

- 2 The circle with equation  $(x-3)^2 + (y-5)^2 = 40$  intersects the  $y$ -axis at points  $A$  and  $B$ .

(a) Find the  $y$ -coordinates of  $A$  and  $B$ , expressing your answers in terms of surds.

[2]

At  $y$ -axis,  $x = 0$ :

$$(0-3)^2 + (y-5)^2 = 40$$

$$9 + y^2 - 10y + 25 = 40$$

$$y^2 - 10y + 34 = 40$$

$$y^2 - 10y - 6 = 0$$

$$y = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-6)}}{2(1)}$$

$$y = 5 + \sqrt{31} \quad \text{or} \quad y = 5 - \sqrt{31}$$

(b) Find the equation of the circle which has  $AB$  as its diameter.

[2]

$$A: (0, 5 + \sqrt{31}) \quad B: (0, 5 - \sqrt{31})$$

$$\begin{aligned} \text{Centre of circle} &= \text{mid-point of } AB: \left( \frac{0+0}{2}, \frac{5+\sqrt{31}+5-\sqrt{31}}{2} \right) \\ &= (0, 5) \end{aligned}$$

$$\rightarrow (x-0)^2 + (y-5)^2 = r^2$$

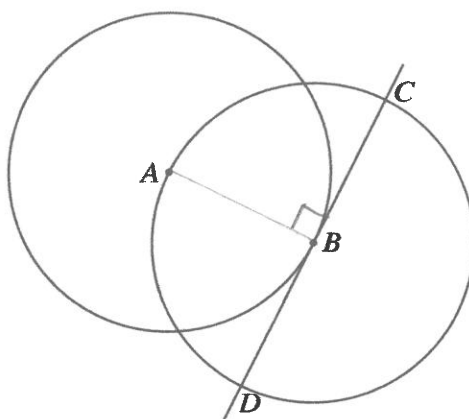
Sub. A:

$$(0)^2 + (5 + \sqrt{31} - 5)^2 = r^2$$

$$(\sqrt{31})^2 = r^2$$

$$31 = r^2$$

$$\rightarrow x^2 + (y-5)^2 = 31$$



The diagram shows a circle with centre  $A$  passing through the point  $B$ . A second circle has centre  $B$  and passes through  $A$ . The tangent at  $B$  to the first circle intersects the second circle at  $C$  and  $D$ .

The coordinates of  $A$  are  $(-1, 4)$  and the coordinates of  $B$  are  $(3, 2)$ .

- (a) Find the equation of the tangent  $CBD$ .

[2]

$$\text{Gradient of } AB = \frac{2-4}{3-(-1)}$$

$$= \frac{-2}{4} = -\frac{1}{2}$$

$$\text{Gradient of tangent} = 2$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 2(x - 3)$$

$$y - 2 = 2x - 6$$

$$\underline{y = 2x - 4}$$

- (b) Find an equation of the circle with centre B.

[3]

$$(x-3)^2 + (y-2)^2 = r^2$$

sub.  $(-1, 4)$ :

$$(-1-3)^2 + (4-2)^2 = r^2$$

$$(-4)^2 + 2^2 = r^2$$

$$16 + 4 = r^2$$

$$r^2 = 20$$

$$\rightarrow \underline{(x-3)^2 + (y-2)^2 = 20}$$

- (c) Find, by calculation, the x-coordinates of C and D.

[3]

sub.  $y = 2x - 4$  into eqn of circle:

$$(x-3)^2 + (2x-4-2)^2 = 20$$

$$(x-3)^2 + (2x-6)^2 = 20$$

$$x^2 - 6x + 9 + 4x^2 - 24x + 36 = 20$$

$$5x^2 - 30x + 25 = 0$$

$$x^2 - 6x + 5 = 0$$

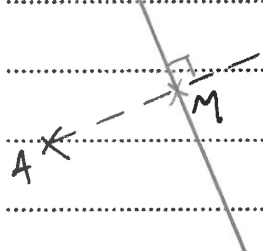
$$(x-5)(x-1) = 0$$

$$\underline{x=5} \text{ or } \underline{x=1}$$

- 10 (a) The coordinates of two points  $A$  and  $B$  are  $(-7, 3)$  and  $(5, 11)$  respectively.

Show that the equation of the perpendicular bisector of  $AB$  is  $3x + 2y = 11$ .

[4]



$$\begin{aligned}
 M &= \text{mid-point of } AB \\
 &= \left( \frac{-7+5}{2}, \frac{3+11}{2} \right) \\
 &= (-1, 7)
 \end{aligned}$$

$$\begin{aligned}
 \text{Gradient of } AB &= \frac{11-3}{5-(-7)} \\
 &= \frac{8}{12} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\text{Gradient of perpendicular bisector} = -\frac{3}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - 7 = -\frac{3}{2}(x - (-1))$$

$$y - 7 = -\frac{3}{2}(x + 1)$$

$$2y - 14 = -3(x + 1)$$

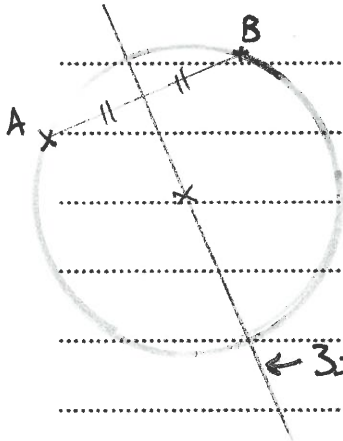
$$2y - 14 = -3x - 3$$

$$\underline{3x + 2y = 11} \quad \text{QED}$$

- (b) A circle passes through A and B and its centre lies on the line  $12x - 5y = 70$ .

Find an equation of the circle.

[5]



Centre of circle lies on both  
 $3x + 2y = 11$  and  $12x - 5y = 70$

Solve simultaneously:

←  $3x + 2y = 11$

$3x + 2y = 11 \times 5: 15x + 10y = 55$  ①

$12x - 5y = 70 \times 2: 24x - 10y = 140$  ②

① + ②:  $39x = 195$

$x = 5$

→  $3x + 2y = 11$

$3(5) + 2y = 11$

$15 + 2y = 11$

$2y = -4$

$y = -2$

→ Centre =  $(5, -2)$

$(x - 5)^2 + (y + 2)^2 = r^2$

Sub. A or B:  $(5, 11): (5 - 5)^2 + (11 + 2)^2 = r^2$

$0^2 + 13^2 = r^2$

$r^2 = 169$

→  $(x - 5)^2 + (y + 2)^2 = 169$

- 9 The equation of a circle is  $x^2 + y^2 + 6x - 2y - 26 = 0$ .

