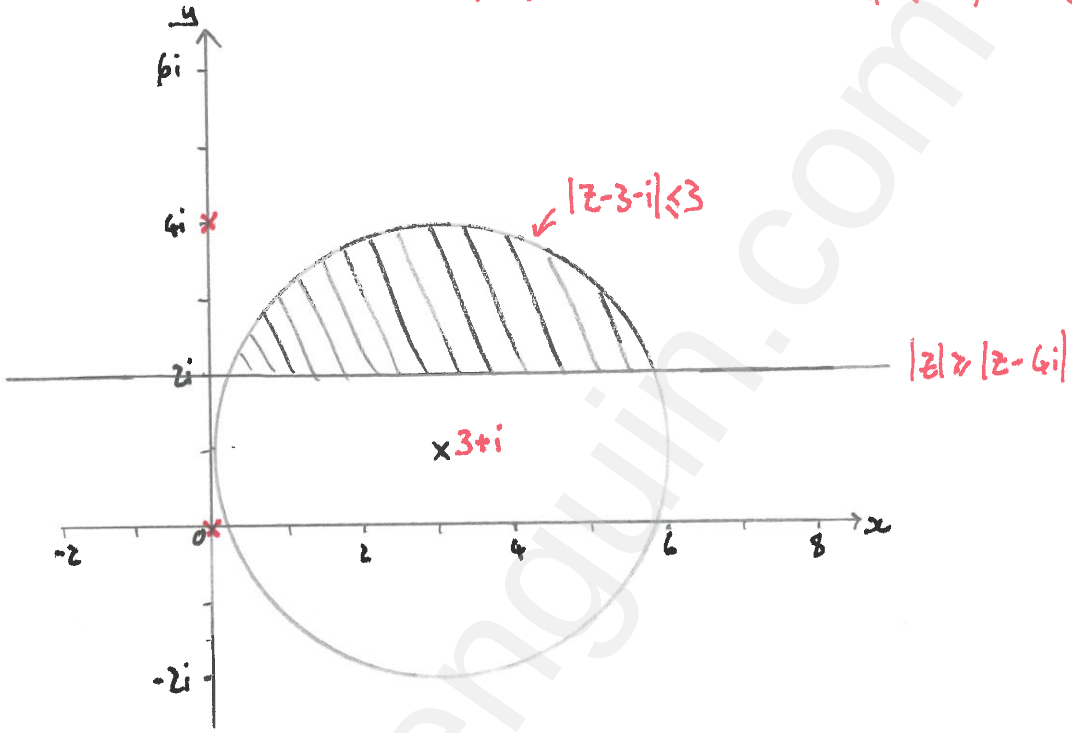
3 On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z-3-i| \leqslant 3$ and $|z| \geqslant |z-4i|$. [4]

↖ perpendicular bisector of (0,0) and (0,4)



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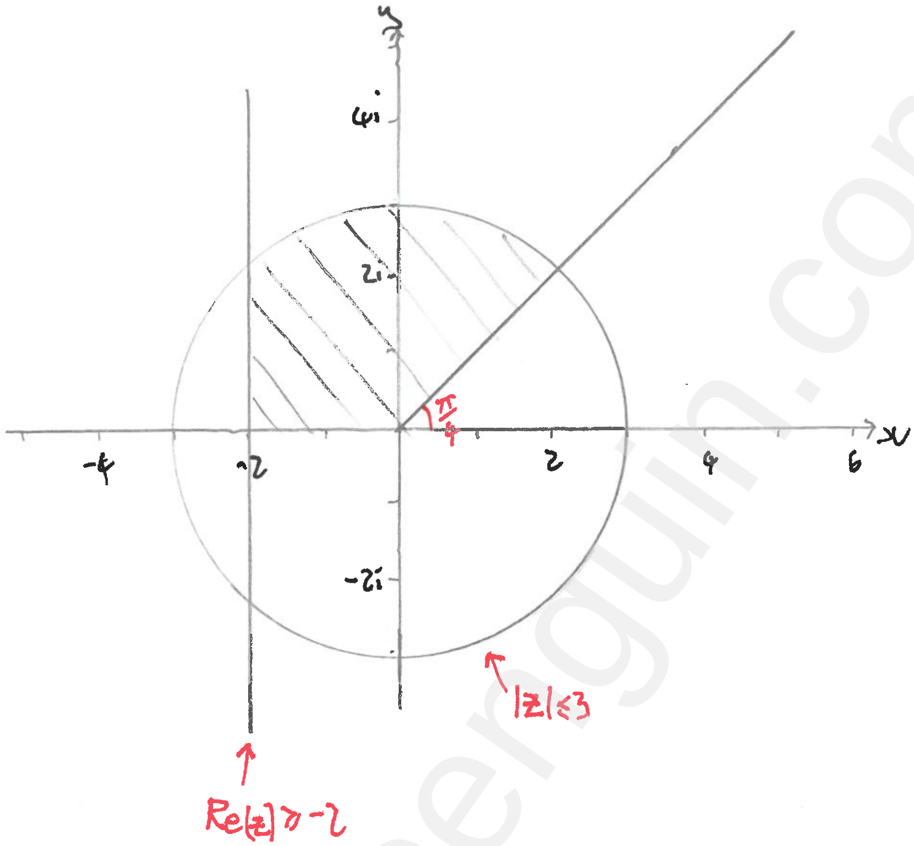
2 On a sketch of an Argand diagram shade the region whose points represent complex numbers z satisfying the inequalities $|z| \leqslant 3$, $\operatorname{Re} z \geqslant -2$ and $\frac{1}{4}\pi \leqslant \arg z \leqslant \pi$. [4]



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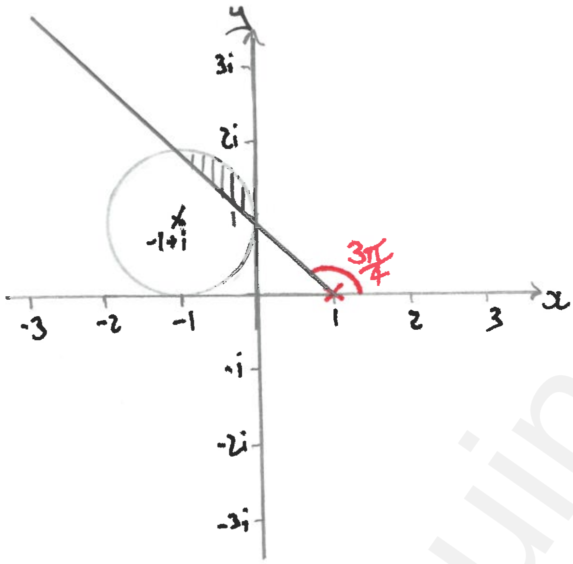
2 On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 1 - \mathrm{i}| \leqslant 1$ and $\arg(z - 1) \leqslant \frac{3}{4}\pi$. [4]



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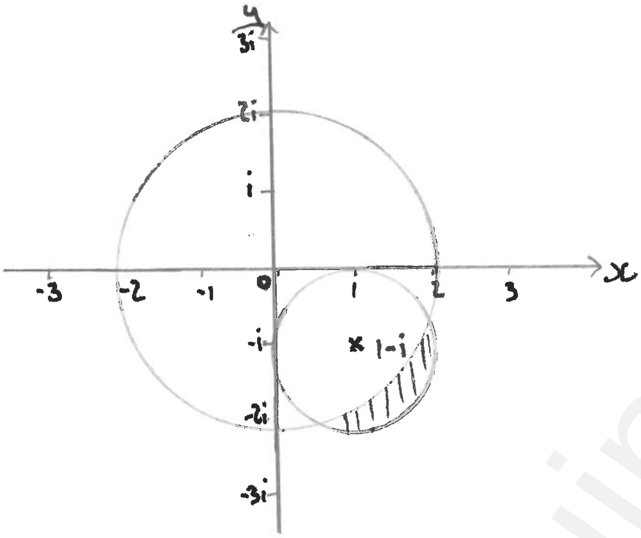
2 On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z| \geqslant 2$ and $|z - 1 + \mathrm{i}| \leqslant 1$. [4]



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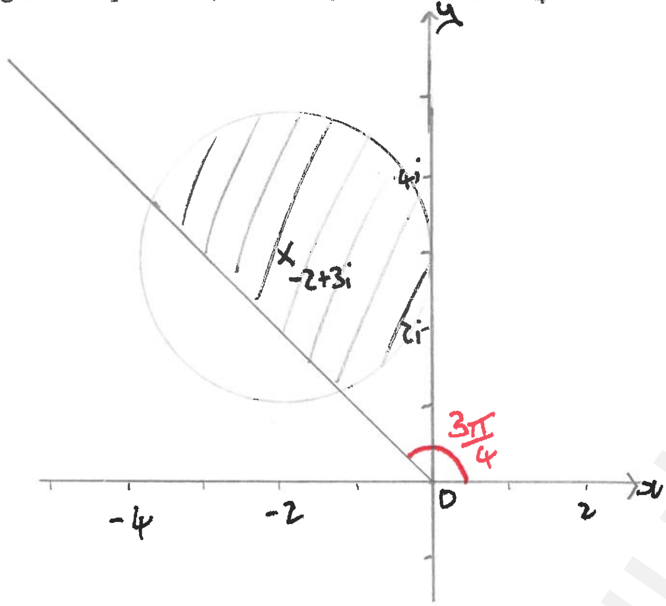
2 On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2 - 3i| \leqslant 2$ and $\arg z \leqslant \frac{3}{4}\pi$. [4]



