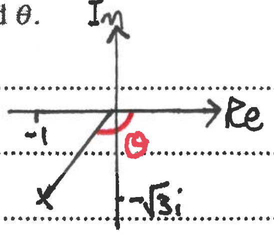
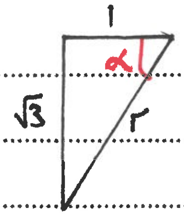
4 The complex number u is given by $u = -1 - i\sqrt{3}$.

(a) Express u in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ. [2]





$\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \frac{\pi}{3}$

$\theta = -\pi + \frac{\pi}{3}$

$= \frac{-2\pi}{3}$

$r = \sqrt{1^2 + (\sqrt{3})^2}$

$= \sqrt{4}$

$= 2$

$\rightarrow u = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$

The complex number v is given by $v = 5\left(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi\right)$.

(b) Express the complex number $\frac{v}{u}$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. [2]

$u = 2e^{-\frac{2\pi i}{3}}$ $\quad v = 5e^{\frac{\pi i}{6}}$

$\frac{v}{u} = \frac{5e^{\pi i/6}}{2e^{-\frac{2\pi i}{3}}}$

$r = \frac{5}{2}$ $\quad \theta = \frac{\pi}{6} - -\frac{2\pi}{3}$

$= \frac{5\pi}{6}$

$\rightarrow \frac{v}{u} = \frac{5}{2}e^{\frac{5\pi i}{6}}$

4 Find the complex number z satisfying the equation

$$\frac{z-3i}{z+3i} = \frac{2-9i}{5}.$$

Give your answer in the form $x+iy$, where x and y are real. [5]

$5(z-3i) = (z+3i)(2-9i)$

Sub. $z = x+iy$:

$5(x+iy-3i) = (x+iy+3i)(2-9i)$

$5x + 5iy - 15i = 2x + 2iy + 6i - 9ix + 9y + 27$

$5x + (5y-15)i = 2x + 9y + 27 + (2y - 9x + 6)i$

Real components:

$5x = 2x + 9y + 27$

$3x - 9y = 27 \quad \div 3$

$x - 3y = 9 \quad (1)$

Imaginary components:

$5y - 15 = 2y - 9x + 6$

$9x + 3y = 21 \quad (2)$

$(1)+(2)$: $10x = 30$

$x = 3$

$\rightarrow (2)$: $9(3) + 3y = 21$

$27 + 3y = 21$

$3y = -6$

$y = -2$

$\rightarrow$ $z = 3 - 2i$

5 The complex number $2 + yi$ is denoted by a, where y is a real number and $y < 0$. It is given that $f(a) = a^3 - a^2 - 2a$.

(a) Find a simplified expression for $f(a)$ in terms of y. **[3]**

$(2+iy)^3 - (2+iy)^2 - 2(2+iy)$

$(2+iy)(4+4iy-y^2) - (4+4iy-y^2) - 4 - 2iy$

$8 + 8iy - 2y^2 + 4iy - 4y^2 - iy^3 - 4 - 4iy + y^2 - 4 - 2iy$

$8 + 12iy - 2y^2 - 4y^2 - iy^3 - 8 - 6iy + y^2$

$6iy - y^2 - 4y^2 - iy^3$

$-y^2 - 4y^2 + 6iy - iy^3$

$-5y^2 + (6y - y^3)i$

(b) Given that $\mathrm{Re}(f(a)) = -20$, find $\arg a$. **[3]**

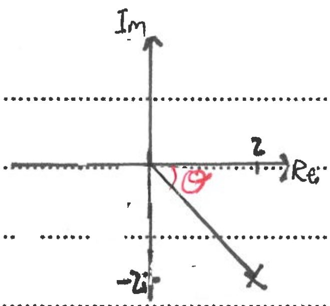
$-5y^2 = -20$

$y^2 = 4$

$y = 2$ or $y = -2$ ✓

$y < 0$

$a: 2 - 2i$



$\theta = -\tan^{-1}\left(\frac{2}{2}\right)$

$= -\frac{\pi}{4}$

4 The complex number u is defined by $u = \dfrac{3+2\mathrm{i}}{a-5\mathrm{i}}$, where a is real.

(a) Express u in the Cartesian form $x + \mathrm{i}y$, where x and y are in terms of a. [3]

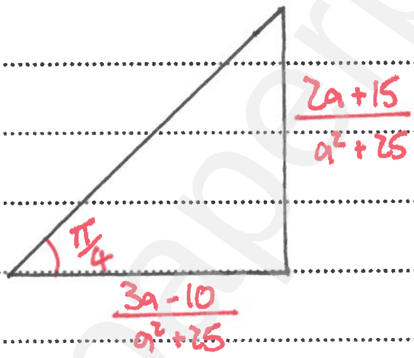
$$u = \frac{3+2i}{a-5i} \times \frac{(a+5i)}{(a+5i)}$$

$$= \frac{3a + 15i + 2ai - 10}{a^2 + 5ai - 5ai + 25}$$

$$= \frac{3a - 10 + 15i + 2ai}{a^2 + 25}$$

$$= \frac{3a-10}{a^2+25} + \frac{2a+15}{a^2+25}i$$

(b) Given that $\arg u = \frac{1}{4}\pi$, find the value of a. [2]



$$\tan\left(\frac{\pi}{4}\right) = \frac{\frac{2a+15}{a^2+25}}{\frac{3a-10}{a^2+25}}$$

$$1 = \frac{2a+15}{3a-10}$$

$$3a - 10 = 2a + 15$$

$$a - 10 = 15$$

$$a = 25$$

3 The square roots of $6-8\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $6-8\mathrm{i}$ in exact Cartesian form. [5]

