

- 1 Find the set of values of m for which the line with equation $y = mx - 3$ and the curve with equation $y = 2x^2 + 5$ do not meet. [3]

Solve simultaneously:

$$2x^2 + 5 = mx - 3$$

$$2x^2 - mx + 8 = 0$$

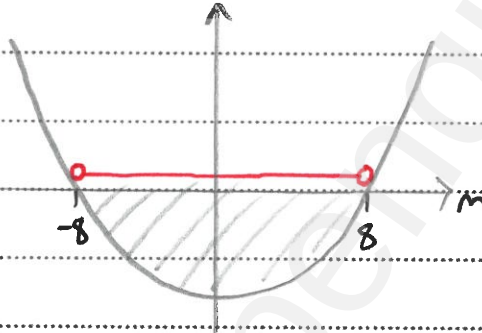
Line and curve do not meet, so $b^2 - 4ac < 0$:

$$(-m)^2 - 4(2)(8) < 0$$

$$m^2 - 64 < 0$$

$$(m + 8)(m - 8) < 0$$

Critical values: $m = -8$, $m = 8$



$$\underline{-8 < m < 8}$$

- 1 Find the set of values of m for which the line with equation $y = mx + 1$ and the curve with equation $y = 3x^2 + 2x + 4$ intersect at two distinct points. [4]

Solve simultaneously:

$$3x^2 + 2x + 4 = mx + 1$$

$$3x^2 + 2x - mx + 3 = 0$$

$$3x^2 + (2-m)x + 3 = 0$$

Intersect at two distinct points, so $b^2 - 4ac > 0$

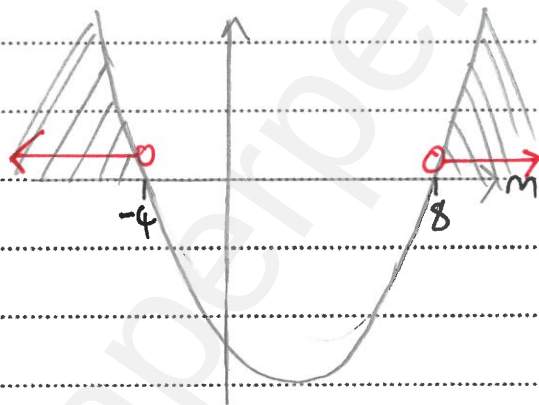
$$(2-m)^2 - 4(3)(3) > 0$$

$$4 - 4m + m^2 - 36 > 0$$

$$m^2 - 4m - 32 > 0$$

$$(m-8)(m+4) > 0$$

critical values: $m=8$, $m=-4$



$$\underline{m < -4} \quad \text{or} \quad \underline{m > 8}$$

- 6 The equation of a curve is $y = 2x^2 + kx + k - 1$, where k is a constant.

(a) Given that the line $y = 2x + 3$ is a tangent to the curve, find the value of k .

[3]

Solve simultaneously:

$$2x^2 + kx + k - 1 = 2x + 3$$

$$2x^2 + kx - 2x + k - 4 = 0$$

$$2x^2 + (k-2)x + k-4 = 0$$

Line is a tangent to curve, so $b^2 - 4ac = 0$

$$(k-2)^2 - 4(2)(k-4) = 0$$

$$k^2 - 4k + 4 - 8k + 32 = 0$$

$$k^2 - 12k + 36 = 0$$

$$(k-6)^2 = 0$$

$$\underline{k = 6}$$

It is now given that $k = 2$. $\rightarrow y = 2x^2 + 2x + 1$

- (b) Express the equation of the curve in the form $y = 2(x+a)^2 + b$, where a and b are constants, and hence state the coordinates of the vertex of the curve. [3]

$$y = 2x^2 + 2x + 1$$

$$= 2[x^2 + x] + 1$$

$$= 2\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}\right] + 1$$

$$= 2\left(x + \frac{1}{2}\right)^2 - \frac{1}{2} + 1$$

$$= 2\left(x + \frac{1}{2}\right)^2 + \frac{1}{2}$$

$\rightarrow$ Coordinates of vertex: $\underline{\underline{\left(-\frac{1}{2}, \frac{1}{2}\right)}}$

- 4 A curve has equation $y = 3x^2 - 4x + 4$ and a straight line has equation $y = mx + m - 1$, where m is a constant.

Find the set of values of m for which the curve and the line have two distinct points of intersection.