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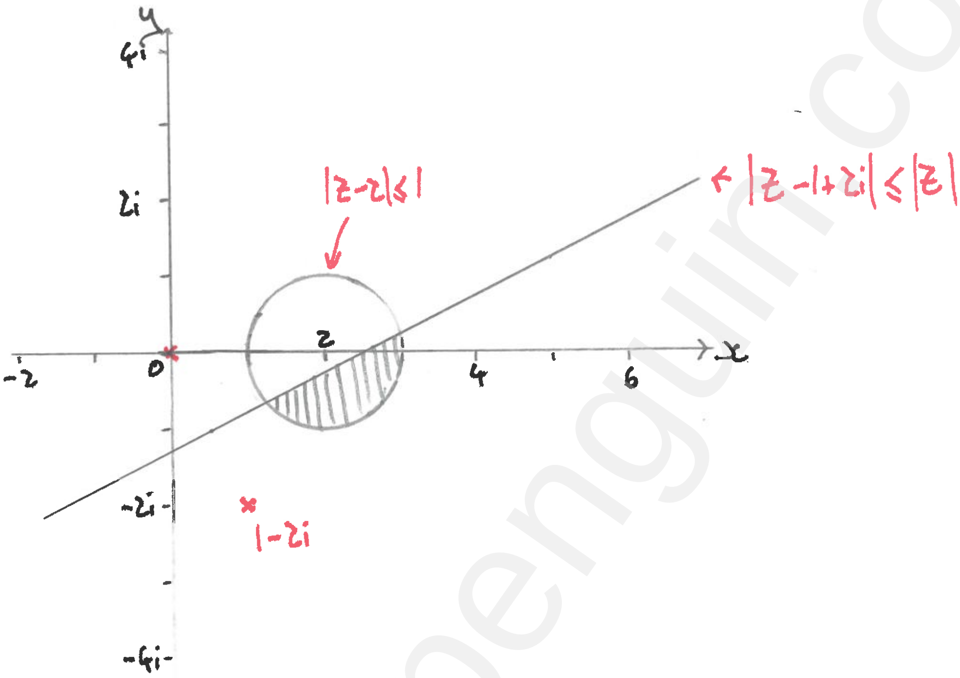
2 On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 1 + 2i| \leqslant |z|$ and $|z - 2| \leqslant 1$. [5]

perpendicular bisector of $(1,-2)$ and $(0,0)$



Optional: Find equation of perpendicular bisector to help plot it:

$$|z - 1 + 2i| = |z|$$
$$|x + iy - 1 + 2i| = |x + iy|$$
$$|(x-1) + i(y+2)| = |x + iy|$$
$$(x-1)^2 + (y+2)^2 = x^2 + y^2$$
$$x^2 - 2x + 1 + y^2 + 4y + 4 = x^2 + y^2$$
$$-2x + 4y + 5 = 0$$
$$4y = 2x - 5$$
$$\underline{y = \tfrac{1}{2}x - \tfrac{5}{4}}$$

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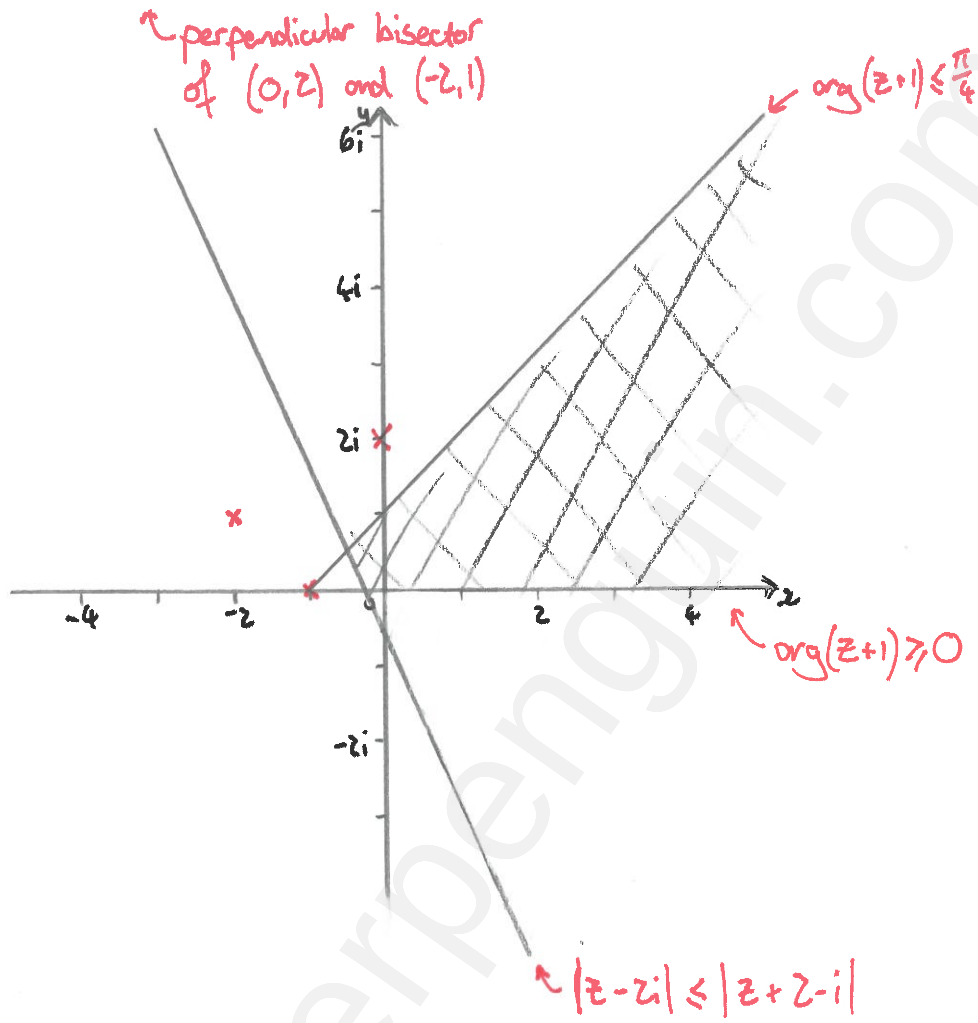
2 On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2i| \leqslant |z + 2 - i|$ and $0 \leqslant \arg(z + 1) \leqslant \frac{1}{4}\pi$. [4]



Optional: Find equation of perpendicular bisector to help plot it:

$$|z - 2i| = |z + 2 - i|$$

$$|x + iy - 2i| = |x + iy + 2 - i|$$

$$|x + i(y-2)| = |(x+2) + i(y-1)|$$

$$x^2 + (y-2)^2 = (x+2)^2 + (y-1)^2$$

$$x^2 + y^2 - 4y + 4 = x^2 + 4x + 4 + y^2 - 2y + 1$$

$$-4y + 4 = 4x - 2y + 5$$

$$-2y = 4x + 1$$

$$\underline{y = -2x - \tfrac{1}{2}}$$

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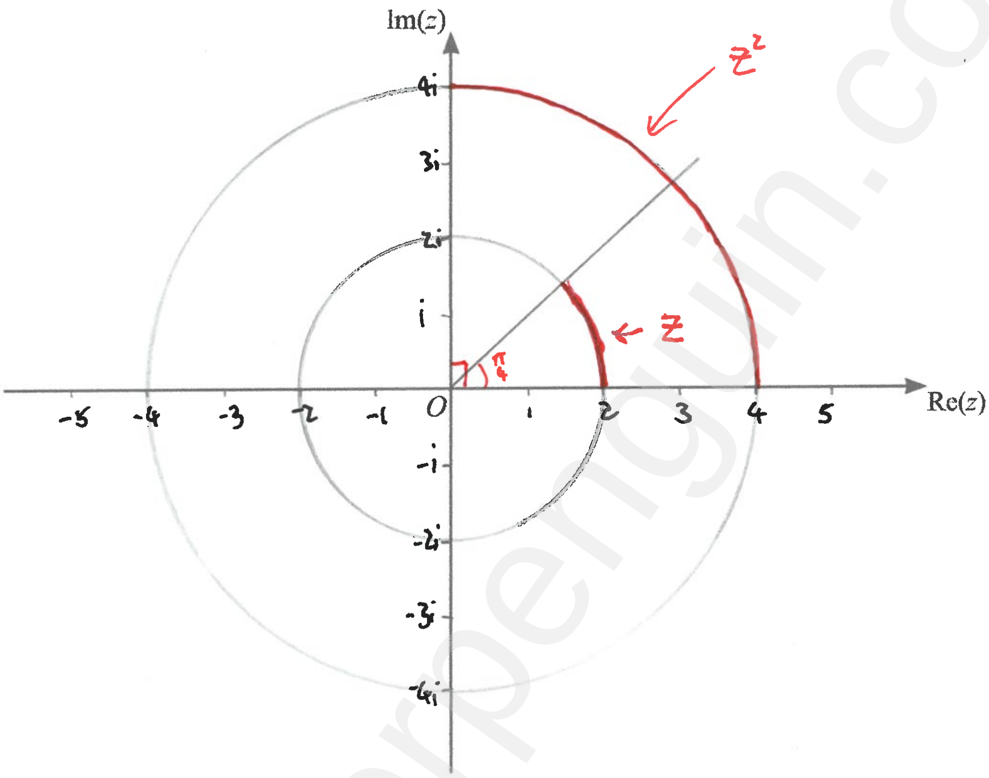
1 The complex number z satisfies $|z| = 2$ and $0 \leqslant \arg z \leqslant \frac{1}{4}\pi$.