$\sqrt{6-8i} = x + iy$) square both sides

$6-8i = (x+iy)^2$

$6-8i = x^2 + 2xyi - y^2$

Real component:

$6 = x^2 - y^2$ ①

Imaginary:

$-8 = 2xy$

$y = \dfrac{-4}{x}$ ②

② → ①:

$6 = x^2 - \left(\dfrac{-4}{x}\right)^2$

$6 = x^2 - \dfrac{16}{x^2}$

$\times x^2 \quad \times x^2 \quad \times x^2$

$6x^2 = x^4 - 16$

$x^4 - 6x^2 - 16 = 0$

Let $Y = x^2$:

$Y^2 - 6Y - 16 = 0$

$(Y-8)(Y+2) = 0$

$Y = 8 \qquad Y = -2$

$x^2 = 8 \qquad x^2 = -2$

× No real sol^s

$x = \pm\sqrt{8}$

$= \pm 2\sqrt{2}$

→ ②: $x = 2\sqrt{2}$:

$y = \dfrac{-4}{2\sqrt{2}}$

$= \dfrac{-2}{\sqrt{2}} = -\sqrt{2}$

→ $x = -2\sqrt{2}$:

$y = \dfrac{-4}{-2\sqrt{2}}$

$= \dfrac{2}{\sqrt{2}}$

$= \sqrt{2}$

→ $2\sqrt{2} - \sqrt{2}i$ or $-2\sqrt{2} + \sqrt{2}i$

4 Solve the equation

$$\frac{5z}{1+2i} - zz^* + 30 + 10i = 0,$$

giving your answers in the form $x + iy$, where x and y are real. [5]

Substitute $z = x + iy$ and $z^* = x - iy$:

$$\frac{5(x+iy)}{1+2i} - (x+iy)(x-iy) + 30 + 10i = 0$$

$$\frac{5x + 5iy}{1+2i} \times \frac{(1-2i)}{(1-2i)} - (x^2 + y^2) + 30 + 10i = 0$$

$$\frac{(5x+5iy)(1-2i)}{(1+2i)(1-2i)} - x^2 - y^2 + 30 + 10i = 0$$

$$\frac{5x - 10ix + 5iy + 10y}{1 - \cancel{2i} + \cancel{2i} + 4} - x^2 - y^2 + 30 + 10i = 0$$

$$\frac{5x + 10y - 10ix + 5iy}{5} - x^2 - y^2 + 30 + 10i = 0$$

$$x + 2y - 2ix + iy - x^2 - y^2 + 30 + 10i = 0$$

$$x + 2y - x^2 - y^2 + 30 + (-2x + y + 10)i = 0$$

Re: $x + 2y - x^2 - y^2 + 30 = 0$ (1) Im: $-2x + y + 10 = 0$

$y = 2x - 10$ (2)

Sub. (2) into (1):

$$x + 2(2x-10) - x^2 - (2x-10)^2 + 30 = 0$$

$$x + 4x - 20 - x^2 - (4x^2 - 40x + 100) + 30 = 0$$

$$5x + 10 - x^2 - 4x^2 + 40x - 100 = 0$$

$$-5x^2 + 45x - 90 = 0$$

$$x^2 - 9x + 18 = 0$$

$$(x-6)(x-3) = 0$$

$x = 6$ or $x = 3$

(2): $x = 6$: $y = 12 - 10 = 2$

$\rightarrow$ $z = 6 + 2i$

$x = 3$: $y = 6 - 10 = -4$

$\rightarrow$ $z = 3 - 4i$

5 **(a)** Solve the equation $z^2 - 6iz - 12 = 0$, giving the answers in the form $x + iy$, where x and y are real and exact. [3]

$$z = \frac{-(-6i) \pm \sqrt{(-6i)^2 - 4(1)(-12)}}{2(1)}$$

$$= \frac{6i \pm \sqrt{-36 + 48}}{2}$$

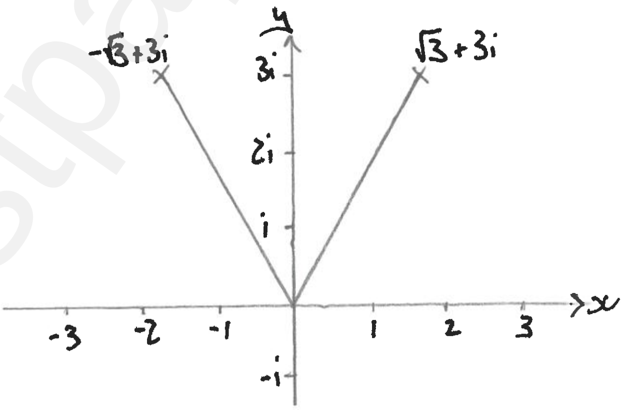
$$= \frac{6i \pm \sqrt{12}}{2}$$

$$= \frac{6i \pm 2\sqrt{3}}{2}$$

$$= 3i \pm \sqrt{3}$$

$z_1 = \sqrt{3} + 3i$ $\quad$ $z_2 = -\sqrt{3} + 3i$

(b) On a sketch of an Argand diagram with origin O, show points A and B representing the roots of the equation in part **(a)**. [1]



