1 Solve the equation

$$\ln\left(1-e^{-2x}\right)+3=0.$$

Give your final answer correct to 4 decimal places. [3]

$$\ln(1-e^{-2x}) = -3$$

$$1 - e^{-2x} = e^{-3}$$

$$-e^{-2x} = e^{-3} - 1$$

$$e^{-2x} = -e^{-3} + 1$$

$$-2x = \ln(1-e^{-3})$$

$$x = \frac{-1}{2}\ln(1-e^{-3})$$

$$= 0.0255$$

2 The equation of a curve is $xy^2 + \ln(x + 2y) = 1$.

Find the gradient of the curve at the point where $x = 0$. [5]

differentiate: $\overset{u}{x}\overset{v}{y^2}$

$u = x$ $\qquad v = y^2$

$\frac{du}{dx} = 1$ $\qquad \frac{dv}{dx} = 2y\frac{dy}{dx}$

$\rightarrow 2xy\frac{dy}{dx} + y^2$

differentiate: $\ln(x + 2y)$ (chain rule):

$Y = \ln t$ $\qquad t = x + 2y$

$\frac{dY}{dt} = \frac{1}{t}$ $\qquad \frac{dt}{dx} = 1 + 2\frac{dy}{dx}$

$\frac{dY}{dx} = \frac{1}{t} \times \left(1 + 2\frac{dy}{dx}\right)$

$= \frac{1}{x+2y}\left(1 + 2\frac{dy}{dx}\right)$

Whole thing:

$2xy\frac{dy}{dx} + y^2 + \frac{1}{x+2y}\left(1 + 2\frac{dy}{dx}\right) = 0$

$\frac{1}{x+2y}\left(1 + 2\frac{dy}{dx}\right) = -y^2 - 2xy\frac{dy}{dx}$

$\times(x+2y)$ $\qquad \times(x+2y)$

$1 + 2\frac{dy}{dx} = (x+2y)\left(-y^2 - 2xy\frac{dy}{dx}\right)$

continued...

Need to know y-value when $x=0$ to sub. into gradient:

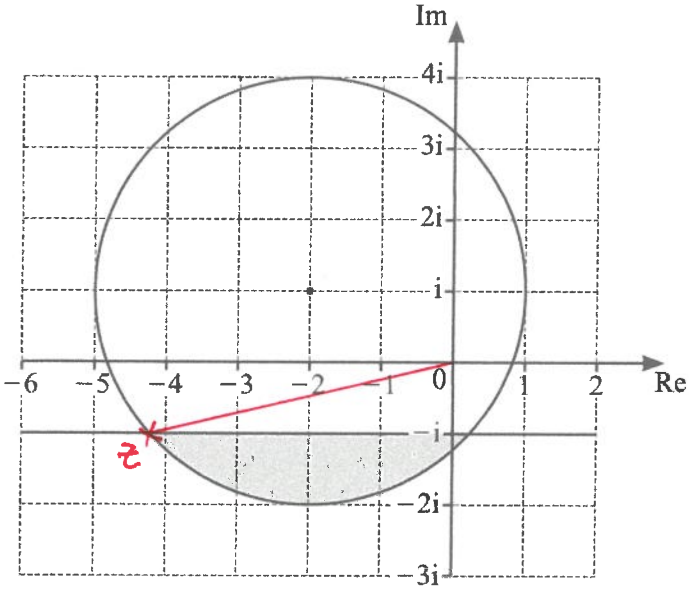
Sub. $x=0$ into original equation to find y:

$$xy^2 + \ln(x+2y) = 1$$
$$0 + \ln 2y = 1$$
$$\ln 2y = 1$$
$$2y = e^1$$
$$y = \frac{e}{2}$$