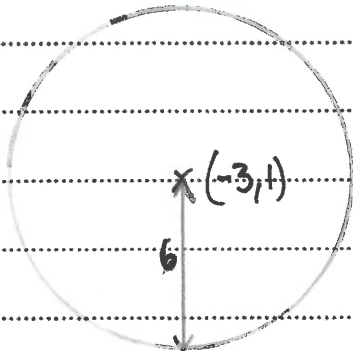
(a) Find the coordinates of the centre of the circle and the radius. Hence find the coordinates of the lowest point on the circle. [4]

$$x^2 + 6x + y^2 - 2y - 26 = 0$$

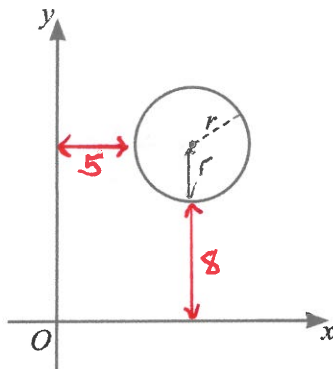
$$(x+3)^2 - 9 + (y-1)^2 - 1 - 26 = 0$$

$$(x+3)^2 + (y-1)^2 = 36$$

Centre =  $(-3, 1)$  radius = 6



Lowest point =  $(-3, -5)$

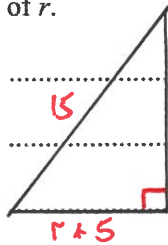


The diagram shows a circle  $C$  of radius  $r$ , where  $x > 0$  and  $y > 0$  for all points on  $C$ . The least distance between any point on  $C$  and the  $x$ -axis is 8 units, and the least distance between any point on  $C$  and the  $y$ -axis is 5 units.

- (a) State the coordinates of the centre of the circle in terms of  $r$ . [1]

$$(r+5, r+8)$$

- (b) Given that the distance between the origin and the centre of the circle is 15 units, find the value of  $r$ . [3]



$$(r+5)^2 + (r+8)^2 = 15^2$$

$$r^2 + 10r + 25 + r^2 + 16r + 64 = 225$$

$$2r^2 + 26r + 89 = 225$$

$$2r^2 + 26r - 136 = 0$$

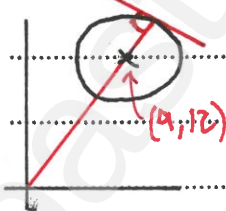
$$r^2 + 13r - 68 = 0$$

$$(r+17)(r-4) = 0$$

$$r = -17 \quad \underline{\underline{r = 4}}$$

- (c) The point on the circle furthest from the origin is denoted by  $P$ .

Find the gradient of the tangent to the circle at  $P$ . [2]



Find gradient of line going through centre:

$$m = \frac{12-0}{9-0} = \frac{4}{3}$$

$$\text{gradient of perpendicular} = \underline{\underline{-\frac{3}{4}}}$$

- 9 A circle has centre at the point  $B(5, 1)$ . The point  $A(-1, -2)$  lies on the circle.

(a) Find the equation of the circle.

[3]

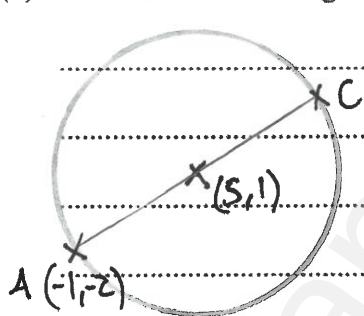
$$\begin{aligned} (x-5)^2 + (y-1)^2 &= r^2 \\ \text{Sub. } (-1, -2): (-1-5)^2 + (-2-1)^2 &= r^2 \\ (-6)^2 + (-3)^2 &= r^2 \\ 36 + 9 &= r^2 \\ r^2 &= 45 \end{aligned}$$

$$\rightarrow \underline{(x-5)^2 + (y-1)^2 = 45}$$

Point  $C$  is such that  $AC$  is a diameter of the circle. Point  $D$  has coordinates  $(5, 16)$ .

(b) Show that  $DC$  is a tangent to the circle.

[4]



$(5, 1)$  is the mid-point of  $AC$ :

$$(5, 1) = \left( \frac{x+(-1)}{2}, \frac{y+(-2)}{2} \right)$$

$$\frac{x-1}{2} = 5 \quad \frac{y-2}{2} = 1$$

$$x-1 = 10 \quad y-2 = 2$$

$$x = 11 \quad y = 4 \rightarrow C = (11, 4)$$

gradient of  $DC = \frac{16-4}{5-11}$

$$= \frac{12}{-6} = -2$$

gradient of  $AC = \frac{4-(-2)}{11-(-1)}$

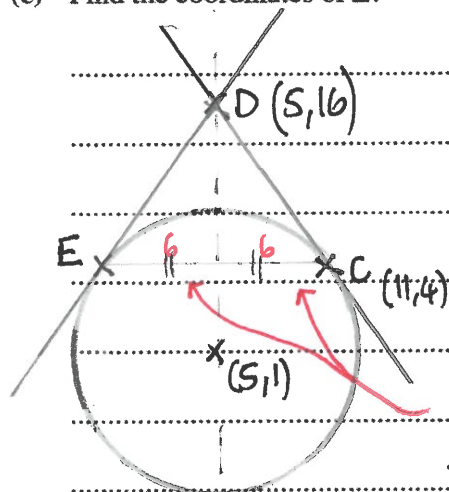
$$= \frac{6}{12} = \frac{1}{2}$$

$-2 \times \frac{1}{2} = -1$ , therefore  $DC$  is perpendicular to  $AC$ ,  
So  $DC$  is a tangent to the circle.

The other tangent from  $D$  to the circle touches the circle at  $E$ .

(c) Find the coordinates of  $E$ .

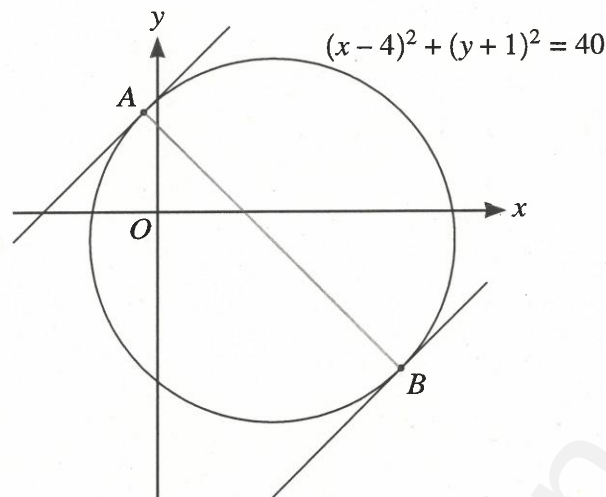
[2]



By symmetry,  $y$ -coordinate of  $E$  is the same as  $y$ -coordinate of  $C$ :  $y = 4$

By symmetry and using the fact that the radius bisects chord  $EC$ ,  
 $x$ -coordinate  $= 5 - 6$   
 $= -1$

$$E = \underline{\underline{(-1, 4)}}$$



The diagram shows the circle with equation  $(x-4)^2 + (y+1)^2 = 40$ . Parallel tangents, each with gradient 1, touch the circle at points A and B.