(c) Find the exact modulus and argument of each root. [3]

$$|z| = \sqrt{(\sqrt{3})^2 + 3^2}$$
$$= \sqrt{3 + 9}$$
$$= \sqrt{12}$$
$$= 2\sqrt{3} \text{ for both } z_1 \text{ and } z_2$$

$$\arg(z_1) = \tan^{-1}\left(\frac{3}{\sqrt{3}}\right) \qquad \arg(z_2) = \pi - \frac{\pi}{3}$$
$$= \frac{\pi}{3} \qquad\qquad = \frac{2\pi}{3}$$

(d) Hence show that the triangle *OAB* is equilateral. [1]

Distance between z_1 and z_2:

$$z_2 - z_1 = -\sqrt{3} + 3i - (\sqrt{3} + 3i)$$
$$= -2\sqrt{3}$$
$$|z_2 - z_1| = \sqrt{(-2\sqrt{3})^2}$$
$$= 2\sqrt{3}$$

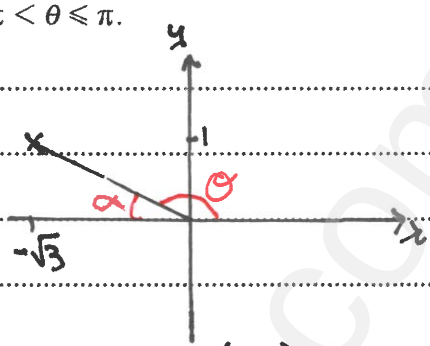
$\therefore$ All three sides of the triangle are $2\sqrt{3}$ so it is an equilateral triangle.

3 It is given that $z = -\sqrt{3} + \mathrm{i}$.

(a) Express z^2 in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

z in $re^{i\theta}$ form:

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$$



α: $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

θ: $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ $\rightarrow z = 2e^{\frac{5\pi i}{6}}$

$\rightarrow z^2 = 2^2 e^{\frac{5\pi i}{3}} = 4e^{\frac{5\pi i}{3}}$ but $-\pi < \theta < \pi$, so: $4e^{\frac{-\pi i}{3}}$

(b) The complex number ω is such that $z^2\omega$ is real and $\left|\frac{z^2}{\omega}\right| = 12$.

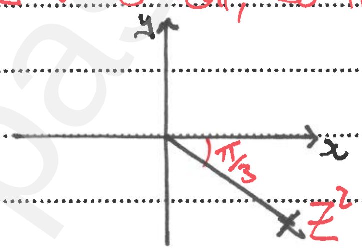
Find the two possible values of ω, giving your answers in the form $Re^{\mathrm{i}\alpha}$, where $R > 0$ and $-\pi < \alpha \leqslant \pi$. [3]

$\left|\frac{z^2}{w}\right| = 12$: $\frac{|z^2|}{|w|} = 12$

$\frac{4}{|w|} = 12$

$|w| = \frac{1}{3}$

$z^2 w$ is real, so it must lie on real axis.



$\arg(z^2) = \frac{-\pi}{3}$

$\frac{-\pi}{3} + \alpha = 0$ or $\frac{-\pi}{3} + \alpha = -\pi$

$\alpha = \frac{\pi}{3}$ $\qquad \alpha = \frac{-2\pi}{3}$

$w = \frac{1}{3}e^{\frac{\pi i}{3}}$ or $w = \frac{1}{3}e^{\frac{-2\pi i}{3}}$

6 The complex number u is defined by

$$u = \frac{7+i}{1-i}.$$

(a) Express u in the form $x + iy$, where x and y are real. [3]