[5]

Solve simultaneously:

$$3x^2 - 4x + 4 = mx + m - 1$$

$$3x^2 - mx - 4x + 5 - m = 0$$

$$3x^2 + (-m-4)x + (5-m) = 0$$

Two points of intersection, so $b^2 - 4ac > 0$:

$$(-m-4)^2 - 4(3)(5-m) > 0$$

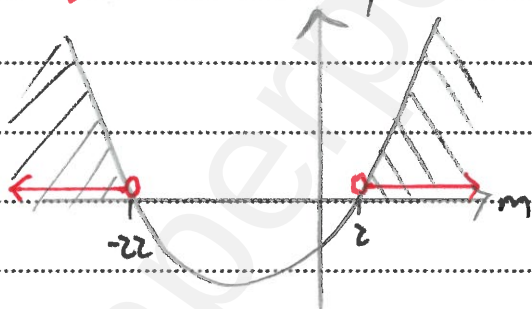
$$m^2 + 8m + 16 - 12(5-m) > 0$$

$$m^2 + 8m + 16 - 60 + 12m > 0$$

$$m^2 + 20m - 44 > 0$$

$$(m+22)(m-2) > 0$$

Critical values: $m = -22$, $m = 2$



$$\underline{m < -22} \quad \text{or} \quad \underline{m > 2}$$

- 6 The equation of a curve is $y = (2k - 3)x^2 - kx - (k - 2)$, where k is a constant. The line $y = 3x - 4$ is a tangent to the curve.

Find the value of k .

[5]

Solve simultaneously:

$$(2k - 3)x^2 - kx - (k - 2) = 3x - 4$$

$$(2k - 3)x^2 - kx - 3x - k + 2 + 4 = 0$$

$$(2k - 3)x^2 + (-k - 3)x + (-k + 6) = 0$$

Line is a tangent to the curve, so $b^2 - 4ac = 0$:

$$(-k - 3)^2 - 4(2k - 3)(-k + 6) = 0$$

$$k^2 + 6k + 9 - 4(-2k^2 + 12k + 3k - 18) = 0$$

$$k^2 + 6k + 9 - 4(-2k^2 + 15k - 18) = 0$$

$$k^2 + 6k + 9 + 8k^2 - 60k + 72 = 0$$

$$9k^2 - 54k + 81 = 0$$

$$k^2 - 6k + 9 = 0$$

$$(k - 3)^2 = 0$$

$$\underline{k = 3}$$

- 4 A line has equation $y = 3x + k$ and a curve has equation $y = x^2 + kx + 6$, where k is a constant.

Find the set of values of k for which the line and curve have two distinct points of intersection. [5]

solve simultaneously:

$$x^2 + kx + 6 = 3x + k$$

$$x^2 + kx - 3x + 6 - k = 0$$

$$x^2 + (k-3)x + 6-k = 0$$

Two distinct points of intersection, so $b^2 - 4ac > 0$:

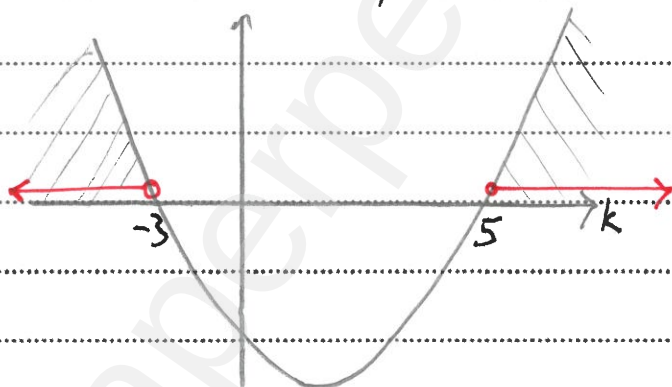
$$(k-3)^2 - 4(1)(6-k) > 0$$

$$k^2 - 6k + 9 - 24 + 4k > 0$$

$$k^2 - 2k - 15 > 0$$

$$(k-5)(k+3) > 0$$

critical values: $k=5, k=-3$



so $k < -3$, $k > 5$

- 6 A line has equation $y = 6x - c$ and a curve has equation $y = cx^2 + 2x - 3$, where c is a constant. The line is a tangent to the curve at point P .

Find the possible values of c and the corresponding coordinates of P .