- (a) Find the equation of the line AB, giving the answer in the form  $y = mx + c$ . [3]

- AB must be a diameter, so it passes through the centre  $(4, -1)$ .
- Gradient of AB =  $-\frac{1}{1} = -1$

$$\begin{aligned}
 \rightarrow y - y_1 &= m(x - x_1) \\
 y - (-1) &= -1(x - 4) \\
 y + 1 &= -x + 4 \\
 \underline{y} &= \underline{-x + 3}
 \end{aligned}$$

- (b) Find the coordinates of A, giving each coordinate in surd form.

[4]

Solve  $y = -x + 3$  and equation of circle simultaneously:

$$(x - 4)^2 + (-x + 3 + 1)^2 = 40$$

$$(x - 4)^2 + (-x + 4)^2 = 40$$

$$x^2 - 8x + 16 + x^2 - 8x + 16 = 40$$

$$2x^2 - 16x + 32 = 40$$

$$2x^2 - 16x - 8 = 0$$

$$x^2 - 8x - 4 = 0$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-4)}}{2(1)}$$

$$x = 4 + 2\sqrt{5} \quad \text{or} \quad x = 4 - 2\sqrt{5}$$

point B (on the right)

✓ (point A)

Sub. into  $y = -x + 3$ :

$$y = -(4 - 2\sqrt{5}) + 3$$

$$= -4 + 2\sqrt{5} + 3$$

$$= -1 + 2\sqrt{5}$$

$$\rightarrow (4 - 2\sqrt{5}, -1 + 2\sqrt{5})$$

- (c) Find the equation of the tangent at A, giving the answer in the form  $y = mx + c$ , where  $c$  is in surd form. [2]

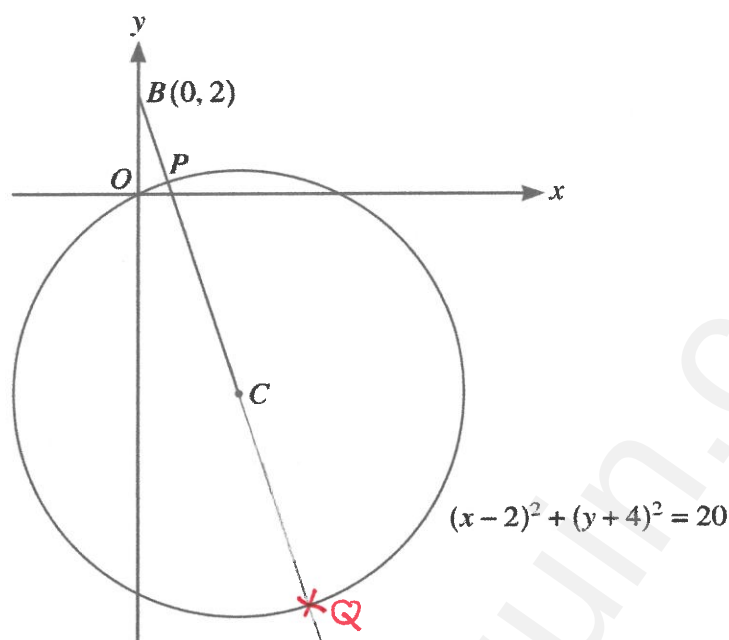
$$y - y_1 = m(x - x_1)$$

$$y - (-1 + 2\sqrt{5}) = 1(x - (4 - 2\sqrt{5}))$$

$$y + 1 - 2\sqrt{5} = x - 4 + 2\sqrt{5}$$

$$y = x - 5 + 4\sqrt{5}$$

7



The diagram shows the circle with equation  $(x-2)^2 + (y+4)^2 = 20$  and with centre  $C$ . The point  $B$  has coordinates  $(0, 2)$  and the line segment  $BC$  intersects the circle at  $P$ .

(a) Find the equation of  $BC$ .

[2]

$$\text{Centre } C = (2, -4)$$

$$\text{Gradient of } BC = \frac{-4-2}{2-0}$$

$$= \frac{-6}{2}$$

$$= -3$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -3(x - 0)$$

$$y - 2 = -3x$$

$$\underline{y = -3x + 2}$$

(b) Hence find the coordinates of  $P$ , giving your answer in exact form.

[5]

Sub.  $y = -3x + 2$  into eqn of circle:

$$(x-2)^2 + (-3x+2+4)^2 = 20$$

$$(x-2)^2 + (-3x+6)^2 = 20$$

$$x^2 - 4x + 4 + 9x^2 - 36x + 36 = 20$$

$$10x^2 - 40x + 40 = 20$$

$$10x^2 - 40x + 20 = 0$$

$$x^2 - 4x + 2 = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$

$$x = 2 + \sqrt{2} \quad \text{or} \quad x = 2 - \sqrt{2}$$

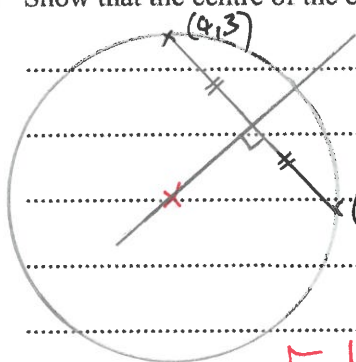
Further to the right, so  
this would be point Q  
on the diagram.

$$\begin{aligned} y &= -3x + 2 \\ &= -3(2 - \sqrt{2}) + 2 \\ &= -6 + 3\sqrt{2} + 2 \\ &= -4 + 3\sqrt{2} \end{aligned}$$

$$\rightarrow \underline{(2 - \sqrt{2}, 3\sqrt{2} - 4)}$$

- 10 Points  $A$  and  $B$  have coordinates  $(4, 3)$  and  $(8, -5)$  respectively. A circle with radius 10 passes through the points  $A$  and  $B$ .

(a) Show that the centre of the circle lies on the line  $y = \frac{1}{2}x - 4$ . [4]



Find mid-point of  $(4, 3)$  and  $(8, -5)$ :

$$\left( \frac{4+8}{2}, \frac{3+(-5)}{2} \right) = (6, -1)$$

Find gradient between  $A$  and  $B$ :

$$m = \frac{-5-3}{8-4}$$

$$= \frac{-8}{4}$$

$$= -2$$

Gradient of perpendicular:

$$m_{\text{perp}} = \frac{1}{2}$$

Eqn of line with  $m = \frac{1}{2}$  and point  $(6, -1)$ :

$$y - y_1 = m(x - x_1)$$

$$y - -1 = \frac{1}{2}(x - 6)$$

$$y + 1 = \frac{1}{2}x - 3$$

$$\underline{y = \frac{1}{2}x - 4} \quad \text{QED}$$

(b) Find the two possible equations of the circle.