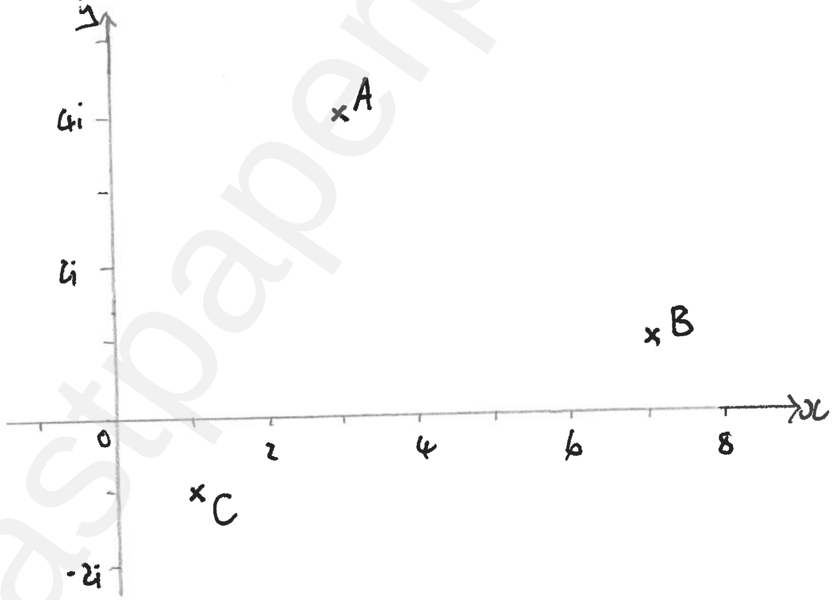
$$\frac{7+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{7+7i+i-1}{1+i-i+1}$$

$$= \frac{6+8i}{2}$$

$$= \underline{3+4i}$$

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7 + i$ and $1 - i$ respectively. [2]

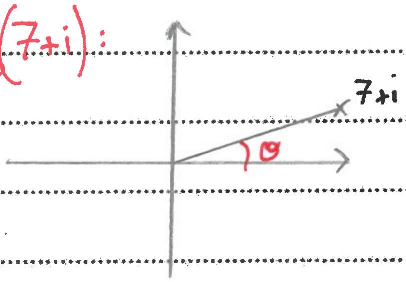


(c) By considering the arguments of $7 + \mathrm{i}$ and $1 - \mathrm{i}$, show that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = \tan^{-1}\left(\tfrac{1}{7}\right) + \tfrac{1}{4}\pi.$$

[3]

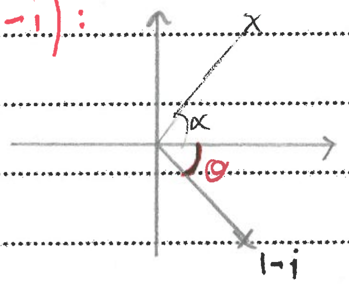
arg(7+i):



$\tan\theta = \frac{1}{7}$

$\theta = \tan^{-1}\left(\frac{1}{7}\right)$

arg(1-i):



$\tan\alpha = \frac{1}{1}$

$\alpha = \tan^{-1}(1)$

$= \frac{\pi}{4}$

$\theta = -\alpha$

$= \frac{-\pi}{4}$

from part (a): $u = \frac{7+i}{1-i}$

so $\arg(u) = \arg(7+i) - \arg(1-i)$

$\rightarrow \underbrace{\tan^{-1}\left(\frac{4}{3}\right)}_{\arg(u)} = \underbrace{\tan^{-1}\left(\frac{1}{7}\right)}_{\arg(7+i)} - \underbrace{\frac{-\pi}{4}}_{\arg(1-i)}$

so $\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{7}\right) + \frac{\pi}{4}$ QED

5 The square roots of $-4+6\sqrt{5}\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $-4+6\sqrt{5}\mathrm{i}$ in exact Cartesian form. [5]

$\sqrt{-4+6\sqrt{5}i} = x + iy$

$-4+6\sqrt{5}i = (x+iy)^2$

$-4+6\sqrt{5}i = x^2 + 2xyi - y^2$

Re: $x^2 - y^2 = -4$ ①

Im: $2xy = 6\sqrt{5}$

$y = \frac{3\sqrt{5}}{x}$ ②

Sub. ② into ①: $x^2 - \left(\frac{3\sqrt{5}}{x}\right)^2 = -4$

$x^2 - \frac{45}{x^2} = -4$ $\times x^2$

$x^4 - 45 = -4x^2$

$x^4 + 4x^2 - 45 = 0$

Let $Y = x^2$:

$Y^2 + 4Y - 45 = 0$

$(Y+9)(Y-5) = 0$

$Y = -9$ or $Y = 5$

$x^2 = -9$ $x^2 = 5$

× No real sol^ns $x = \pm\sqrt{5}$

Sub. $x = \sqrt{5}$ into ②:

$y = \frac{3\sqrt{5}}{\sqrt{5}}$

$= 3$

Sub. $x = -\sqrt{5}$ into ②:

$y = \frac{3\sqrt{5}}{-\sqrt{5}}$

$= -3$

$\rightarrow z = \sqrt{5} + 3i$ or $z = -\sqrt{5} - 3i$

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5 The complex number $3 - \mathrm{i}$ is denoted by u.

(a) Show, on an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively.