[7]

Solve simultaneously:

$$cx^2 + 2x - 3 = 6x - c$$

$$cx^2 - 4x + c - 3 = 0 \quad (1)$$

Line is a tangent to the curve, so $b^2 - 4ac = 0$:

$$(-4)^2 - 4(c)(c-3) = 0$$

$$16 - 4c^2 + 12c = 0$$

$$-4c^2 + 12c + 16 = 0 \quad \div -4$$

$$c^2 - 3c - 4 = 0$$

$$(c-4)(c+1) = 0$$

$$c = 4 \quad \text{or} \quad c = -1$$

$c = 4$: sub into (1):

$$4x^2 - 4x + 1 = 0$$

$$(2x-1)(2x-1) = 0$$

$$x = \frac{1}{2}$$

Sub. into $y = 6x - c$:

$$y = 6x - 4$$

$$= 6\left(\frac{1}{2}\right) - 4$$

$$= -1$$

$c = -1$: sub into (1):

$$-x^2 - 4x - 4 = 0$$

$$x^2 + 4x + 4 = 0$$

$$(x+2)(x+2) = 0$$

$$x = -2$$

sub into $y = 6x - c$

$$y = 6x + 1$$

$$= 6(-2) + 1$$

$$= -11$$

When $c = 4$, $P: \left(\frac{1}{2}, -1\right)$

When $c = -1$, $P: (-2, -11)$

- 5 The line with equation $y = kx - k$, where k is a positive constant, is a tangent to the curve with equation $y = -\frac{1}{2x}$.

Find, in either order, the value of k and the coordinates of the point where the tangent meets the curve. [5]

Solve simultaneously:

$$kx - k = -\frac{1}{2x}$$

$$2kx^2 - 2kx = -1$$

$$2kx^2 - 2kx + 1 = 0$$

Line is a tangent, so $b^2 - 4ac = 0$:

$$(-2k)^2 - 4(2k)(1) = 0$$

$$4k^2 - 8k = 0$$

$$4k(k - 2) = 0$$

k is positive constant $\rightarrow$ $4k = 0$ or $k = 2$
 $k = 0$ ✗

Sub. $k = 2$:

$$y = 2x - 2$$

Solve simultaneously:

$$2x - 2 = -\frac{1}{2x}$$

$$4x^2 - 4x = -1$$

$$4x^2 - 4x + 1 = 0$$

$$(2x - 1)^2 = 0$$

$$2x - 1 = 0$$

$$x = \frac{1}{2}$$

Sub into $y = 2x - 2$:

$$y = 2\left(\frac{1}{2}\right) - 2$$

$$= 1 - 2$$

$$= -1$$

$$\rightarrow \left(\frac{1}{2}, -1\right)$$



- 4 Show that the curve with equation $x^2 - 3xy - 40 = 0$ and the line with equation $3x + y + k = 0$ meet for all values of the constant k . [5]

②: $y = -3x - k$

Sub into ①: $x^2 - 3x(-3x - k) - 40 = 0$

$$x^2 + 9x^2 + 3kx - 40 = 0$$

$$10x^2 + 3kx - 40 = 0$$

$$a = 10$$

$$b = 3k$$

$$c = -40$$

$b^2 - 4ac$:

$$(3k)^2 - 4(10)(-40)$$

$$9k^2 + 1600$$

↑
positive for all
values of k

↑
positive

$b^2 - 4ac$ is > 0 for all values of k ,
so the curve and the line must meet.



- 3 The equation of a curve is $y = 2x^2 + m(2x + 1)$, where m is a constant, and the equation of a line is $y = 6x + 4$.

Show that, for all values of m , the line intersects the curve at two distinct points.

[5]

Solve simultaneously:

$$2x^2 + 2mx + m = 6x + 4$$

$$2x^2 + 2mx - 6x + m - 4 = 0$$

$$2x^2 + (2m - 6)x + m - 4 = 0$$

Find $b^2 - 4ac$ and show that it is > 0 :

$$(2m - 6)^2 - 4(2)(m - 4)$$

$$4m^2 - 24m + 36 - 8(m - 4)$$

$$4m^2 - 24m + 36 - 8m + 32$$

$$4m^2 - 32m + 68$$

To show this is positive, complete the square:

$$4[m^2 - 8m] + 68$$

$$4[(m - 4)^2 - 16] + 68$$

$$4(m - 4)^2 - 64 + 68$$

$$4(m - 4)^2 + 4$$

Always positive because it's

being squared, so $b^2 - 4ac > 0$,

meaning line intersects curve at two distinct points.

- 6 Points A and B have coordinates (8, 3) and (p, q) respectively. The equation of the perpendicular bisector of AB is $y = -2x + 4$.

Find the values of p and q.