[5]

Centre of circle lies on  $y = \frac{1}{2}x - 4$ :

centre:  $(x_1, \frac{1}{2}x_1 - 4)$

Equation of circle:

$$(x - x_1)^2 + (y - (\frac{1}{2}x_1 - 4))^2 = 10^2$$

$$(x - x_1)^2 + (y - \frac{1}{2}x_1 + 4)^2 = 100$$

Sub. a point on circumference (4, 3):

$$(4 - x_1)^2 + (3 - \frac{1}{2}x_1 + 4)^2 = 100$$

$$(4 - x_1)^2 + (7 - \frac{1}{2}x_1)^2 = 100$$

$$16 - 8x_1 + x_1^2 + 49 - 7x_1 + \frac{1}{4}x_1^2 = 100$$

$$\frac{5}{4}x_1^2 - 15x_1 + 65 = 100$$

$$\frac{5}{4}x_1^2 - 15x_1 - 35 = 0$$

$$5x_1^2 - 60x_1 - 140 = 0$$

$$x_1^2 - 12x_1 - 28 = 0$$

$$(x_1 - 14)(x_1 + 2) = 0$$

$$x_1 = 14 \quad \text{or} \quad x_1 = -2$$

 $x_1 = 14$ :

$$y = \frac{1}{2}x - 4$$

$$= \frac{1}{2}(14) - 4$$

$$= 7 - 4$$

$$= 3 \rightarrow \text{centre } (14, 3)$$

 $x = -2$ :

$$y = \frac{1}{2}x - 4$$

$$= \frac{1}{2}(-2) - 4$$

$$= -1 - 4$$

$$= -5 \rightarrow \text{centre } (-2, -5)$$

$$\rightarrow \underline{(x - 14)^2 + (y - 3)^2 = 100} \quad \text{or} \quad \underline{(x + 2)^2 + (y + 5)^2 = 100}$$

- 7 The point A has coordinates (1, 5) and the line  $l$  has gradient  $-\frac{2}{3}$  and passes through A. A circle has centre (5, 11) and radius  $\sqrt{52}$ .

(a) Show that  $l$  is the tangent to the circle at A. [2]

eq<sup>n</sup> of circle:  $(x-5)^2 + (y-11)^2 = 52$  ①

eq<sup>n</sup> of line:  $y - y_1 = m(x - x_1)$

$$y - 5 = -\frac{2}{3}(x - 1)$$

$$y - 5 = -\frac{2}{3}x + \frac{2}{3}$$

$$y = -\frac{2}{3}x + \frac{17}{3}$$
 ②

Sub. ② into ①:  $(x-5)^2 + (-\frac{2}{3}x + \frac{17}{3} - 11)^2 = 52$

$$(x-5)^2 + (-\frac{2}{3}x - \frac{16}{3})^2 = 52$$

$$x^2 - 10x + 25 + \frac{4}{9}x^2 + \frac{64}{9}x + \frac{256}{9} = 52$$

$$9x^2 - 90x + 225 + 4x^2 + 64x + 256 = 468$$

$$13x^2 - 26x + 13 = 0 \rightarrow b^2 - 4ac: (-2)^2 - 4(1)(1)$$

$$x^2 - 2x + 1 = 0 \quad = 4 - 4 = 0$$

discriminant = 0  
so there is only  
one intersection, so  
 $l$  is a tangent.

(b) Find the equation of the other circle of radius  $\sqrt{52}$  for which  $l$  is also the tangent at A. [3]

A is the mid-point of (5, 11) and C:

$(1, 5) = \left( \frac{5+x}{2}, \frac{11+y}{2} \right)$

$$\frac{5+x}{2} = 1 \quad \frac{11+y}{2} = 5$$

$$5+x = 2 \quad 11+y = 10$$

$$x = -3 \quad y = -1$$

$\rightarrow$  Centre:  $(-3, -1)$

$\rightarrow \underline{(x+3)^2 + (y+1)^2 = 52}$

- 12 A diameter of a circle  $C_1$  has end-points at  $(-3, -5)$  and  $(7, 3)$ .

(a) Find an equation of the circle  $C_1$ .

[3]

$$\text{mid-point: } \left( \frac{-3+7}{2}, \frac{-5+3}{2} \right) = (2, -1)$$

$$\rightarrow (x-2)^2 + (y+1)^2 = r^2$$

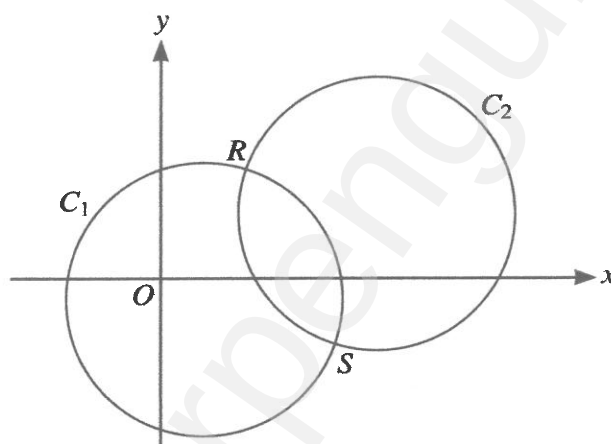
sub.  $(7, 3)$ :  $(7-2)^2 + (3+1)^2 = r^2$

$$5^2 + 4^2 = r^2$$

$$25 + 16 = r^2$$

$$r^2 = 41$$

$$(x-2)^2 + (y+1)^2 = 41$$



The circle  $C_1$  is translated by  $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$  to give circle  $C_2$ , as shown in the diagram.

(b) Find an equation of the circle  $C_2$ .

[2]

Translate centre by  $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$ : Centre of  $C_2$ :  $(10, 3)$

Radius is unchanged:

$$(x-10)^2 + (y-3)^2 = 41$$

The two circles intersect at points  $R$  and  $S$ .

- (c) Show that the equation of the line  $RS$  is  $y = -2x + 13$ .

[4]

Solve  $C_1$  and  $C_2$  simultaneously:

$$(x-2)^2 + (y+1)^2 = (x-10)^2 + (y-3)^2$$

(we can do this because they are both = 41)

$$x^2 - 4x + 4 + y^2 + 2y + 1 = x^2 - 20x + 100 + y^2 - 6y + 9$$

$$-4x + 2y + 5 = -20x - 6y + 109$$

$$8y = -16x + 104$$

$$y = -2x + 13 \quad \text{QED}$$

- (d) Hence show that the  $x$ -coordinates of  $R$  and  $S$  satisfy the equation  $5x^2 - 60x + 159 = 0$ .