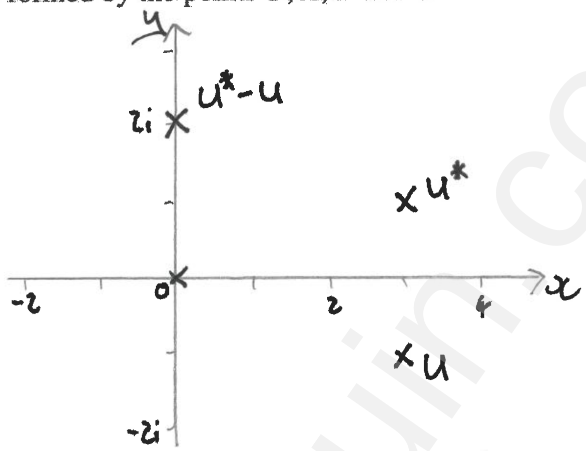
State the type of quadrilateral formed by the points O, A, B and C. [3]

$u = 3 - i$

$u^* = 3 + i$

$u^* - u = 3 + i - (3 - i)$

$= \underline{2i}$



OABC forms a parallelogram.

(b) Express $\frac{u^*}{u}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

$\frac{u^*}{u} = \frac{3+i}{3-i} \times \frac{3+i}{3+i}$

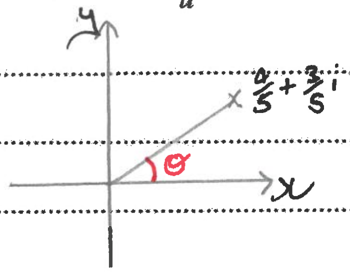
$= \frac{9 + 6i - 1}{9 + 3i - 3i + 1}$

$= \frac{8 + 6i}{10}$

$= \underline{\frac{4}{5} + \frac{3}{5}i}$

(c) By considering the argument of $\dfrac{u^*}{u}$, or otherwise, prove that $\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right)$. [2]

$\arg\left(\dfrac{u^*}{u}\right)$:



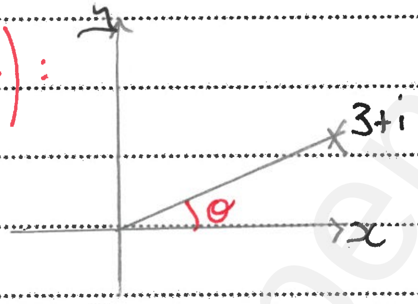
$$\tan\theta = \left(\frac{3/5}{4/5}\right)$$

$$\tan\theta = \left(\frac{3}{4}\right)$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\arg\left(\frac{u^*}{u}\right) = \arg(u^*) - \arg(u)$$

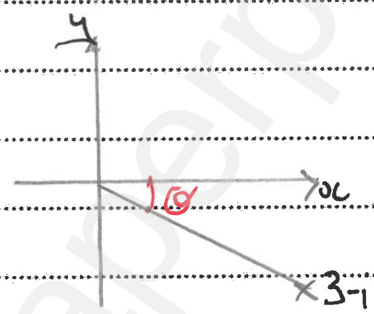
$\arg(u^*)$:



$$\tan\theta = \left(\frac{1}{3}\right)$$

$$\theta = \tan^{-1}\left(\frac{1}{3}\right)$$

$\arg(u)$:



$$\theta = -\tan^{-1}\left(\frac{1}{3}\right)$$

$$\rightarrow \arg\left(\frac{u^*}{u}\right) = \arg(u^*) - \arg(u)$$

$$\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{1}{3}\right) - -\tan^{-1}\left(\frac{1}{3}\right)$$

$$\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right) \quad \text{QED}$$

8 The complex numbers u and v are defined by $u = -4 + 2\mathrm{i}$ and $v = 3 + \mathrm{i}$.

(a) Find $\frac{u}{v}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

$$\frac{-4+2i}{3+i} \times \frac{(3-i)}{(3-i)}$$

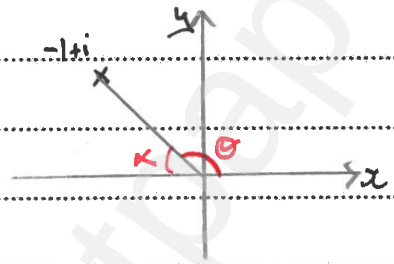
$$= \frac{-12+4i+6i+2}{9-3i+3i+1}$$

$$= \frac{-10+10i}{10}$$

$$= -1+i$$

(b) Hence express $\frac{u}{v}$ in the form $re^{\mathrm{i}\theta}$, where r and θ are exact. [2]

$$r = \sqrt{(-1)^2 + 1^2}$$
$$= \sqrt{2}$$



$$\tan\alpha = \frac{1}{1}$$
$$\alpha = \tan^{-1}(1)$$
$$= \frac{\pi}{4}$$
$$\theta = \pi - \alpha$$
$$= \pi - \frac{\pi}{4}$$
$$= \frac{3\pi}{4}$$

$$\rightarrow \frac{u}{v} = \sqrt{2}e^{\frac{3\pi i}{4}}$$

In an Argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $2u + v$ respectively.

(c) State fully the geometrical relationship between OA and BC. [2]

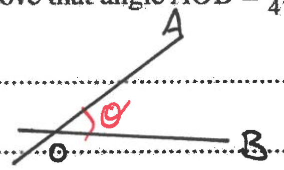
$\vec{BC} = \underline{c} - \underline{b}$ $\qquad\qquad \vec{OA} = u$

$= 2u + v - v$

$= 2u$

So BC is parallel to OA and twice the magnitude.