[4]

If l is the perpendicular bisector of AB, it passes through the mid-point of AB, and has gradient $= \frac{-1}{m_{AB}}$

gradient:

$$\text{gradient of } l = -2$$

$$\text{so gradient of } AB = \frac{1}{2}$$

$$\frac{1}{2} = \frac{q-3}{p-8}$$

$$p-8 = 2q-6$$

$$p = 2q + 2 \quad (1)$$

$y = -2x + 4$ passes through mid-point of AB:

$$\text{mid-point} = \left(\frac{8+p}{2}, \frac{3+q}{2} \right)$$

Sub into $y = -2x + 4$:

$$\frac{3+q}{2} = -2 \left(\frac{8+p}{2} \right) + 4$$

$$\frac{3+q}{2} = -8 - p + 4$$

$$\frac{3+q}{2} = -4 - p$$

$$3+q = -8-2p$$

$$2p = -q - 11 \quad (2)$$

$(1) \times 2$: $2p = 4q + 4 \rightarrow$ solve simultaneously with (2) :

$$4q + 4 = -q - 11$$

$$5q = -15$$

$$q = -3$$

$$\begin{aligned} \text{Sub into } (1): p &= 2(-3) + 2 \\ &= -6 + 2 \\ &= -4 \end{aligned}$$

$$\rightarrow \underline{(-4, -3)} \quad [\text{Turn over}]$$



- 9 The equation of a curve is $y = \frac{1}{2}k^2x^2 - 2kx + 2$ and the equation of a line is $y = kx + p$, where k and p are constants with $0 < k < 1$.

(a) It is given that one of the points of intersection of the curve and the line has coordinates $(\frac{5}{2}, \frac{1}{2})$.

Find the values of k and p , and find the coordinates of the other point of intersection. [7]

Sub. $(\frac{5}{2}, \frac{1}{2})$ into ①: $\frac{1}{2} = \frac{1}{2}k^2 \times (\frac{5}{2})^2 - 2k(\frac{5}{2}) + 2$

$$\frac{1}{2} = \frac{1}{2}k^2 \times \frac{25}{4} - 5k + 2$$

$$\frac{1}{2} = \frac{25}{8}k^2 - 5k + 2$$

$$4 = 25k^2 - 40k + 16$$

$$25k^2 - 40k + 12 = 0$$

ac = 300 two numbers with product 300 and sum -40 : -30, -10

$$25k^2 - 30k - 10k + 12 = 0$$

$$5k(5k - 6) - 2(5k - 6) = 0$$

$$(5k - 2)(5k - 6) = 0$$

$$k = \frac{2}{5} \text{ or } k = \frac{6}{5}$$

$$0 < k < 1$$

Sub. $(\frac{5}{2}, \frac{1}{2})$ into ②: $\frac{1}{2} = k(\frac{5}{2}) + p$

$$\uparrow k = \frac{2}{5}$$

$$\frac{1}{2} = \frac{2}{5} \times \frac{5}{2} + p$$

$$\frac{1}{2} = 1 + p$$

$$p = -\frac{1}{2}$$

eqn ①: $y = \frac{1}{2}(\frac{2}{5})^2x^2 - 2(\frac{2}{5})x + 2$

$$= \frac{1}{2} \times \frac{4}{25}x^2 - \frac{4}{5}x + 2$$

$$y = \frac{2}{25}x^2 - \frac{4}{5}x + 2$$

eqn ②: $y = \frac{2}{5}x - \frac{1}{2}$

Solve simultaneously: $\frac{2}{25}x^2 - \frac{4}{5}x + 2 = \frac{2}{5}x - \frac{1}{2}$

$$4x^2 - 40x + 100 = 20x - 25$$

$$4x^2 - 60x + 125 = 0$$

$ac = 500$. $\rightarrow$ two numbers w/ product 500 and sum -60 : $-50, -10$

$$4x^2 - 10x - 50x + 125 = 0$$

$$2x(2x - 5) - 25(2x - 5) = 0$$

$$(2x - 25)(2x - 5) = 0$$

$$x = \frac{25}{2} \checkmark \quad x = \frac{5}{2} \times \text{(already used)}$$

Sub. $x = \frac{25}{2}$ into ②:

$$y = \frac{2}{5} \times \frac{25}{2} - \frac{1}{2}$$

$$= 5 - \frac{1}{2} = \frac{9}{2}$$

$$\left(\frac{25}{2}, \frac{9}{2} \right)$$

- (b) It is given instead that the line and the curve do **not** intersect.

Find the set of possible values of p .

[3]

Solve simultaneously:

$$\frac{1}{2}k^2x^2 - 2kx + 2 = kx + p$$

$$\frac{1}{2}k^2x^2 - 3kx + 2 - p = 0$$

$$a = \frac{1}{2}k^2 \quad b = -3k \quad c = 2 - p$$

No intersection, so $b^2 - 4ac < 0$:

$$(-3k)^2 - 4\left(\frac{1}{2}k^2\right)(2 - p) < 0$$

$$9k^2 - 2k^2(2 - p) < 0$$

$$9k^2 - 4k^2 + 2k^2p < 0$$

$$5k^2 + 2k^2p < 0$$

$$2k^2p < -5k^2$$

$$p < \frac{-5k^2}{2k^2}$$

$$p < \underline{\underline{\frac{-5}{2}}}$$