[2]

Sub.  $y = -2x + 13$  into  $C_1$  or  $C_2$ :

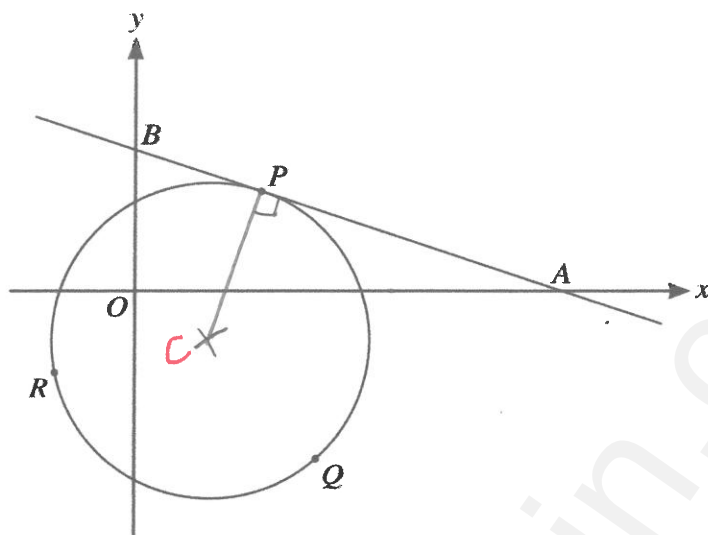
$$(x-2)^2 + (-2x+13+1)^2 = 41$$

$$(x-2)^2 + (-2x+14)^2 = 41$$

$$x^2 - 4x + 4 + 4x^2 - 56x + 196 = 41$$

$$5x^2 - 60x + 200 = 41$$

$$5x^2 - 60x + 159 = 0 \quad \text{QED}$$



The diagram shows the circle with equation  $x^2 + y^2 - 6x + 4y - 27 = 0$  and the tangent to the circle at the point  $P(5, 4)$ .

- (a) The tangent to the circle at  $P$  meets the  $x$ -axis at  $A$  and the  $y$ -axis at  $B$ .

Find the area of triangle  $OAB$ , where  $O$  is the origin.

[5]

find C:  $x^2 - 6x + y^2 + 4y - 27 = 0$   
 $(x-3)^2 - 9 + (y+2)^2 - 4 - 27 = 0$   
 $(x-3)^2 + (y+2)^2 = 40$   
 Centre =  $(3, -2)$

Gradient of  $CP = \frac{4 - (-2)}{5 - 3} = \frac{6}{2} = 3$

Gradient of tangent =  $-\frac{1}{3}$   $\rightarrow$   $0 = -\frac{1}{3}x + \frac{17}{3}$

$y - y_1 = m(x - x_1)$

$y - 4 = -\frac{1}{3}(x - 5)$

$y - 4 = -\frac{1}{3}x + \frac{5}{3}$

$y = -\frac{1}{3}x + \frac{17}{3}$

$\rightarrow B = (0, \frac{17}{3})$

$\frac{1}{3}x = \frac{17}{3}$

$x = 17$

$\rightarrow A = (17, 0)$

Area =  $\frac{1}{2} \times b \times h$

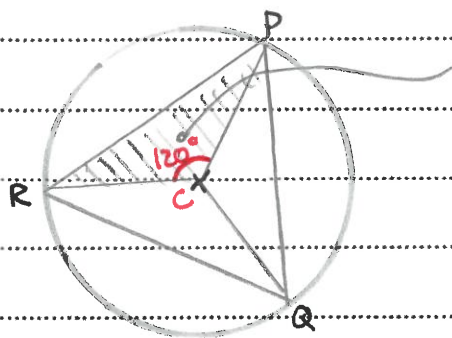
$= \frac{1}{2} \times 17 \times \frac{17}{3}$

$= \frac{289}{6}$

- (b) Points  $Q$  and  $R$  also lie on the circle, such that  $PQR$  is an equilateral triangle.

Find the exact area of triangle  $PQR$ .

[3]



$$\begin{aligned} \text{Area } \triangle PQR &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} \times \sqrt{40} \times \sqrt{40} \sin(120) \\ &= 10\sqrt{3} \end{aligned}$$

$$3 \times 10\sqrt{3} = \underline{\underline{30\sqrt{3}}}$$

- 10 The equation of a circle is  $(x-3)^2 + y^2 = 18$ . The line with equation  $y = mx + c$  passes through the point  $(0, -9)$  and is a tangent to the circle.

Find the two possible values of  $m$  and, for each value of  $m$ , find the coordinates of the point at which the tangent touches the circle. [8]

Straight line:  $y = mx + c$

intercept is  $(0, -9)$ :  $y = mx - 9$

Solve simultaneously:

$$(x-3)^2 + (mx-9)^2 = 18$$

$$x^2 - 6x + 9 + m^2x^2 - 18mx + 81 = 18$$

$$m^2x^2 + x^2 - 18mx - 6x + 72 = 0$$

$$(m^2 + 1)x^2 + (-18m - 6)x + 72 = 0 \quad (1)$$

line is a tangent to the circle, so  $b^2 - 4ac = 0$ :

$$(-18m - 6)^2 - 4(m^2 + 1)(72) = 0$$

$$324m^2 + 216m + 36 - 288m^2 - 288 = 0$$

$$36m^2 + 216m - 252 = 0$$

$$m^2 + 6m - 7 = 0$$

$$(m + 7)(m - 1) = 0$$

$$\underline{m = -7} \text{ or } \underline{m = 1}$$

Sub.  $m = -7$  into (1):

$$((-7)^2 + 1)x^2 + (-18(-7) - 6)x + 72 = 0$$

$$50x^2 + 120x + 72 = 0$$

$$25x^2 + 60x + 36 = 0$$

$$x = \frac{-60 \pm \sqrt{60^2 - 4(25)(36)}}{2(25)}$$

$$\underline{x = \frac{-6}{5}}$$

$$\begin{aligned}
 & \text{y-coord: } y = -7x - 9 \\
 & \quad = -7\left(\frac{-6}{5}\right) - 9 \\
 & \quad = \underline{\underline{\frac{-3}{5}}}
 \end{aligned}$$

so when  $m = -7$ , coordinate =  $\left(\frac{-6}{5}, \frac{-3}{5}\right)$

---

Sub.  $m = 1$  into ①:

$$\begin{aligned}
 (1^2 + 1)x^2 + (-18(1) - 6)x + 72 &= 0 \\
 2x^2 - 24x + 72 &= 0 \\
 x^2 - 12x + 36 &= 0 \\
 (x - 6)^2 &= 0 \\
 \underline{x = 6}
 \end{aligned}$$

$$\begin{aligned}
 & \text{y-coord: } y = 1x - 9 \\
 & \quad = 6 - 9 \\
 & \quad = \underline{\underline{-3}}
 \end{aligned}$$

so when  $m = 1$ , coordinate =  $(6, -3)$

---

11 The equation of a circle with centre  $C$  is  $x^2 + y^2 - 8x + 4y - 5 = 0$ .

(a) Find the radius of the circle and the coordinates of  $C$ .

[3]

$$x^2 - 8x + y^2 + 4y - 5 = 0$$

$$(x - 4)^2 - 16 + (y + 2)^2 - 4 - 5 = 0$$

$$(x - 4)^2 + (y + 2)^2 - 25 = 0$$

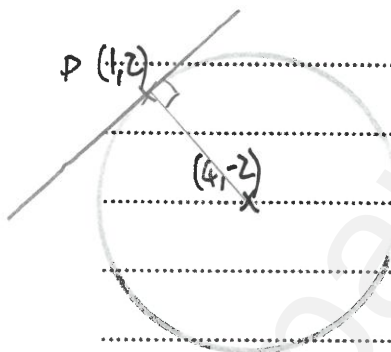
$$(x - 4)^2 + (y + 2)^2 = 25$$

Centre = (4, -2)      Radius = 5

The point  $P(1, 2)$  lies on the circle.