(d) Prove that angle $AOB = \frac{3}{4}\pi$. [2]



angle $AOB = \arg(\vec{OA}) - \arg(\vec{OB})$

$= \arg(u) - \arg(v)$

$= \arg\left(\frac{u}{v}\right)$

$= \frac{3\pi}{4}$ from part (b)

5 **(a)** The complex number u is given by

$$u = \frac{(\cos\frac{1}{7}\pi + i\sin\frac{1}{7}\pi)^4}{\cos\frac{1}{7}\pi - i\sin\frac{1}{7}\pi}.$$

Find the exact value of arg u. [2]

TOP: When multiplying complex numbers in mod-arg form, we add the arguments: $\frac{\pi}{7} + \frac{\pi}{7} + \frac{\pi}{7} + \frac{\pi}{7} = \frac{4\pi}{7}$

$\rightarrow \cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)$

Bottom: Cosine is an even function, so $\cos\left(\frac{\pi}{7}\right) = \cos\left(\frac{-\pi}{7}\right)$

Sine is an odd function, so $\sin\left(\frac{-\pi}{7}\right) = -\sin\left(\frac{\pi}{7}\right)$

so: $\cos\left(\frac{\pi}{7}\right) - i\sin\left(\frac{\pi}{7}\right) = \cos\left(\frac{-\pi}{7}\right) + i\sin\left(\frac{-\pi}{7}\right)$

$\rightarrow \dfrac{\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)}{\cos\left(\frac{-\pi}{7}\right) + i\sin\left(\frac{-\pi}{7}\right)}$ dividing, so subtract the arguments:

$\rightarrow \cos\left(\frac{5\pi}{7}\right) + i\sin\left(\frac{5\pi}{7}\right)$ $\rightarrow$ arg $= \frac{5\pi}{7}$

(b) The complex numbers u and u^* are plotted on an Argand diagram.

Describe the single geometrical transformation that maps u onto u^* and state the exact value of arg u^*. [2]

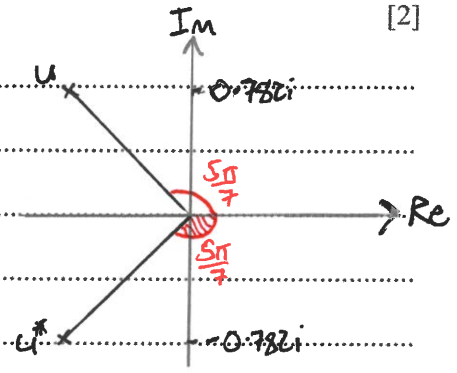
optional

$u = \cos\left(\frac{5\pi}{7}\right) + i\sin\left(\frac{5\pi}{7}\right)$

$= -0.623 + 0.782i$

$u^* = \cos\left(\frac{5\pi}{7}\right) - i\sin\left(\frac{5\pi}{7}\right)$

$= -0.623 - 0.782i$



$\rightarrow$ u is mapped onto u^* by a reflection in the Real axis.

arg $u^* = \frac{-5\pi}{7}$

9 The complex numbers z and ω are defined by $z = 1 - \mathrm{i}$ and $\omega = -3 + 3\sqrt{3}\mathrm{i}$.

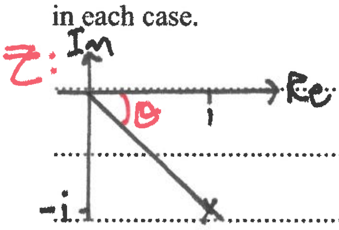
(a) Express $z\omega$ in the form $a + b\mathrm{i}$, where a and b are real and in exact surd form. [1]

$zw = (1 - i)(-3 + 3\sqrt{3}i)$

$= -3 + 3\sqrt{3}i + 3i + 3\sqrt{3}$

$= \underline{(3\sqrt{3} - 3) + (3\sqrt{3} + 3)i}$

(b) Express z and ω in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ in each case. [4]