(a) On the Argand diagram below, sketch the locus of the points representing z. [2]

(b) On the **same diagram**, sketch the locus of the points representing z^2. [2]



(b) $|z| = 2$

Square both sides

$|z^2| = 4$

for argument:

$\arg(z \times z) = \arg(z) + \arg(z)$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$

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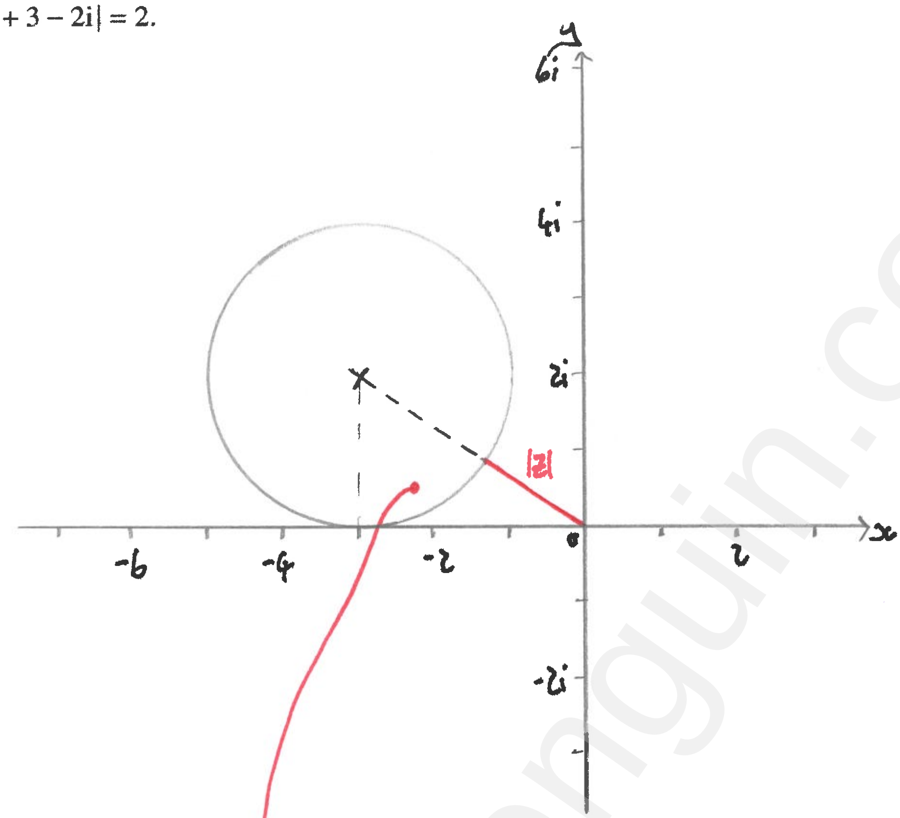
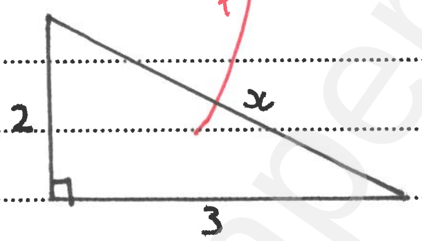
3 **(a)** On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z + 3 - 2i| = 2$. [2]



(b) Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [2]

Least value of |z| is closest point on circle to origin.



$x^2 = 2^2 + 3^2$

$x^2 = 13$

$x = \sqrt{13}$

$|z| = \sqrt{13}$ - radius of circle

$= \underline{\sqrt{13} - 2}$

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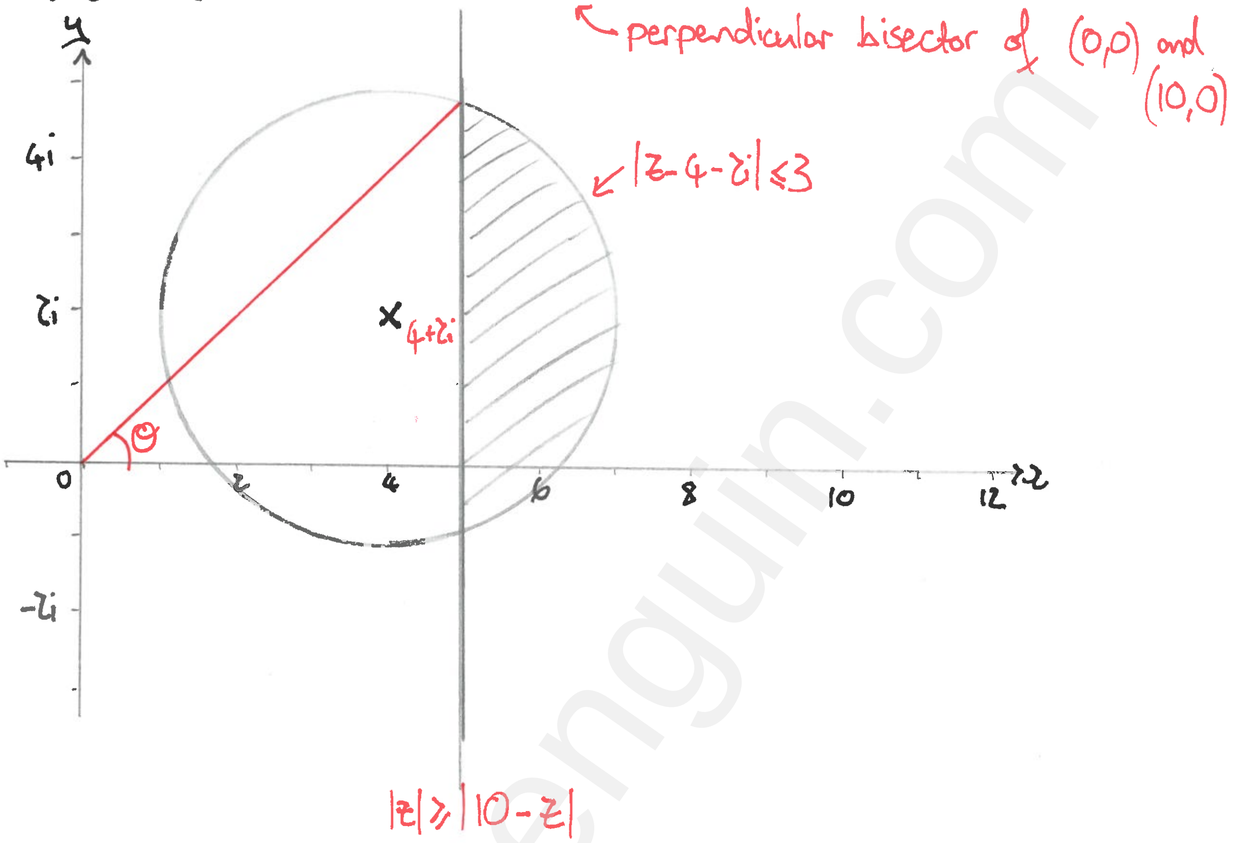
5 **(a)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z-4-2\mathrm{i}| \leqslant 3$ and $|z| \geqslant |10-z|$. [4]



(b) Find the greatest value of $\arg z$ for points in this region. [2]

Greatest value of $\arg z$ = θ on diagram.

Find cartesian equation of circle:

$|x+iy-4-2i| = 3$

$|(x-4)+i(y-2)| = 3$

$(x-4)^2 + (y-2)^2 = 9$

sub. $x=5$:

$(5-4)^2 + (y-2)^2 = 9$

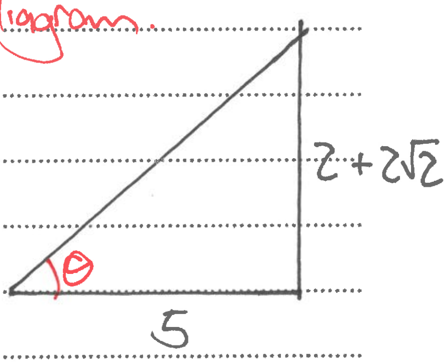
$1 + (y-2)^2 = 9$

$(y-2)^2 = 8$

$y-2 = \pm 2\sqrt{2}$

$y = 2 \pm 2\sqrt{2}$

from diagram, we want $y = 2+2\sqrt{2}$:



$\tan\theta = \dfrac{2+2\sqrt{2}}{5}$

$\theta = 0.768$ rad

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9 (a) The complex numbers u and w are such that

$$u - w = 2\mathrm{i} \quad \text{and} \quad uw = 6.$$

Find u and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

$u - w = 2i$ $\qquad uw = 6$ ②

$u = 2i + w$ ①

Sub. ① into ②:

$(2i + w)w = 6$

$2iw + w^2 = 6$

$w^2 + 2iw - 6 = 0$

$w = \dfrac{-2i \pm \sqrt{(2i)^2 - 4(1)(-6)}}{2(1)}$

$= \dfrac{-2i \pm \sqrt{-4 + 24}}{2}$

$= \dfrac{-2i \pm \sqrt{20}}{2}$

$= \dfrac{-2i \pm 2\sqrt{5}}{2}$

$= -i + \sqrt{5}$ or $-i - \sqrt{5}$

$= \underline{\sqrt{5} - i}$ $\qquad = \underline{-\sqrt{5} - i}$

sub. $\sqrt{5} - i$ into ①:

$u = 2i + \sqrt{5} - i$

$= \underline{\sqrt{5} + i}$

sub. $-\sqrt{5} - i$ into ①:

$u = 2i - \sqrt{5} - i$

$= \underline{-\sqrt{5} + i}$

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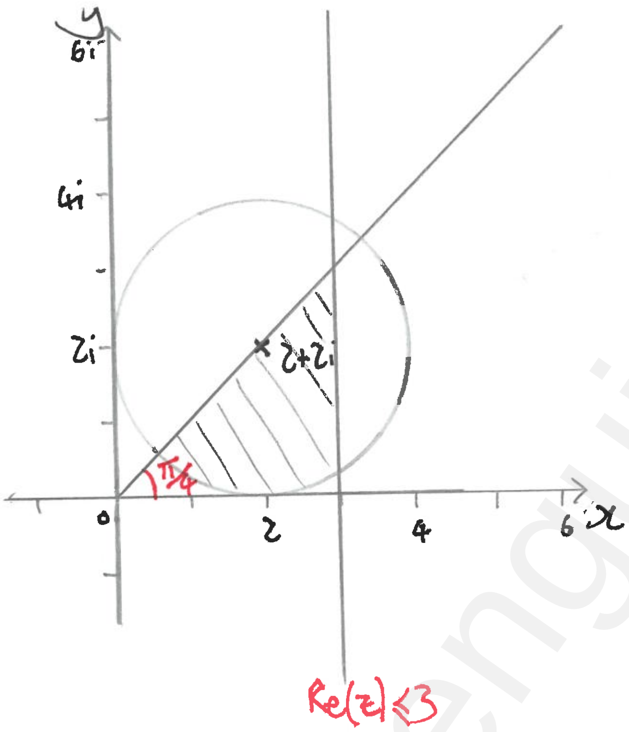
(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities

$$|z - 2 - 2i| \leqslant 2, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{4}\pi \quad \text{and} \quad \operatorname{Re} z \leqslant 3.$$ [5]