(b) Show that the equation of the tangent to the circle at  $P$  is  $4y = 3x + 5$ .

[3]



$$\text{Gradient of radius} = \frac{-2 - 2}{4 - 1}$$

$$= \frac{-4}{3}$$

$$\text{Gradient of tangent} = \frac{3}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{3}{4}(x - 1)$$

$$4y - 8 = 3(x - 1)$$

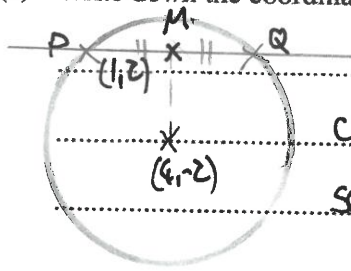
$$4y - 8 = 3x - 3$$

$$\underline{4y = 3x + 5} \quad \textcircled{1}$$

The point  $Q$  also lies on the circle and  $PQ$  is parallel to the  $x$ -axis.

- (c) Write down the coordinates of  $Q$ .

[2]



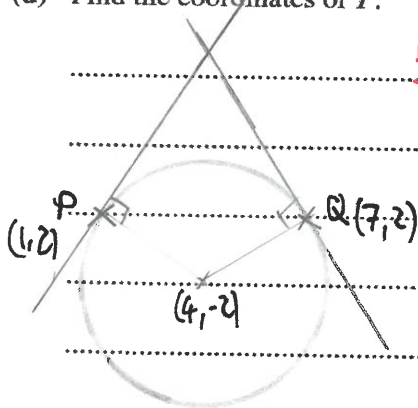
$y$ -coord of  $Q = y$ -coord of  $P = 2$   
 chord  $PQ$  bisects radius so  $PM = MQ = 3$   
 so  $x$ -coord of  $Q = 7$

$$\rightarrow Q = (7, 2)$$

The tangents to the circle at  $P$  and  $Q$  meet at  $T$ .

- (d) Find the coordinates of  $T$ .

[3]



Eqn of tangent at  $Q$ :

$$\text{gradient of radius} = \frac{2-2}{7-4} = \frac{4}{3}$$

$$\text{gradient of tangent} = -\frac{3}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{3}{4}(x - 7)$$

$$4y - 8 = -3(x - 7)$$

$$4y - 8 = -3x + 21$$

$$4y = -3x + 29 \quad (2)$$

Solve (1) + (2) simultaneously:

$$(1) = (2): \quad 3x + 5 = -3x + 29$$

$$6x = 24$$

$$x = 4$$

$$\text{sub into (1): } 4y = 3(4) + 5 \rightarrow 4y = 17$$

$$4y = 12 + 5 \rightarrow y = \frac{17}{4} \quad \left(4, \frac{17}{4}\right)$$

- 11 The coordinates of points A, B and C are A (5, -2), B (10, 3) and C (2p, p), where p is a constant.

(a) Given that AC and BC are equal in length, find the value of the fraction p. [3]

$$\begin{aligned} |AC| &= \sqrt{(2p-5)^2 + (p+2)^2} \\ &= \sqrt{4p^2 - 20p + 25 + p^2 + 4p + 4} \\ &= \sqrt{5p^2 - 16p + 29} \end{aligned}$$

$$\begin{aligned} |BC| &= \sqrt{(2p-10)^2 + (p-3)^2} \\ &= \sqrt{4p^2 - 40p + 100 + p^2 - 6p + 9} \\ &= \sqrt{5p^2 - 46p + 109} \end{aligned}$$

$$\begin{aligned} |AC| &= |BC|: \sqrt{5p^2 - 16p + 29} = \sqrt{5p^2 - 46p + 109} \\ 5p^2 - 16p + 29 &= 5p^2 - 46p + 109 \\ 30p &= 80 \\ p &= \frac{8}{3} \end{aligned}$$

(b) It is now given instead that AC is perpendicular to BC and that p is an integer.

(i) Find the value of p. [4]

$$\text{gradient of AC} = \frac{p+2}{2p-5} = \frac{p+2}{2p-5}$$

$$\text{gradient of BC} = \frac{p-3}{2p-10}$$

AC and BC are perpendicular, so  $m_{AC} \times m_{BC} = -1$ :

$$\frac{p+2}{2p-5} \times \frac{p-3}{2p-10} = -1$$

$$\frac{p^2 - p - 6}{4p^2 - 30p + 50} = -1$$

$$p^2 - p - 6 = -4p^2 + 30p - 50$$

$$5p^2 - 31p + 44 = 0$$

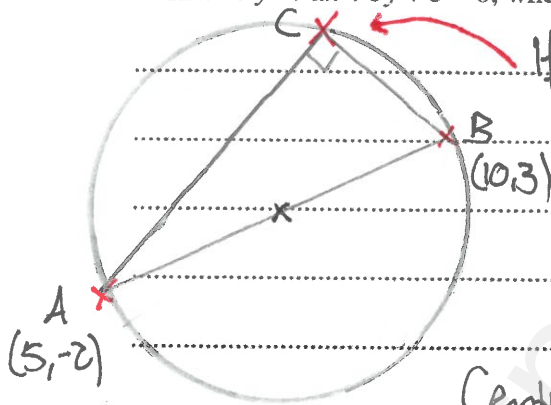
Cont...

$$p = \frac{-(-31) \pm \sqrt{(-31)^2 - 4(5)(44)}}{2(5)}$$

$$p = 4 \quad \text{or} \quad p = \frac{11}{5}$$

$p$  is an integer, so  $p = 4$

- (ii) Find the equation of the circle which passes through  $A$ ,  $B$  and  $C$ , giving your answer in the form  $x^2 + y^2 + ax + by + c = 0$ , where  $a$ ,  $b$  and  $c$  are constants. [4]



If  $p = 4$ , coordinates of  $C = (8, 4)$

Given  $\angle ACB$  is a right-angle,  
 $AB$  is a diameter.

$$\begin{aligned} \text{Centre} &= \text{mid-point of } AB = \left( \frac{5+10}{2}, \frac{-2+3}{2} \right) \\ &= \left( \frac{15}{2}, \frac{1}{2} \right) \end{aligned}$$