$r = \sqrt{1^2 + (-1)^2}$

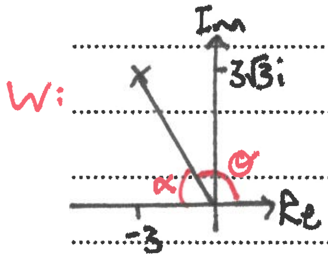
$= \sqrt{2}$

$\tan\theta = \frac{1}{1}$

$\theta = \frac{\pi}{4}$

below x-axis, so $\theta = \frac{-\pi}{4}$

$\underline{z = \sqrt{2}e^{\frac{-\pi i}{4}}}$



$r = \sqrt{(-3)^2 + (3\sqrt{3})^2}$

$= \sqrt{9 + 27}$

$= 6$

$\tan\alpha = \frac{3\sqrt{3}}{3}$

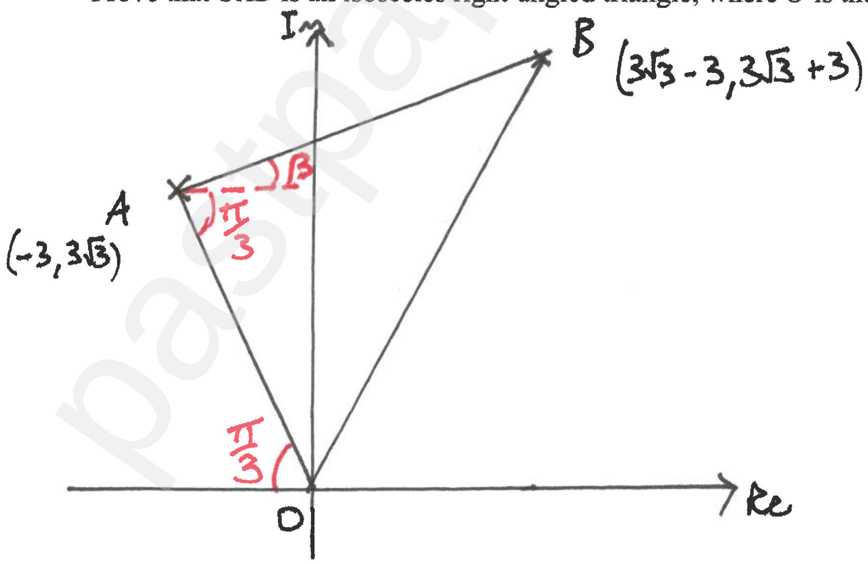
$\alpha = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\underline{w = 6e^{\frac{2\pi i}{3}}}$

(c) On an Argand diagram, the points representing ω and $z\omega$ are A and B respectively.

Prove that OAB is an isosceles right-angled triangle, where O is the origin. [2]



$|\vec{OA}| = 6$ (from part (b)) Looks more likely that $|\vec{AB}| = |\vec{OA}|$:

$\vec{AB} = \underline{b} - \underline{a} = zw - w$

$= (3\sqrt{3} - 3) + (3\sqrt{3} + 3)i - (-3 + 3\sqrt{3}i)$

$= 3\sqrt{3} + 3i$

$|\vec{AB}| = \sqrt{(3\sqrt{3})^2 + 3^2}$

$= 6 = |\vec{OA}|$, therefore isosceles.

Angle OAB $= \frac{\pi}{3} + \beta$ (see diagram)

$\beta = \arg(\vec{AB})$

$= \tan^{-1}\left(\frac{3}{3\sqrt{3}}\right)$

$= \frac{\pi}{6}$

$\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$ QED

(d) Using your answers to part **(b)**, prove that $\tan\frac{5}{12}\pi = \frac{\sqrt{3}+1}{\sqrt{3}-1}$. [3]

Through a bit of trial and error, $\frac{5\pi}{12} = \frac{2\pi}{3} + \frac{-\pi}{4}$:

$\frac{5\pi}{12} = \arg(w) + \arg(z)$

$\frac{5\pi}{12} = \arg(wz)$ $(= \arg(zw))$

$\frac{5\pi}{12} = \tan^{-1}\left(\frac{3\sqrt{3} + 3}{3\sqrt{3} - 3}\right)$

$\tan\left(\frac{5\pi}{12}\right) = \frac{3\sqrt{3} + 3}{3\sqrt{3} - 3}$

$\tan\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$ QED

7 (a) Verify that $-1+\sqrt{5}\mathrm{i}$ is a root of the equation $2x^3 + x^2 + 6x - 18 = 0$. [3]

Use factor theorem, substituting $-1+\sqrt{5}i$:

$2(-1+\sqrt{5}i)^3 + (-1+\sqrt{5}i)^2 + 6(-1+\sqrt{5}i) - 18$

$= 2(-1+\sqrt{5}i)(1-2\sqrt{5}i-5) + (1-2\sqrt{5}i-5) - 6 + 6\sqrt{5}i - 18$

$= 2(-1+\sqrt{5}i)(-4-2\sqrt{5}i) + (-4-2\sqrt{5}i) - 24 + 6\sqrt{5}i$

$= 2(4+2\sqrt{5}i-4\sqrt{5}i+10) - 4 - 2\sqrt{5}i - 24 + 6\sqrt{5}i$

$= 2(14-2\sqrt{5}i) - 28 + 4\sqrt{5}i$

$= 28 - 4\sqrt{5}i - 28 + 4\sqrt{5}i$

$= \underline{0}$

Therefore $-1+\sqrt{5}i$ is a root.

(b) Find the other roots of this equation. [4]

Coefficients are all real, so if $u_1 = -1+\sqrt{5}i$:

$u_2 = u_1^*$

$= \underline{-1-\sqrt{5}i}$

$(x-(-1+\sqrt{5}i))(x-(-1-\sqrt{5}i))$

$x^2-(-1-\sqrt{5}i)x-(-1+\sqrt{5}i)x+(-1+\sqrt{5}i)(-1-\sqrt{5}i)$

$x^2+x+\cancel{\sqrt{5}ix}+x-\cancel{\sqrt{5}ix}+1+\cancel{\sqrt{5}i}-\cancel{\sqrt{5}i}+5$

x^2+2x+6

$$\begin{array}{r|l} & 2x - 3 \\ \hline x^2+2x+6 & 2x^3+x^2+6x-18 \\ & \underline{2x^3+4x^2+12x} \\ & \quad -3x^2-6x-18 \\ & \quad \underline{-3x^2-6x-18} \end{array}$$

