{1}{2}}$$

$$= 4 - 4 \times 2$$

$$= 4 - 8$$

$$= -4 \rightarrow \underline{(1, -4)}$$

Sub. $x=1$ into $\frac{d^2y}{dx^2}$:

$$9(3(1)+1)^{-\frac{3}{2}}$$

$$= 9(4)^{-\frac{3}{2}}$$

$$= \frac{9}{8}$$

$\frac{d^2y}{dx^2} > 0$, so minimum point

- 5 The equation of a curve is $y = 2x^2 - \frac{1}{2x} + 3$.

(a) Find the coordinates of the stationary point. [3]

$$y = 2x^2 - \frac{1}{2}x^{-1} + 3$$

$$\frac{dy}{dx} = 4x + \frac{1}{2}x^{-2}$$

SP: $\frac{dy}{dx} = 0: 4x + \frac{1}{2}x^{-2} = 0$

$$4x + \frac{1}{2x^2} = 0$$

$$\begin{array}{ccc} \times 2x^2 & \times 2x^2 & \times 2x^2 \\ 8x^3 + 1 = 0 & & \end{array}$$

$$8x^3 = -1$$

$$x^3 = -\frac{1}{8}$$

$$x = -\frac{1}{2}$$

$$y = 2\left(-\frac{1}{2}\right)^2 - \frac{1}{2\left(-\frac{1}{2}\right)} + 3$$

$$= 2\left(\frac{1}{4}\right) - \frac{1}{-1} + 3$$

$$= \frac{1}{2} + 1 + 3$$

$$= \frac{9}{2}$$

$$\left(-\frac{1}{2}, \frac{9}{2}\right)$$

(b) Determine the nature of the stationary point. [2]

$$\frac{d^2y}{dx^2} = 4 - x^{-3}$$

Sub. $x = -\frac{1}{2}: 4 - \left(-\frac{1}{2}\right)^{-3}$

$$= 4 - (-2)^3$$

$$= 4 - 8$$

$$= -4$$

$\frac{d^2y}{dx^2} < 0$, so maximum point

(c) For positive values of x , determine whether the curve shows a function that is increasing, decreasing or neither. Give a reason for your answer. [2]

$$\frac{dy}{dx} = 4x + \frac{1}{2x^2}$$

When x is positive, both terms in this expression are positive, so $\frac{dy}{dx}$ is always positive, so it's an increasing function.

- 10 The gradient of a curve at the point (x, y) is given by $\frac{dy}{dx} = 2(x+3)^{\frac{1}{2}} - x$. The curve has a stationary point at $(a, 14)$, where a is a positive constant.

(a) Find the value of a .

[3]

$$\text{SP: } \frac{dy}{dx} = 0: \quad 2(x+3)^{\frac{1}{2}} - x = 0$$

$$2(x+3)^{\frac{1}{2}} = x$$

$$(x+3)^{\frac{1}{2}} = \frac{x}{2}$$

$$\sqrt{x+3} = \frac{x}{2}$$

$$x+3 = \frac{x^2}{4}$$

$$4x+12 = x^2$$

$$x^2 - 4x - 12 = 0$$

$$(x-6)(x+2) = 0$$

$$\underline{x=6} \quad \text{or} \quad x=-2$$

a is positive

$$\rightarrow \underline{a=6}$$

(b) Determine the nature of the stationary point.

[3]

$$\frac{d^2y}{dx^2} = (x+3)^{-\frac{1}{2}} - 1$$

$$\text{Sub. } x=6: (6+3)^{-\frac{1}{2}} - 1$$

$$= 9^{-\frac{1}{2}} - 1$$

$$= \frac{1}{3} - 1$$

$$= -\frac{2}{3}$$

$$\frac{d^2y}{dx^2} < 0, \text{ so } \underline{\underline{\text{maximum point}}}$$

- 11 The equation of a curve is

$$y = k\sqrt{4x+1} - x + 5,$$

where k is a positive constant.

- (a) Find $\frac{dy}{dx}$.

[2]

$$\begin{aligned} y &= k(4x+1)^{\frac{1}{2}} - x + 5 \\ \frac{dy}{dx} &= \frac{1}{2}k \times 4(4x+1)^{-\frac{1}{2}} - 1 \\ &= 2k(4x+1)^{-\frac{1}{2}} - 1 \end{aligned}$$

- (b) Find the x -coordinate of the stationary point in terms of k .

[2]

$$\begin{aligned} \text{SP: } \frac{dy}{dx} &= 0: \quad 2k(4x+1)^{-\frac{1}{2}} - 1 = 0 \\ 2k(4x+1)^{-\frac{1}{2}} &= 1 \\ (4x+1)^{-\frac{1}{2}} &= \frac{1}{2k} \\ \frac{1}{\sqrt{4x+1}} &= \frac{1}{2k} \\ 2k &= \sqrt{4x+1} \\ 4k^2 &= 4x+1 \\ 4k^2 - 1 &= 4x \\ \frac{4k^2 - 1}{4} &= x \\ x &= \frac{4k^2 - 1}{4} \end{aligned}$$

- 8 The equation of a curve is such that $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$. The curve passes through the point (3, 5).