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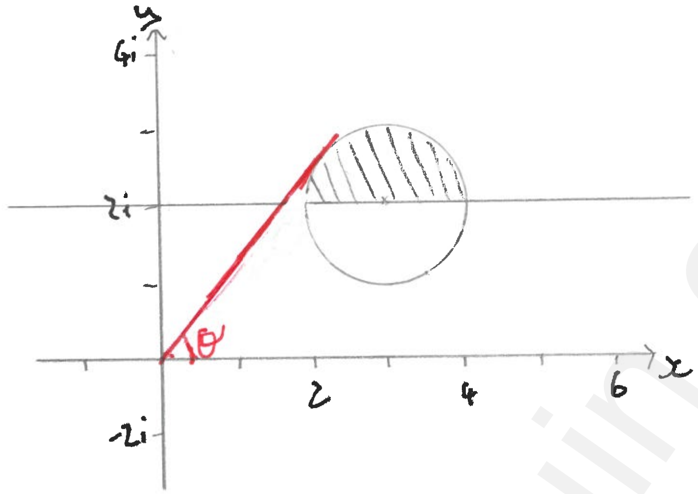
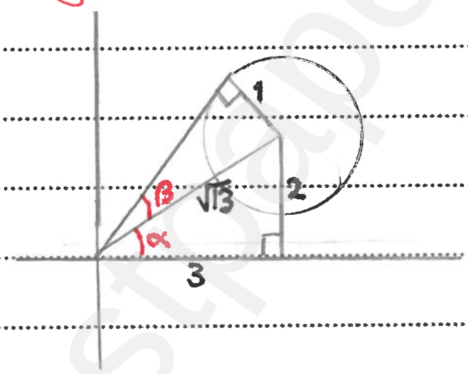
5 (a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - 2i| \leqslant 1$ and $\text{Im}\, z \geqslant 2$. [4]



(b) Find the greatest value of $\arg z$ for points in the shaded region, giving your answer in degrees. [3]

greatest value of θ is a tangent to the circle from the origin, as shown above.



$\tan\alpha = \frac{2}{3}$

$\alpha = \tan^{-1}\left(\frac{2}{3}\right)$

$\sin\beta = \frac{1}{\sqrt{13}}$

$\beta = \sin^{-1}\left(\frac{1}{\sqrt{13}}\right)$

$\theta = \tan^{-1}\left(\frac{2}{3}\right) + \sin^{-1}\left(\frac{1}{\sqrt{13}}\right)$

$= \underline{49.8^\circ}$

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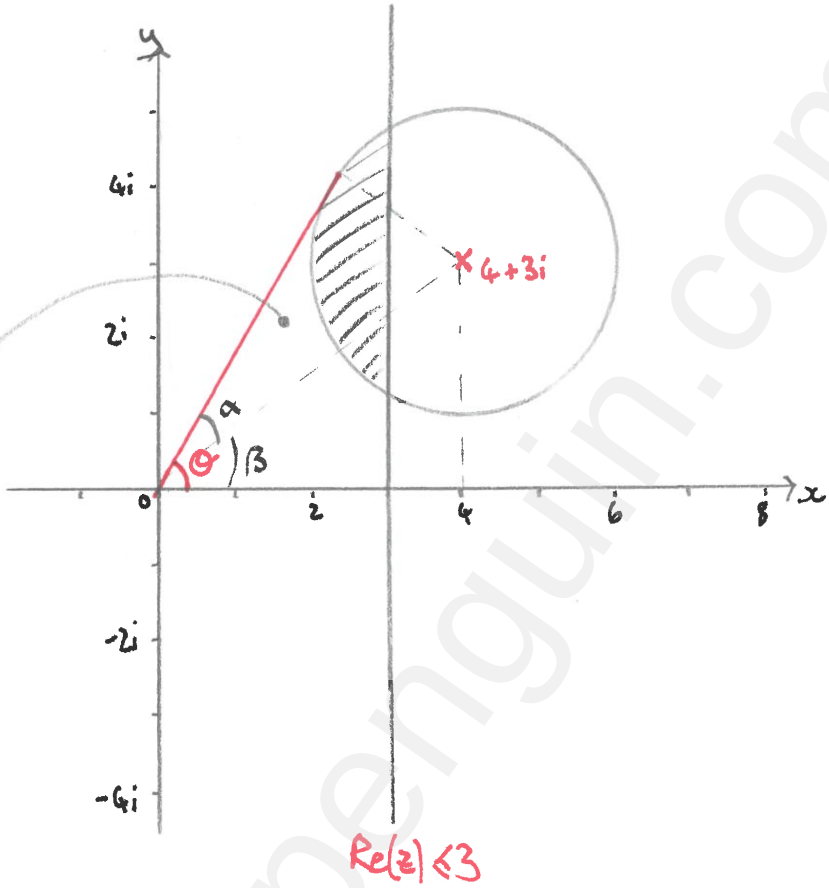
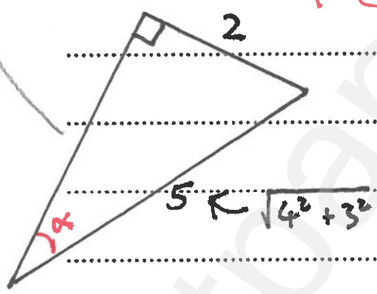
4 **(a)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 4 - 3i| \leqslant 2$ and $\text{Re}\, z \leqslant 3$. [4]



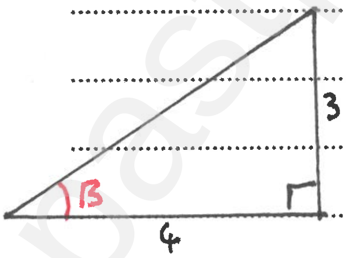
(b) Find the greatest value of $\arg z$ for points in this region. [2]

Greatest value of arg(z) is at tangent to circle as shown.



$\sin\alpha = \frac{2}{5}$

$\alpha = 0.4115$ STO



$\tan\beta = \frac{3}{4}$

$\beta = 0.6435$ STO

$\theta = \alpha + \beta$

$= \underline{1.06}$

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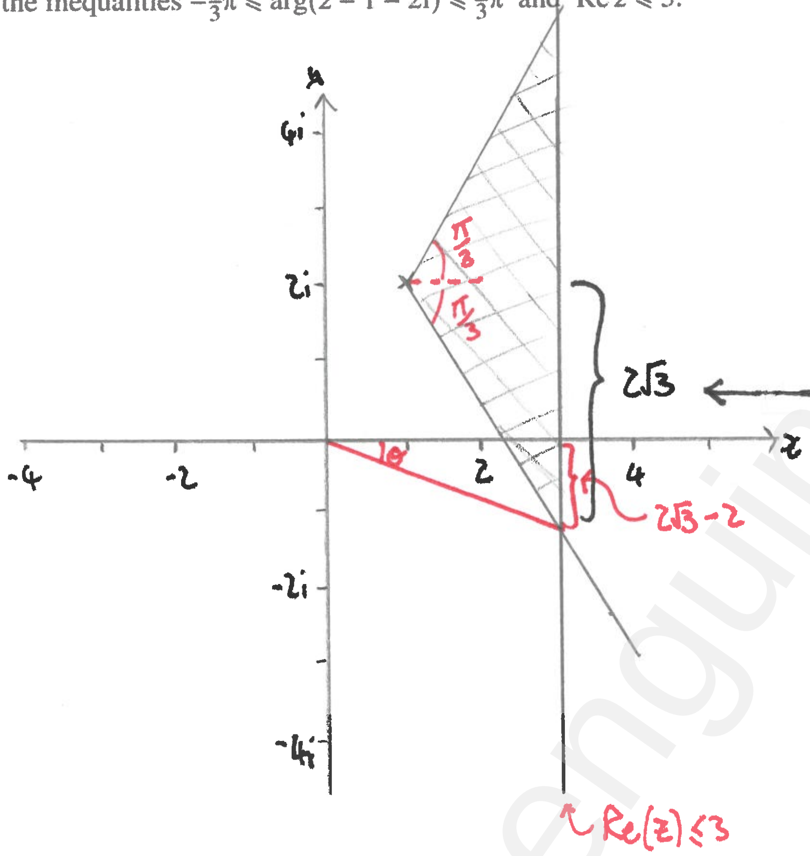
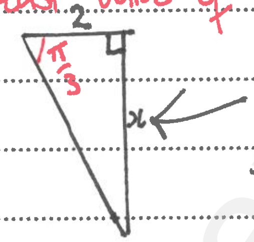
2 **(a)** On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leqslant \arg(z-1-2\mathrm{i}) \leqslant \frac{1}{3}\pi$ and $\operatorname{Re} z \leqslant 3$. [3]



(b) Calculate the least value of $\arg z$ for points in the region from **(a)**. Give your answer in radians correct to 3 decimal places. [2]

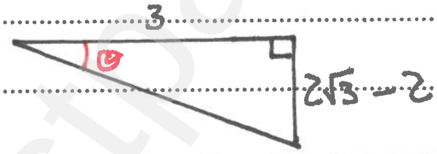
Least value of $\arg(z)$ is lowest point in region (θ).