→ Another factor is $2x-3$

so root $= \underline{\frac{3}{2}}$

10 The polynomial $x^3 + 5x^2 + 31x + 75$ is denoted by $p(x)$.

(a) Show that $(x + 3)$ is a factor of $p(x)$. [2]

Sub. $x = -3$:

$(-3)^3 + 5(-3)^2 + 31(-3) + 75$

$-27 + 5 \times 9 - 93 + 75$

$-27 + 45 - 93 + 75$

$18 - 18 = 0$

So $(x+3)$ is a factor.

(b) Show that $z = -1 + 2\sqrt{6}\mathrm{i}$ is a root of $p(z) = 0$. [3]

Sub. $z = -1 + 2\sqrt{6}i$:

$(-1 + 2\sqrt{6}i)^3 + 5(-1+2\sqrt{6}i)^2 + 31(-1+2\sqrt{6}i) + 75$

$(-1+2\sqrt{6}i)(1 - 4\sqrt{6}i - 24) + 5(1 - 4\sqrt{6}i - 24) - 31 + 62\sqrt{6}i + 75$

$(-1+2\sqrt{6}i)(-23 - 4\sqrt{6}i) + 5(-23 - 4\sqrt{6}i) + 44 + 62\sqrt{6}i$

$23 + 4\sqrt{6}i - 46\sqrt{6}i + 48 - 115 - 20\sqrt{6}i + 44 + 62\sqrt{6}i$

$71 - 42\sqrt{6}i - 115 - 20\sqrt{6}i + 44 + 62\sqrt{6}i$

$-44 - 62\sqrt{6}i + 44 + 62\sqrt{6}i = 0$

So $z = -1 + 2\sqrt{6}i$ is a root. QED

(c) Hence find the complex numbers z which are roots of $p(z^2) = 0$. [7]

From parts (a) and (b), roots of $p(z)$ are $z=-3$, $z=-1+2\sqrt{6}i$ and $z=-1-2\sqrt{6}i$ (because coefficients of p are real, so z^* is also a root).

Replace z with z^2 for each root:

① $z^2 = -3$

$z = \pm\sqrt{-3}$

$\underline{z = \sqrt{3}i}$ or $\underline{z = -\sqrt{3}i}$

② $z^2 = -1 + 2\sqrt{6}i$

$(x+iy)^2 = -1 + 2\sqrt{6}i$

$x^2 + 2ixy - y^2 = -1 + 2\sqrt{6}i$

Re: $x^2 - y^2 = -1$ ① Im: $2xy = 2\sqrt{6}$

$y = \frac{\sqrt{6}}{x}$ ②

sub. ② into ①: $x^2 - \left(\frac{\sqrt{6}}{x}\right)^2 = -1$

$x^2 - \frac{6}{x^2} = -1$

$x^4 - 6 = -x^2$

$x^4 + x^2 - 6 = 0$

$(x^2+3)(x^2-2) = 0$

$x^2 = -3$ $x^2 = 2$

no real sol^{n}s $x = \pm\sqrt{2}$

→ ②: $x=\sqrt{2}$: $y = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$

$x = -\sqrt{2}$: $y = \frac{\sqrt{6}}{-\sqrt{2}} = -\sqrt{3}$

$\underline{z = \sqrt{2}+\sqrt{3}i}$ or $\underline{z = -\sqrt{2}-\sqrt{3}i}$

continued

3 $z^2 = -1 - 2\sqrt{6}i$

$(x+iy)^2 = -1 - 2\sqrt{6}i$

$x^2 + 2ixy - y^2 = -1 - 2\sqrt{6}i$

Re: $x^2 - y^2 = -1$ ① Im: $2xy = -2\sqrt{6}$

$y = \frac{-\sqrt{6}}{x}$ ②

sub. ② into ①: $x^2 - \left(\frac{-\sqrt{6}}{x}\right)^2 = -1$

$x^2 - \frac{6}{x^2} = -1$

same quadratic as before

$\rightarrow x^2 = -3$ no real sol^{n}s $\quad x^2 = 2$

$x = \pm\sqrt{2}$

→②: $x = \sqrt{2}$: $y = \frac{-\sqrt{6}}{\sqrt{2}} = -\sqrt{3}$

$x = -\sqrt{2}$: $y = \frac{-\sqrt{6}}{-\sqrt{2}} = \sqrt{3}$

$\underline{z = \sqrt{2} - \sqrt{3}i}$ or $\underline{z = -\sqrt{2} + \sqrt{3}i}$

All six answers:

$z_1 = \sqrt{3}i$, $z_2 = -\sqrt{3}i$, $z_3 = \sqrt{2} + \sqrt{3}i$, $z_4 = -\sqrt{2} - \sqrt{3}i$

$z_5 = \sqrt{2} - \sqrt{3}i$, $z_6 = -\sqrt{2} + \sqrt{3}i$