(b) Find the x -coordinate of the stationary point.

[2]

$$\begin{aligned}\frac{dy}{dx} &= 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} \\ \text{SP: } \frac{dy}{dx} &= 0: \quad 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} = 0 \\ 3x^{\frac{1}{2}} &= 3x^{-\frac{1}{2}} \\ 3\sqrt{x} &= \frac{3}{\sqrt{x}} \\ \times \sqrt{x} \quad \times \sqrt{x} \\ 3x &= 3 \\ x &= 1\end{aligned}$$

(c) State the set of values of x for which y increases as x increases.

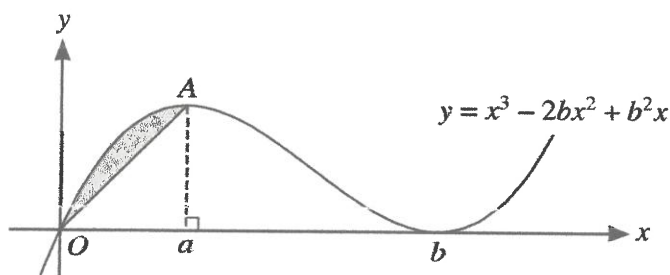
[1]

If the minimum point is at $x = 1$, this means that to the left of this point, y increases but x decreases, and to the right, y increases as x increases.

So $x > 1$

x decreases $x = 1$ x increases 9709/12/01/22

11



The diagram shows part of the curve with equation $y = x^3 - 2bx^2 + b^2x$ and the line OA , where A is the maximum point on the curve. The x -coordinate of A is a and the curve has a minimum point at $(b, 0)$, where a and b are positive constants.

(a) Show that $b = 3a$.

[4]

$$y = x^3 - 2bx^2 + b^2x$$

$$\frac{dy}{dx} = 3x^2 - 4bx + b^2$$

SP: $\frac{dy}{dx} = 0$:

$$3x^2 - 4bx + b^2 = 0$$

Factorise: $ac = 3b^2$

two numbers that multiply to $3b^2$ and sum to $-4b$:
 $-3b$ and $-b$

split the middle:

$$3x^2 - 3bx - bx + b^2 = 0$$

$$3x(x - b) - b(x - b) = 0$$

$$(3x - b)(x - b) = 0$$

$$3x - b = 0 \quad \text{or} \quad x - b = 0$$

$$x = \frac{b}{3} \quad \text{or} \quad x = b$$

$x = a$ on graph

$x = b$ on graph

$$a = \frac{b}{3}$$

$$\underline{b = 3a}$$

11 It is given that a curve has equation $y = k(3x - k)^{-1} + 3x$, where k is a constant.

(a) Find, in terms of k , the values of x at which there is a stationary point.

[4]

$$y = k(3x - k)^{-1} + 3x$$

$$\frac{dy}{dx} = -k \times 3(3x - k)^{-2} + 3$$

$$= -3k(3x - k)^{-2} + 3$$

SP: $\frac{dy}{dx} = 0$:

$$-3k(3x - k)^{-2} + 3 = 0$$

$$3k(3x - k)^{-2} = 3$$

$$k(3x - k)^{-2} = 1$$

$$\frac{k}{(3x - k)^2} = 1$$

$$k = (3x - k)^2$$

$$(3x - k)^2 = k$$

$$3x - k = \pm\sqrt{k}$$

$$3x = k \pm \sqrt{k}$$

$$x = \frac{k \pm \sqrt{k}}{3}$$

The function f has a stationary value at $x = a$ and is defined by

$$f(x) = 4(3x - 4)^{-1} + 3x \quad \text{for } x \geq \frac{3}{2}.$$

(b) Find the value of a and determine the nature of the stationary value.

[3]

from (a): $x = \frac{k \pm \sqrt{k}}{3}$

$k = 4$: $x = \frac{4 \pm \sqrt{4}}{3}$

$x = \frac{6}{3}$ or $x = \frac{2}{3}$

$x = 2$

$x \geq \frac{3}{2}$

so $a = 2$

$f'(x) = -4 \times 3(3x - 4)^{-2} + 3$ → sub. $x = 2$:

$= -12(3x - 4)^{-2} + 3$

$f''(x) = 72(2)^{-3}$

$f''(x) = 24 \times 3(3x - 4)^{-3}$

$= 9$

$= 72(3x - 4)^{-3}$

$f''(x) > 0$, so minimum point

(c) The function g is defined by $g(x) = -(3x + 1)^{-1} + 3x$ for $x \geq 0$.

Determine, making your reasoning clear, whether g is an increasing function, a decreasing function or neither.

[2]

$g'(x) = 3(3x + 1)^{-2} + 3$

$= \frac{3}{(3x + 1)^2} + 3$

↑ always positive because it is squared

$g'(x) > 0$, so increasing function