$\tan\left(\frac{\pi}{3}\right) = \frac{x}{2}$

$x = 2\tan\left(\frac{\pi}{3}\right)$

$= 2\sqrt{3}$



$\tan\theta = \frac{2\sqrt{3}-2}{3}$

$\theta = 0.454$

but below x-axis, so:

$\theta = -0.454$

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[Turn over

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8 **(a)** Solve the equation $(1+2i)w + iw^* = 3 + 5i$. Give your answer in the form $x + iy$, where x and y are real. [4]

Sub. $w = x + iy$ and $w^* = x - iy$:

$$(1+2i)(x+iy) + i(x-iy) = 3+5i$$

$$x + iy + 2ix - 2y + ix + y = 3 + 5i$$

$$x - y + i(3x + y) = 3 + 5i$$

Equate real components:

$$x - y = 3 \quad (1)$$

Equate imaginary components:

$$3x + y = 5 \quad (2)$$

(1) + (2): $4x = 8$

$\underline{x = 2}$

Sub into (2): $3(2) + y = 5$

$6 + y = 5$

$\underline{y = -1}$

$\underline{w = 2 - i}$

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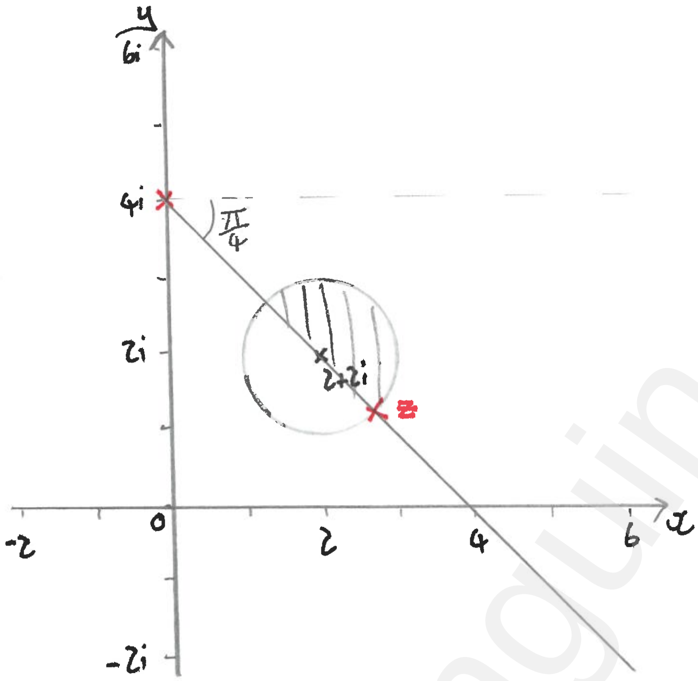
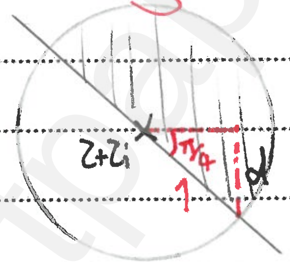
(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2 - 2\text{i}| \leqslant 1$ and $\arg(z - 4\text{i}) \geqslant -\frac{1}{4}\pi$. [4]



(ii) Find the least value of $\text{Im}\,z$ for points in this region, giving your answer in an exact form. [2]

Least value of $\text{Im}(z)$ means the lowest y-coordinate in the region - see point z on the diagram.



$\leftarrow \sin\left(\frac{\pi}{4}\right) = \frac{d}{1}$

$d = \sin\left(\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}$

y-coordinate $= 2 - \frac{1}{\sqrt{2}}$

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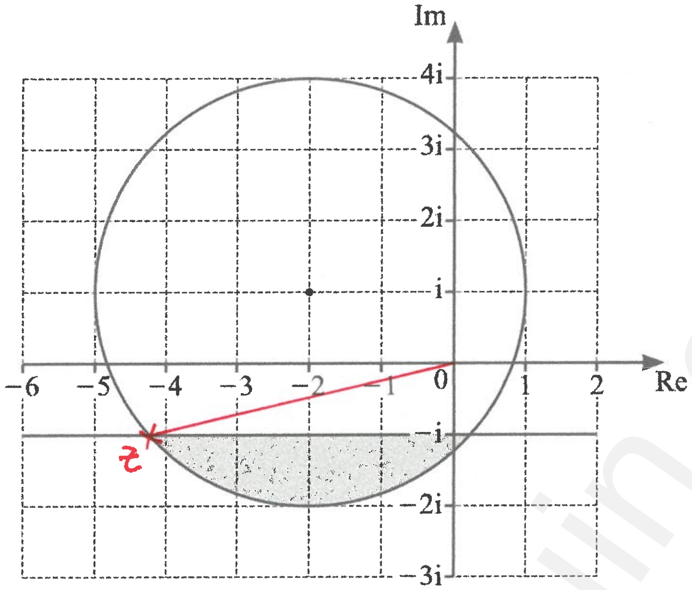
3



The shaded region on the Argand diagram shows points representing complex numbers z defined by two inequalities. The shaded region is bounded by a circle and a line parallel to the real axis. The boundaries of the region are included in the shaded region.

(a) Find two inequalities in terms of z that define the shaded region. [3]

line: $\operatorname{Im}(z) \leqslant -1$

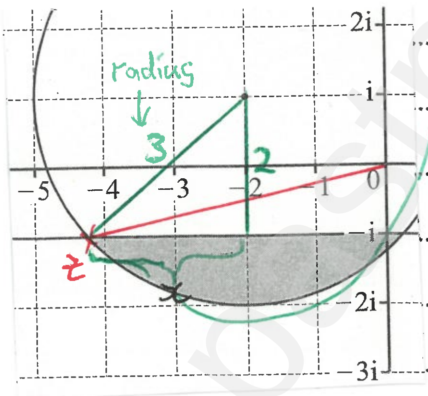
centre radius

circle: $|z - (-2+i)| \leqslant 3$

$|z + 2 - i| \leqslant 3$

(b) Find the greatest value of $|z|$ for points in this region. [3]

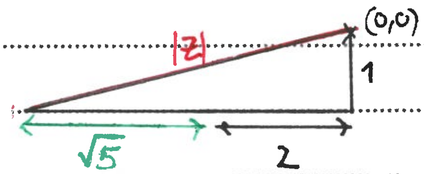
Greatest value of $|z|$ is at point shown on diagram.



$x^2 = 3^2 - 2^2$

$x = \sqrt{9-4}$

$x = \sqrt{5}$



$|z| = \sqrt{1^2 + (2+\sqrt{5})^2}$

$= \sqrt{1 + 4 + 4\sqrt{5} + 5}$

$= \sqrt{10 + 4\sqrt{5}}$

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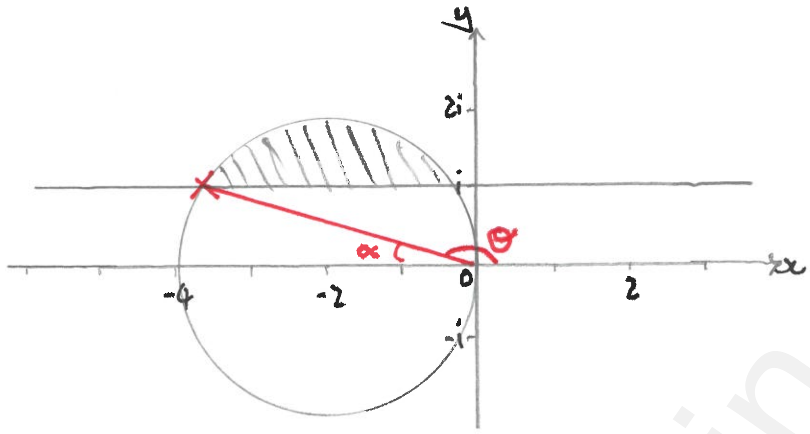
5 (a) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z + 2| \leqslant 2$ and $\operatorname{Im} z \geqslant 1$. [4]



(b) Find the greatest value of $\arg z$ for points in the shaded region. [2]

Greatest value of arg z is θ on diagram.

Cartesian equation of circle:

$$(x+2)^2 + y^2 = 2^2$$

y=1: $(x+2)^2 + 1 = 4$

$$(x+2)^2 = 3$$

$$x+2 = \pm\sqrt{3}$$

$$x = -2 \pm \sqrt{3}$$

$x = -2+\sqrt{3}$ or $x = -2-\sqrt{3}$

other point on circle where y=1

$$\alpha = \tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) = \frac{\pi}{12}$$

$$\theta = \pi - \frac{\pi}{12} = \frac{11\pi}{12}$$

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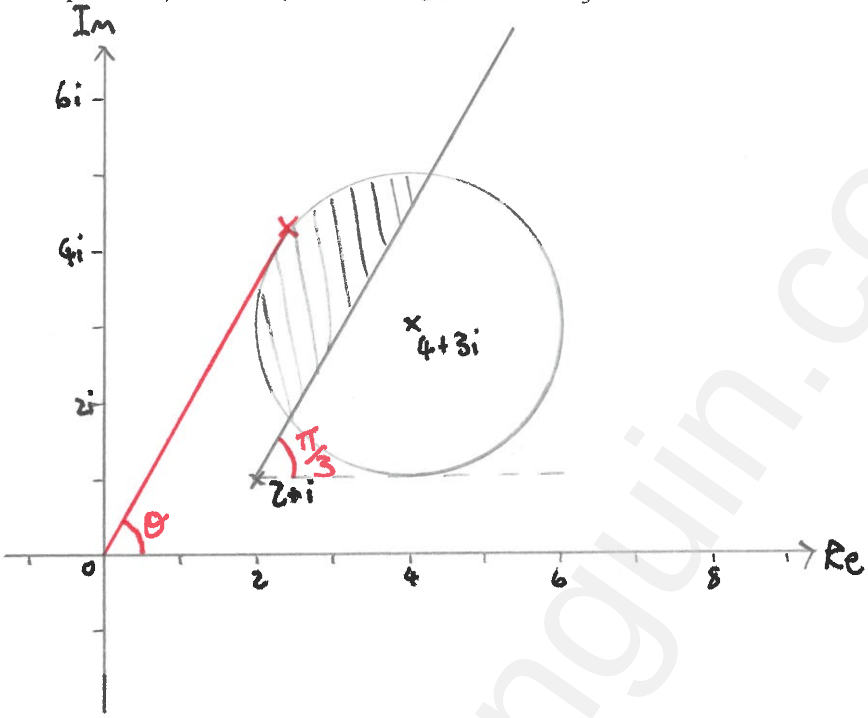
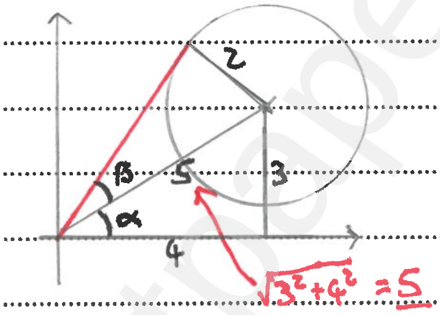
6 **(a)** On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z-4-3\mathrm{i}| \leqslant 2$ and $\arg(z-2-\mathrm{i}) \geqslant \frac{1}{3}\pi$. [5]



(b) Calculate the greatest value of $\arg z$ for points in this region. [2]

Greatest value of arg(z) is θ on the diagram, where z is a tangent to the circle.