$$\left( x - \frac{15}{2} \right)^2 + \left( y - \frac{1}{2} \right)^2 = r^2$$

Sub.  $A, B$  or  $C: (8, 4):$

$$\left( 8 - \frac{15}{2} \right)^2 + \left( 4 - \frac{1}{2} \right)^2 = r^2$$

$$\left( \frac{1}{2} \right)^2 + \left( \frac{7}{2} \right)^2 = r^2$$

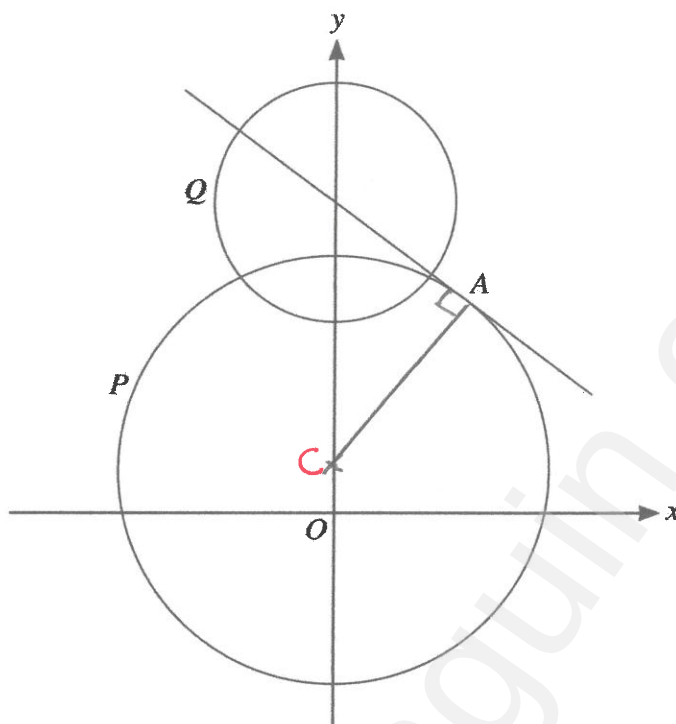
$$\frac{1}{4} + \frac{49}{4} = r^2$$

$$r^2 = \frac{50}{4}$$

$$\left( x - \frac{15}{2} \right)^2 + \left( y - \frac{1}{2} \right)^2 = \frac{50}{4}$$

$$x^2 - 15x + \frac{225}{4} + y^2 - y + \frac{1}{4} = \frac{50}{4}$$

$$\underline{x^2 + y^2 - 15x - y + 44 = 0}$$



The diagram shows a circle  $P$  with centre  $(0, 2)$  and radius 10 and the tangent to the circle at the point  $A$  with coordinates  $(6, 10)$ . It also shows a second circle  $Q$  with centre at the point where this tangent meets the  $y$ -axis and with radius  $\frac{5}{2}\sqrt{5}$ .

- (a) Write down the equation of circle  $P$ .

[1]

$$(x - 0)^2 + (y - 2)^2 = 10^2$$

$$x^2 + (y - 2)^2 = 100 \quad \textcircled{1}$$

- (b) Find the equation of the tangent to the circle  $P$  at  $A$ .

[2]

$$\begin{aligned} \text{Gradient of radius } AC &= \frac{10 - 2}{6 - 0} \\ &= \frac{8}{6} = \frac{4}{3} \end{aligned}$$

$$\text{Gradient of tangent} = -\frac{3}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y - 10 = -\frac{3}{4}(x - 6)$$

$$y - 10 = -\frac{3}{4}x + \frac{9}{2}$$

$$y = -\frac{3}{4}x + \frac{29}{2}$$

- (c) Find the equation of circle  $Q$  and hence verify that the y-coordinates of both of the points of intersection of the two circles are 11. \* intercept of line in part (a) [3]

Centre of  $Q$ :  $(0, \frac{29}{2})$  radius =  $\frac{5}{2}\sqrt{5}$  (given)

$$(x-0)^2 + (y-\frac{29}{2})^2 = (\frac{5}{2}\sqrt{5})^2$$

$$x^2 + (y-\frac{29}{2})^2 = \frac{125}{4} \quad (2)$$

from eqn (1):  $x^2 = 100 - (y-2)^2$

sub into (2):  $100 - (y-2)^2 + (y-\frac{29}{2})^2 = \frac{125}{4}$

$$100 - (y^2 - 4y + 4) + y^2 - 29y + \frac{841}{4} = \frac{125}{4}$$

$$100 - y^2 + 4y - 4 + y^2 - 29y + \frac{841}{4} = \frac{125}{4}$$

$$-25y + \frac{1225}{4} = \frac{125}{4}$$

$$-25y = -275$$

$$y = 11 \quad \text{QED}$$

- (d) Find the coordinates of the points of intersection of the tangent and circle  $Q$ , giving the answers in surd form. [3]

Sub.  $y = \frac{-3}{4}x + \frac{29}{2}$  into:  $x^2 + (y-\frac{29}{2})^2 = \frac{125}{4}$

$$x^2 + (\frac{-3}{4}x + \frac{29}{2} - \frac{29}{2})^2 = \frac{125}{4}$$

$$x^2 + (\frac{-3}{4}x)^2 = \frac{125}{4}$$

$$x^2 + \frac{9}{16}x^2 = \frac{125}{4}$$

$$\frac{25}{16}x^2 = \frac{125}{4}$$

$$x^2 = 20$$

$$x = \pm\sqrt{20}$$

$$x = \sqrt{20}: y = \frac{-3}{4}(\sqrt{20}) + \frac{29}{2}$$

$$= \frac{29}{2} - \frac{3\sqrt{20}}{4}$$

$$(\sqrt{20}, \frac{29}{2} - \frac{3\sqrt{20}}{4})$$

$$x = -\sqrt{20}: y = \frac{-3}{4}(-\sqrt{20}) + \frac{29}{2}$$

$$= \frac{29}{2} + \frac{3\sqrt{20}}{4}$$

$$(-\sqrt{20}, \frac{29}{2} + \frac{3\sqrt{20}}{4})$$

11 A circle with centre  $C$  has equation  $(x - 8)^2 + (y - 4)^2 = 100$ .

(a) Show that the point  $T(-6, 6)$  is outside the circle.

[3]

Sub.  $(-6, 6)$ :  $(-6-8)^2 + (6-4)^2$

$$= (-14)^2 + 2^2$$

$$= 196 + 4$$

$$= 200$$

distance from centre to  $(-6, 6) = \sqrt{200}$

$$\sqrt{200} > 10$$

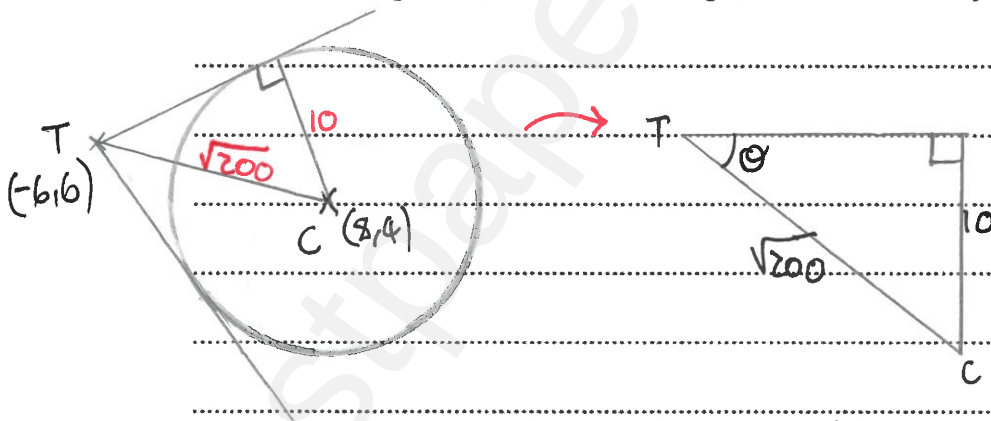
radius of circle

so  $T$  is outside circle.