$\sqrt{3^2+4^2} = \underline{5}$

$\tan\alpha = \frac{3}{4}$

$\alpha = \tan^{-1}\left(\frac{3}{4}\right)$

$\sin\beta = \frac{2}{5}$

$\beta = \sin^{-1}\left(\frac{2}{5}\right)$

$\theta = \tan^{-1}\left(\frac{3}{4}\right) + \sin^{-1}\left(\frac{2}{5}\right)$

$= \underline{1.06}$

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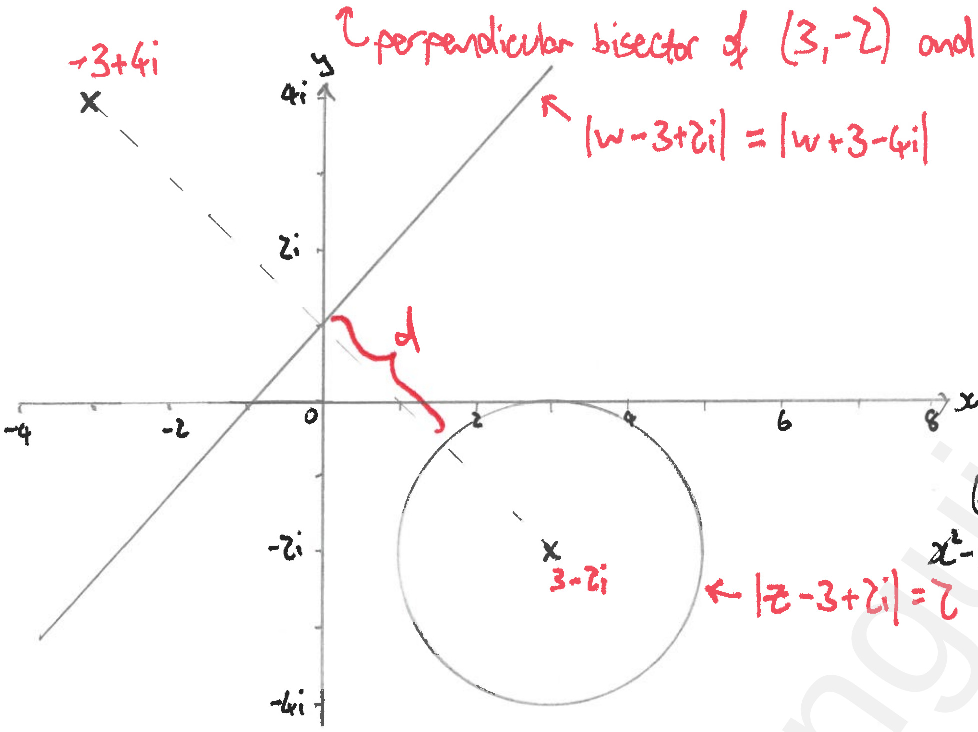
7 (a) On a single Argand diagram sketch the loci given by the equations $|z-3+2i|=2$ and $|w-3+2i|=|w+3-4i|$ where z and w are complex numbers. [4]



Optional: Find equation of perpendicular bisector to help plot it:

$|w-3+2i| = |w+3-4i|$

$|x+iy-3+2i| = |x+iy+3-4i|$

$|(x-3)+i(y+2)| = |(x+3)+i(y-4)|$

$(x-3)^2 + (y+2)^2 = (x+3)^2 + (y-4)^2$

$x^2-6x+9+y^2+4y+4 = x^2+6x+9+y^2-8y+16$

$-12x+12y-12=0$

$-x+y-1=0$

$y = x+1$

(b) Hence find the least value of $|z-w|$ for points on these loci. Give your answer in an exact form. [2]

least value is shortest distance, d as shown on diagram.
d = distance between (0,1) and (3,-2) subtract radius.

distance between (0,1) and (3,-2):

$\sqrt{(3-0)^2+(-2-1)^2} = \sqrt{9+9}$
$= \sqrt{18}$
$= 3\sqrt{2}$

radius of circle $= 2$

$d = 3\sqrt{2}-2$

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10 The complex number $1 + 2i$ is denoted by u. The polynomial $2x^3 + ax^2 + 4x + b$, where a and b are real constants, is denoted by $p(x)$. It is given that u is a root of the equation $p(x) = 0$.

(a) Find the values of a and b. [4]

Use factor theorem, substituting $1+2i$:

$2(1+2i)^3 + a(1+2i)^2 + 4(1+2i) + b = 0$

$2(1+2i)(1+4i-4) + a(1+4i-4) + 4 + 8i + b = 0$

$2(1+2i)(-3+4i) + a(-3+4i) + 4 + 8i + b = 0$

$2(-3+4i-6i-8) - 3a + 4ai + 4 + 8i + b = 0$

$2(-11-2i) - 3a + 4ai + 4 + 8i + b = 0$

$-22 - 4i - 3a + 4ai + 4 + 8i + b = 0$

$-3a + b - 18 + i(4a + 4) = 0$

Im: $4a + 4 = 0$

$4a = -4$

$\underline{a = -1}$

Re: $-3a + b - 18 = 0$

$-3(-1) + b - 18 = 0$

$3 + b - 18 = 0$

$b - 15 = 0$

$\underline{b = 15}$

(b) State a second complex root of this equation. [1]

All coefficients of the polynomial are real, so if $u_1 = 1+2i$:

$u_2 = u_1^*$

$= \underline{1 - 2i}$

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(c) Find the real factors of p(x). [2]

$(x-(1+2i))(x-(1-2i))$

$= x^2 - (1-2i)x - (1+2i)x + (1+2i)(1-2i)$

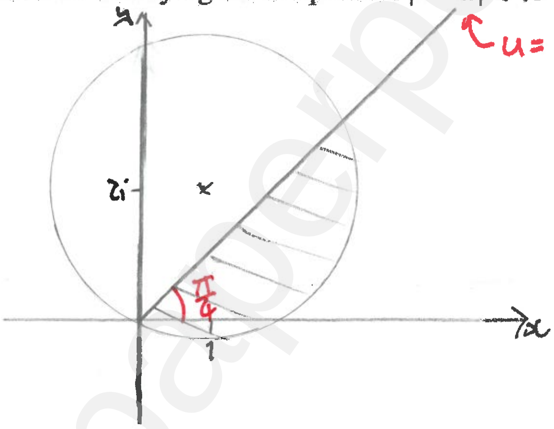
$= x^2 - x + \cancel{2ix} - x - \cancel{2ix} + 1 - \cancel{2i} + \cancel{2i} + 4$

$= x^2 - 2x + 5$

$$\begin{array}{r|l} & \quad 2x + 3 \\ \hline x^2 - 2x + 5 & 2x^3 - x^2 + 4x + 15 \\ & \underline{2x^3 - 4x^2 + 10x} \\ & \qquad\quad 3x^2 - 6x + 15 \\ & \qquad\quad \underline{3x^2 - 6x + 15} \end{array}$$

$\rightarrow$ real factor $= \underline{2x + 3}$

(d) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z-u| \leqslant \sqrt{5}$ and $\arg z \leqslant \frac{1}{4}\pi$. [4]



$u = 1+2i$

Note that $|u| = |1+2i|$

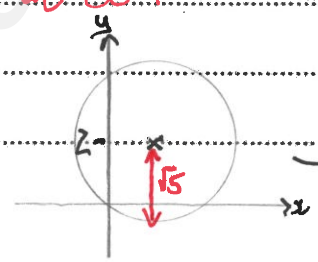
$= \sqrt{1^2+2^2}$

$= \sqrt{5}$

so circle goes through origin.

(ii) Find the least value of Im z for points in the shaded region. Give your answer in an exact form. [1]

least value of Im(z) is the lowest point on the circle, where $x=1$:



$\rightarrow \underline{y = 2 - \sqrt{5}}$

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10 (a) The complex number u is defined by $u = \dfrac{3i}{a+2i}$, where a is real.

(i) Express u in the Cartesian form $x + iy$, where x and y are in terms of a. [3]

$$\frac{3i}{a+2i} \times \frac{a-2i}{a-2i}$$

$$= \frac{3ai + 6}{a^2 - 2ai + 2ai + 4}$$

$$= \frac{6 + 3ai}{a^2 + 4}$$

$$= \frac{6}{a^2+4} + \frac{3ai}{a^2+4}$$

(ii) Find the exact value of a for which $\arg u^* = \frac{1}{3}\pi$. [3]

$$u^* = \frac{6}{a^2+4} - \frac{3ai}{a^2+4}$$

$$\arg(u^*) = \frac{\pi}{3}$$

$$\tan^{-1}\left(\frac{\frac{-3a}{a^2+4}}{\frac{6}{a^2+4}}\right) = \frac{\pi}{3}$$

$$\tan^{-1}\left(\frac{-3a}{6}\right) = \frac{\pi}{3}$$

$$\tan^{-1}\left(\frac{-a}{2}\right) = \frac{\pi}{3}$$

$$\frac{-a}{2} = \tan\left(\frac{\pi}{3}\right)$$

$$\frac{-a}{2} = \sqrt{3}$$

$$-a = 2\sqrt{3}$$

$$\underline{a = -2\sqrt{3}}$$

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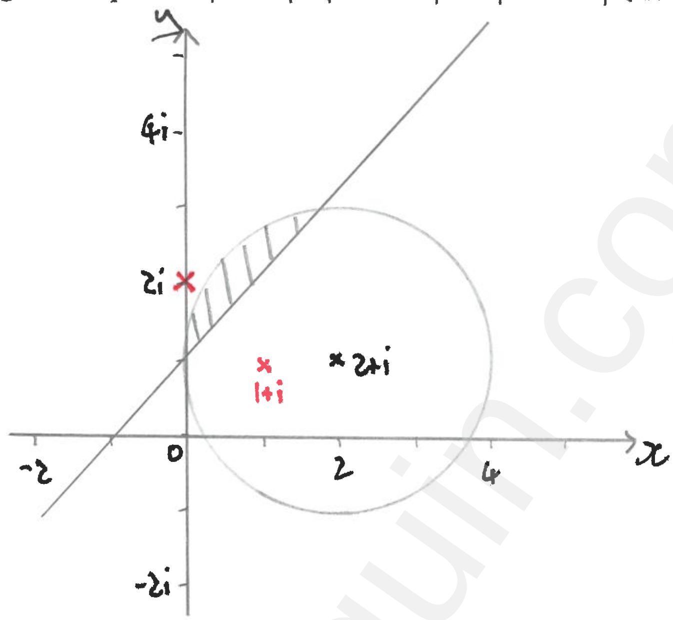
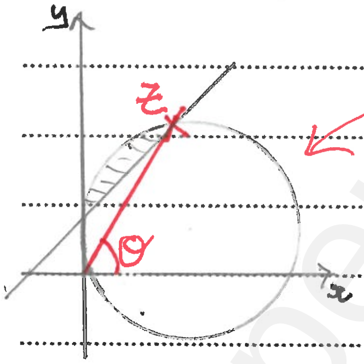
(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2i| \leqslant |z - 1 - i|$ and $|z - 2 - i| \leqslant 2$. [4]



(ii) Calculate the least value of $\arg z$ for points in this region. [2]



least value of arg(z) is θ

z is the point of intersection of the circle and line, which is $2 + 3i$ (the very top of the circle).

$\tan\theta = \frac{3}{2}$

$\theta = \tan^{-1}\left(\frac{3}{2}\right)$

$\approx 56.3^\circ$

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10 **(a)** The complex numbers v and w satisfy the equations

$$v + \mathrm{i}w = 5 \qquad \text{and} \qquad (1 + 2\mathrm{i})v - w = 3\mathrm{i}.$$

Solve the equations for v and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

$V + iW = 5 \qquad (1+2i)V - W = 3i$ (2)

$V = 5 - iW$ (1)

Sub. (1) into (2):

$(1+2i)(5-iW) - W = 3i$

$5 - iW + 10i + 2W - W = 3i$

$W + 5 + i(10 - W) = 3i$

Sub. $W = x + iy$:

$x + iy + 5 + i(10 - x - iy) = 3i$

$x + iy + 5 + 10i - xi + y = 3i$

$x + y + 5 + i(-x + y + 10) = 3i$

Equate real components:

$x + y + 5 = 0$

$x + y = -5$ (3)

Equate imaginary components:

$-x + y + 10 = 3$

$-x + y = -7$ (4)

(3) + (4): $2y = -12$

$\underline{y = -6}$

Sub into (3): $x - 6 = -5$

$\underline{x = 1}$

$\rightarrow \underline{W = 1 - 6i}$

$\rightarrow$ sub. into (1):

$V = 5 - i(1 - 6i)$

$V = 5 - i - 6$

$\underline{V = -1 - i}$

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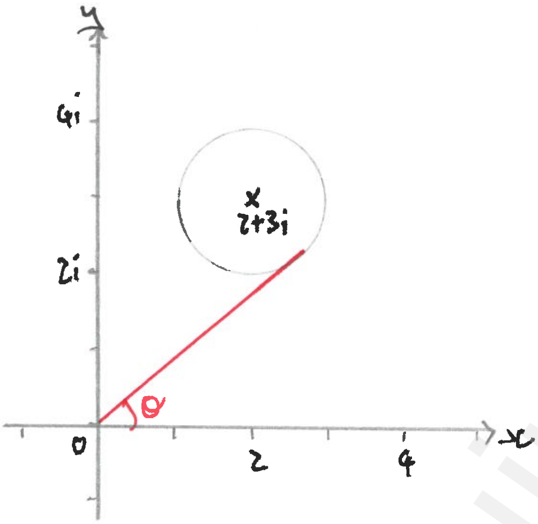
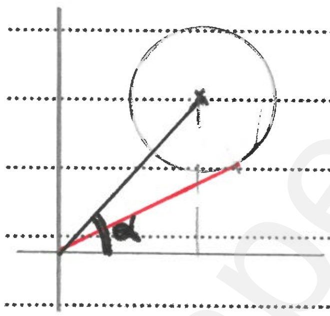
(b) **(i)** On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z-2-3\mathrm{i}|=1$. [2]

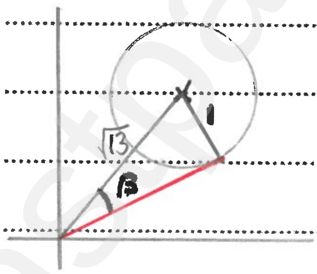


(ii) Calculate the least value of arg z for points on this locus. [2]

Least value of arg z is a tangent to the circle from the origin as shown.



$$\tan\alpha = \frac{3}{2}$$
$$\alpha = \tan^{-1}\left(\frac{3}{2}\right)$$



$$\sin\beta = \frac{1}{\sqrt{13}}$$
$$\beta = \sin^{-1}\left(\frac{1}{\sqrt{13}}\right)$$

$$\theta = \alpha - \beta$$
$$= \tan^{-1}\left(\frac{3}{2}\right) - \sin^{-1}\left(\frac{1}{\sqrt{13}}\right)$$
$$= \underline{40.2^\circ}$$

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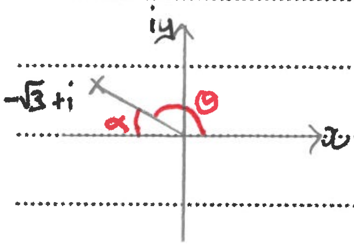
11 The complex number $-\sqrt{3} + \mathrm{i}$ is denoted by u.

(a) Express u in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [2]

$r = \sqrt{(-\sqrt{3})^2 + 1^2}$

$= \sqrt{4}$

$= 2$



$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

$\theta = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

$\rightarrow \underline{2e^{\frac{5\pi i}{6}}}$

(b) Hence show that u^6 is real and state its value. [2]

$u^6 = \left(2e^{\frac{5\pi i}{6}}\right)^6$

$= 64e^{5\pi i}$

$\theta = 5\pi$ is equivalent to $\theta = \pi$, meaning it lies on the negative real axis, so the value of $\underline{u^6 = -64}$

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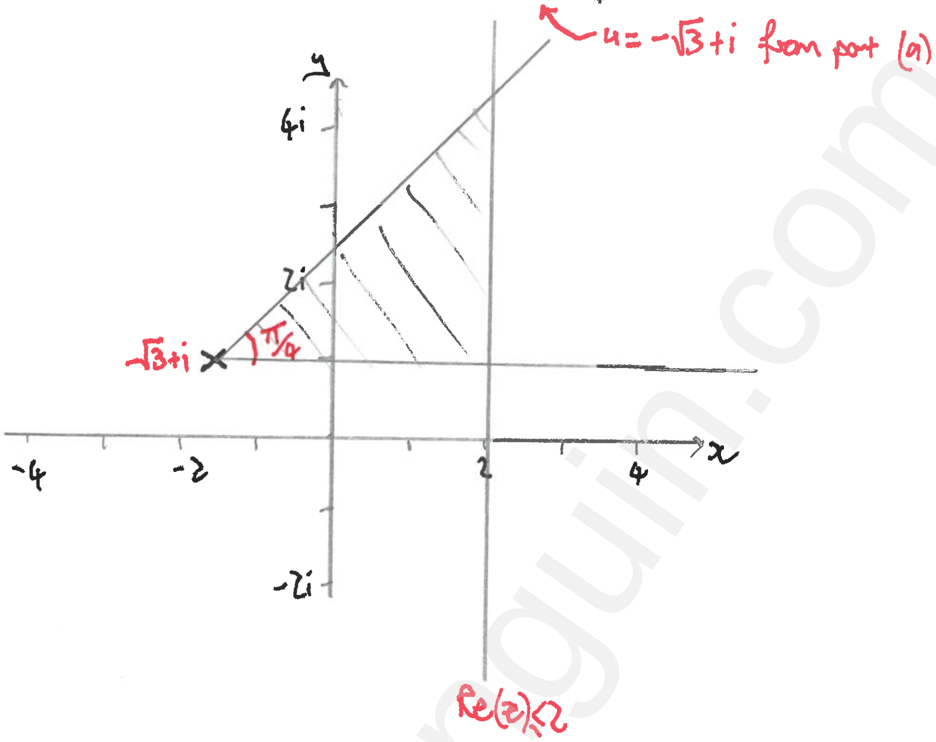
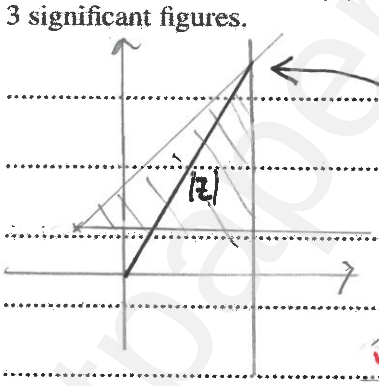
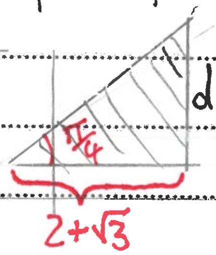
(c) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $0 \leqslant \arg(z-u) \leqslant \frac{1}{4}\pi$ and $\operatorname{Re} z \leqslant 2$. [4]



(ii) Find the greatest value of $|z|$ for points in the shaded region. Give your answer correct to 3 significant figures. [2]



largest value of $|z|$ is furthest point from origin:



$\tan\left(\frac{\pi}{4}\right) = \frac{d}{2+\sqrt{3}}$

$d = (2+\sqrt{3})\tan\left(\frac{\pi}{4}\right)$

$= 2+\sqrt{3}$

so y-coordinate of z is $1+2+\sqrt{3} = \underline{3+\sqrt{3}}$

x-coordinate is 2

$\rightarrow \sqrt{2^2+(3+\sqrt{3})^2} = \underline{5.14}$

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10 The complex number $-1+\sqrt{7}\mathrm{i}$ is denoted by u. It is given that u is a root of the equation

$$2x^3 + 3x^2 + 14x + k = 0,$$

where k is a real constant.