Two tangents from  $T$  to the circle are drawn.

(b) Show that the angle between one of the tangents and  $CT$  is exactly  $45^\circ$ .

[2]



$$\sin \theta = \frac{10}{\sqrt{200}}$$

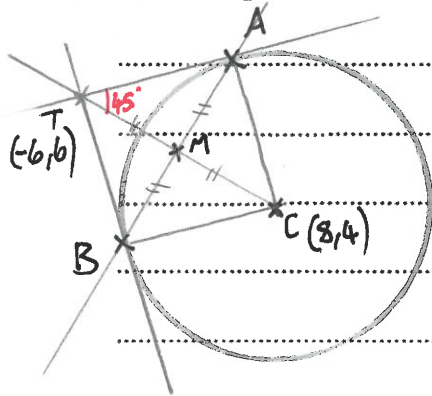
$$\theta = \sin^{-1} \left( \frac{10}{\sqrt{200}} \right)$$

$$= 45^\circ \quad \text{RED}$$

The two tangents touch the circle at A and B.

(c) Find the equation of the line AB, giving your answer in the form  $y = mx + c$ .

[4]



from (b)  $\angle ATB$  is a right-angle,  
so  $ACBT$  is a square.

Find point M:  $\left(\frac{8+(-6)}{2}, \frac{4+6}{2}\right) = (1, 5)$

$$\begin{aligned} \text{Gradient of } CT &= \frac{4-6}{8-(-6)} \\ &= \frac{-2}{14} = -\frac{1}{7} \end{aligned}$$

Gradient of AB = 7

$$y - y_1 = m(x - x_1)$$

$$y - 5 = 7(x - 1)$$

$$y - 5 = 7x - 7$$

$$\underline{y = 7x - 2}$$

(d) Find the x-coordinates of A and B.

[3]

Sub.  $y = 7x - 2$  into eqn of circle:

$$(x - 8)^2 + (7x - 2 - 4)^2 = 100$$

$$(x - 8)^2 + (7x - 6)^2 = 100$$

$$x^2 - 16x + 64 + 49x^2 - 84x + 36 = 100$$

$$50x^2 - 100x + 100 = 100$$

$$50x^2 - 100x = 0$$

$$x^2 - 2x = 0$$

$$x(x - 2) = 0$$

$$\underline{x = 0} \text{ or } \underline{x = 2}$$

- 10 The equation of a circle is  $(x - a)^2 + (y - 3)^2 = 20$ . The line  $y = \frac{1}{2}x + 6$  is a tangent to the circle at the point  $P$ .

(a) Show that one possible value of  $a$  is 4 and find the other possible value.

[5]

Sub.  $y = \frac{1}{2}x + 6$  into eqn of circle:

$$(x - a)^2 + \left(\frac{1}{2}x + 6 - 3\right)^2 = 20$$

$$(x - a)^2 + \left(\frac{1}{2}x + 3\right)^2 = 20$$

$$x^2 - 2ax + a^2 + \frac{1}{4}x^2 + 3x + 9 = 20$$

$$\frac{5}{4}x^2 + 3x - 2ax + a^2 + 9 - 20 = 0$$

$$\frac{5}{4}x^2 + (3 - 2a)x + a^2 - 11 = 0$$

If the line is a tangent,  $b^2 - 4ac = 0$ :

$$(3 - 2a)^2 - 4\left(\frac{5}{4}\right)(a^2 - 11) = 0$$

$$9 - 12a + 4a^2 - 5(a^2 - 11) = 0$$

$$9 - 12a + 4a^2 - 5a^2 + 55 = 0$$

$$-a^2 - 12a + 64 = 0$$

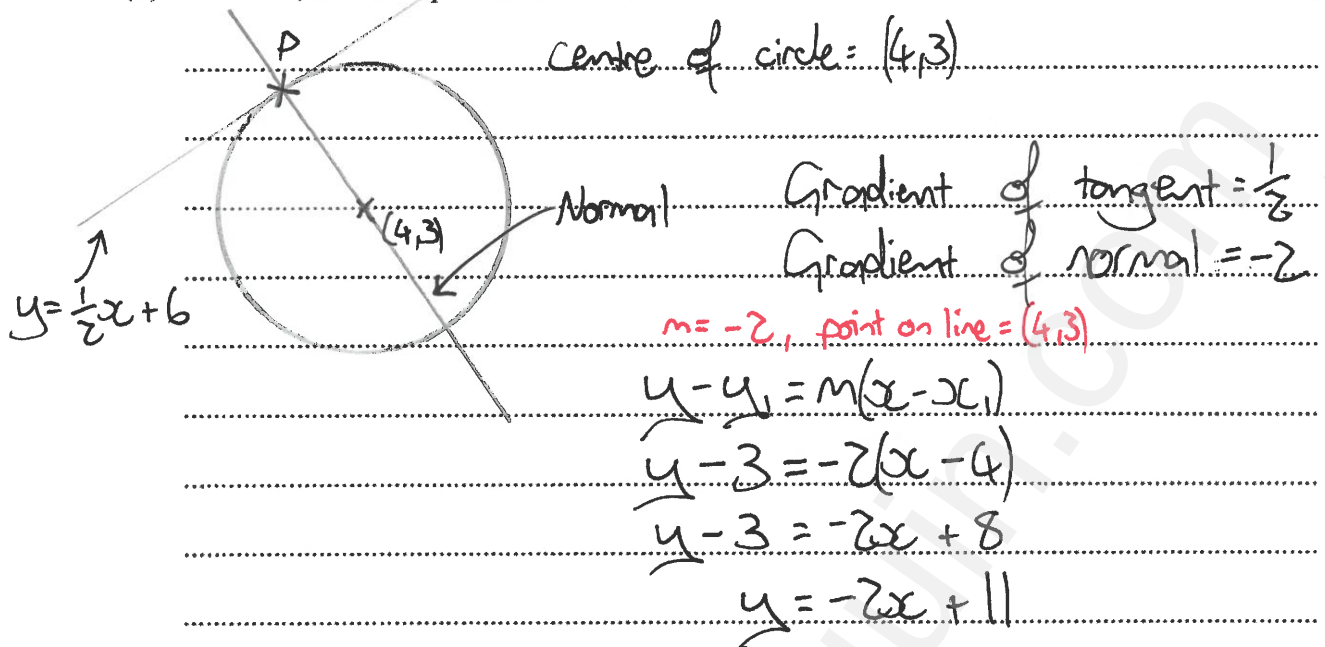
$$a^2 + 12a - 64 = 0$$

$$(a + 16)(a - 4) = 0$$

$$\underline{a = -16} \quad \underline{a = 4}$$

(b) For  $a = 4$ , find the equation of the normal to the circle at  $P$ .

[4]



(c) For  $a = 4$ , find the equations of the two tangents to the circle which are parallel to the normal found in (b).

[4]