(a) Find the value of k. [3]

Use factor theorem, substituting $-1+\sqrt{7}i$:

$2(-1+\sqrt{7}i)^3 + 3(-1+\sqrt{7}i)^2 + 14(-1+\sqrt{7}i) + k = 0$

$2(-1+\sqrt{7}i)(1-2\sqrt{7}i-7) + 3(1-2\sqrt{7}i-7) - 14 + 14\sqrt{7}i + k = 0$

$2(-1+\sqrt{7}i)(-6-2\sqrt{7}i) + 3(-6-2\sqrt{7}i) - 14 + 14\sqrt{7}i + k = 0$

$2(6+2\sqrt{7}i-6\sqrt{7}i+14) - 18 - 6\sqrt{7}i - 14 + 14\sqrt{7}i + k = 0$

$2(20-4\sqrt{7}i) - 32 + 8\sqrt{7}i + k = 0$

$40 - \cancel{8\sqrt{7}i} - 32 + \cancel{8\sqrt{7}i} + k = 0$

$8 + k = 0$

$\underline{k = -8}$

(b) Find the other two roots of the equation. [4]

Coefficients are all real, so if $u_1 = -1+\sqrt{7}i$:

$u_2 = u_1^*$

$= \underline{-1-\sqrt{7}i}$

$(x-(-1+\sqrt{7}i))(x-(-1-\sqrt{7}i))$

$x^2 - (-1-\sqrt{7}i)x - (-1+\sqrt{7}i)x + (-1+\sqrt{7}i)(-1-\sqrt{7}i)$

$x^2 + x + \cancel{\sqrt{7}ix} + x - \cancel{\sqrt{7}ix} + 1 + \cancel{\sqrt{7}i} - \cancel{\sqrt{7}i} + 7$

$x^2 + 2x + 8$

$$\begin{array}{r|l} & 2x - 1 \\ \hline x^2+2x+8 & 2x^3 + 3x^2 + 14x - 8 \\ & \underline{2x^3 + 4x^2 + 16x} \\ & \quad -x^2 - 2x - 8 \\ & \quad \underline{-x^2 - 2x - 8} \end{array}$$

factor $= 2x-1$

So root $= \underline{\tfrac{1}{2}}$

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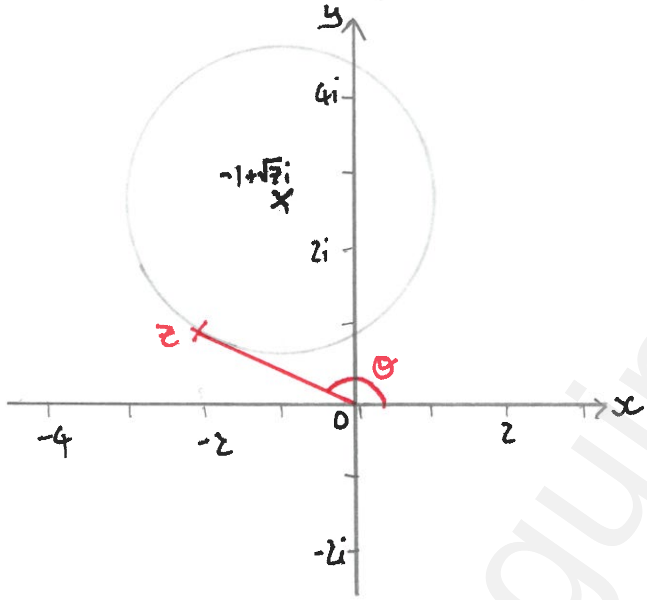
(c) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the equation $|z - u| = 2$. [2]

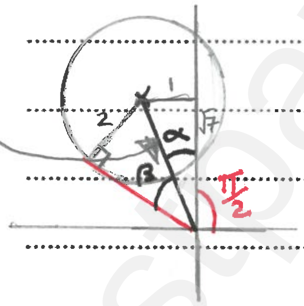
$u = -1 + \sqrt{7}i$, from part (a) $\sqrt{7} \simeq 2.65$



(d) Determine the greatest value of arg z for points on this locus, giving your answer in radians. [2]

Greatest value of arg z is a tangent to the circle from the origin as shown in the diagram.

$\sqrt{(-1)^2 + \sqrt{7}^2}$
$= \sqrt{8}$



$\tan\alpha = \frac{1}{\sqrt{7}}$

$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{7}}\right)$

$\sin\beta = \frac{2}{\sqrt{8}}$

$\beta = \sin^{-1}\left(\frac{2}{\sqrt{8}}\right)$

$\theta = \frac{\pi}{2} + \tan^{-1}\left(\frac{1}{\sqrt{7}}\right) + \sin^{-1}\left(\frac{2}{\sqrt{8}}\right)$

$= 2.72$ radians

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DO NOT WRITE IN THIS MARGIN

8 **(a)** Given that $z = 1 + yi$ and that y is a real number, express $\frac{1}{z}$ in the form $a + bi$, where a and b are functions of y. [2]

$$\frac{1}{z} = \frac{1}{1+yi} \times \frac{(1-yi)}{(1-yi)}$$

$$= \frac{1-yi}{1-\cancel{yi}+\cancel{yi}+y^2}$$

$$= \frac{1-yi}{1+y^2}$$

$$= \frac{1}{1+y^2} + \frac{-y}{1+y^2}i$$

(b) Show that $\left(a-\frac{1}{2}\right)^2 + b^2 = \frac{1}{4}$, where a and b are the functions of y found in part **(a)**. [3]

LHS: $\left(\frac{1}{1+y^2} - \frac{1}{2}\right)^2 + \left(\frac{-y}{1+y^2}\right)^2$

$$\left(\frac{1}{1+y^2} - \frac{1}{2}\right)\left(\frac{1}{1+y^2} - \frac{1}{2}\right) + \left(\frac{-y}{1+y^2}\right)^2$$

$$\frac{1}{(1+y^2)^2} - \frac{1}{2} \times \frac{1}{1+y^2} - \frac{1}{2} \times \frac{1}{1+y^2} + \frac{1}{4} + \frac{y^2}{(1+y^2)^2}$$

$$\frac{1}{(1+y^2)^2} - \frac{1 \times (1+y^2)}{1+y^2 \times (1+y^2)} + \frac{y^2}{(1+y^2)^2} + \frac{1}{4}$$

$$\frac{1}{(1+y^2)^2} - \frac{1+y^2}{(1+y^2)^2} + \frac{y^2}{(1+y^2)^2} + \frac{1}{4}$$

$$\frac{1 - 1 - y^2 + y^2}{(1+y^2)^2} + \frac{1}{4}$$

$$= 0 + \frac{1}{4} = \frac{1}{4} \quad \text{QED}$$

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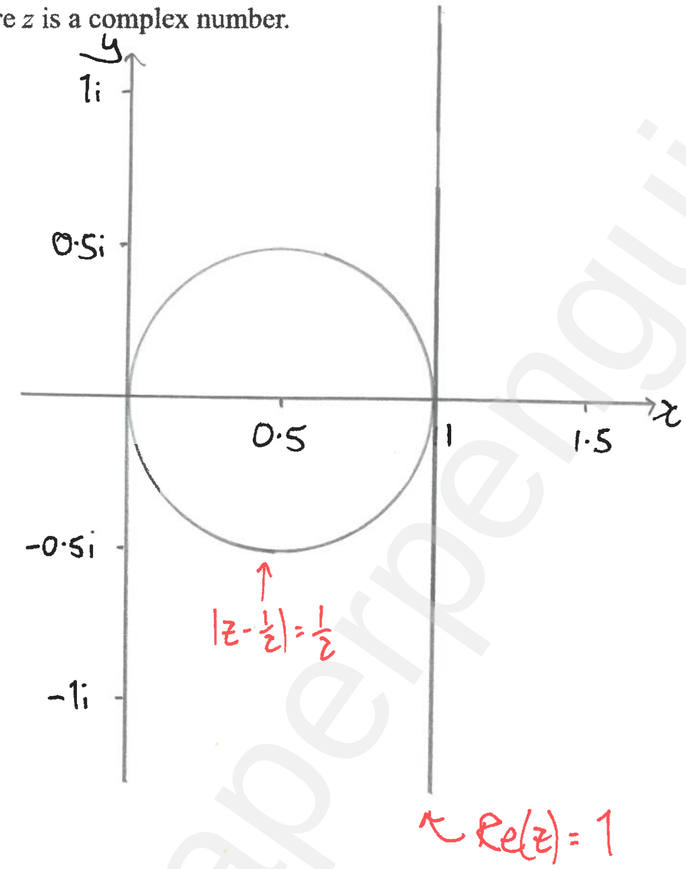
(c) On a single Argand diagram, sketch the loci given by the equations $\mathrm{Re}(z) = 1$ and $\left|z - \frac{1}{2}\right| = \frac{1}{2}$, where z is a complex number. [3]



Cartesian equation (useful for part (d)):

$$\left|x+iy-\tfrac{1}{2}\right| = \tfrac{1}{2}$$

$$\left|\left(x-\tfrac{1}{2}\right)+iy\right| = \tfrac{1}{2}$$

$$\left(x-\tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4} \leftarrow r^2$$

(d) The complex number z is such that $\mathrm{Re}(z) = 1$. Use your answer to part (b) to give a geometrical description of the locus of $\frac{1}{z}$. [1]

This seems really confusing until you realise you have to use (a) and (b).

If $\mathrm{Re}(z) = 1$, complex number is $z = 1 + yi$ (as in part (a))

So $\frac{1}{z} = \frac{1}{1+y^2} + \frac{-y}{1+y^2}i$ (from part (a))

$= a + bi$

From (b), the locus of $\frac{1}{z}$: $\left(a-\frac{1}{2}\right)^2 + b^2 = \frac{1}{4}$

→ This is a circle, centre $\left(\frac{1}{2}, 0\right)$, radius $= \frac{1}{2}$.

For what it's worth, to properly understand this requires "Transformations on the Complex Plane" which is not on the syllabus.

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