Now substitute $x=0$ and $y=\frac{e}{2}$ into gradient formula:

$$1 + 2\frac{dy}{dx} = (x+2y)\left(-y^2 - 2xy\frac{dy}{dx}\right)$$
$$1 + 2\frac{dy}{dx} = \left(0 + 2\times\frac{e}{2}\right)\left(-\left(\frac{e}{2}\right)^2 - 0\right)$$
$$1 + 2\frac{dy}{dx} = e \times -\frac{e^2}{4}$$
$$1 + 2\frac{dy}{dx} = -\frac{e^3}{4}$$
$$2\frac{dy}{dx} = \frac{-e^3}{4} - 1$$
$$\frac{dy}{dx} = \frac{-e^3}{8} - \frac{1}{2}$$

3



The shaded region on the Argand diagram shows points representing complex numbers z defined by two inequalities. The shaded region is bounded by a circle and a line parallel to the real axis. The boundaries of the region are included in the shaded region.

(a) Find two inequalities in terms of z that define the shaded region. [3]

line: $\operatorname{Im}(z) \leqslant -1$

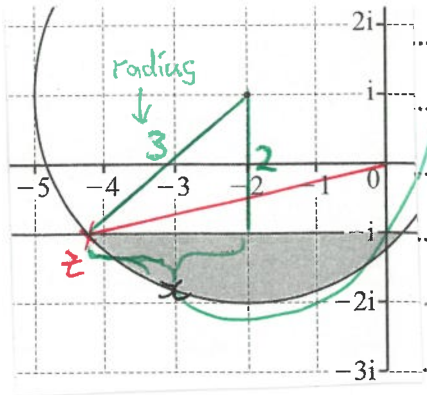
centre radius

Circle: $|z - (-2+i)| \leqslant 3$

$|z + 2 - i| \leqslant 3$

(b) Find the greatest value of $|z|$ for points in this region. [3]

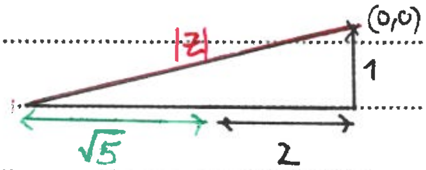
Greatest value of $|z|$ is at point shown on diagram.



$x^2 = 3^2 - 2^2$

$x = \sqrt{9-4}$

$x = \sqrt{5}$



$|z| = \sqrt{1^2 + (2+\sqrt{5})^2}$

$= \sqrt{1 + 4 + 4\sqrt{5} + 5}$

$= \sqrt{10 + 4\sqrt{5}}$

4 By first expressing the equation $\tan(x-60°) = 2\cot x$ as a quadratic equation in $\tan x$, solve the equation for $0° \leqslant x \leqslant 180°$. [6]

$\tan(x-60) = 2 \times \frac{1}{\tan x}$

$\frac{\tan x - \tan 60}{1 + \tan x \tan 60} = \frac{2}{\tan x}$

$\frac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x} = \frac{2}{\tan x}$

$\tan x(\tan x - \sqrt{3}) = 2(1 + \sqrt{3}\tan x)$

$\tan^2 x - \sqrt{3}\tan x = 2 + 2\sqrt{3}\tan x$

$\tan^2 x - 3\sqrt{3}\tan x - 2 = 0$

Let $Y = \tan x$:

$Y^2 - 3\sqrt{3}Y - 2 = 0$

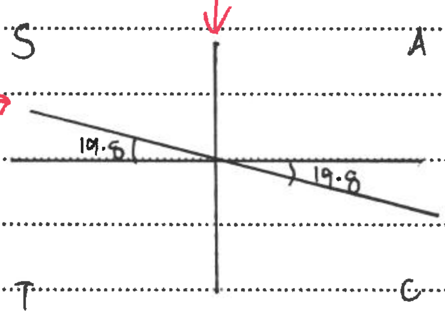
$Y = \frac{3\sqrt{3} \pm \sqrt{(-3\sqrt{3})^2 - 4(1)(-2)}}{2(1)}$

$Y = 5.556..$ or $Y = -0.3599..$

$\tan x = 5.556..$ $\quad$ $\tan x = -0.3599..$

$x = 79.8°$ $\quad$ $x = -19.8°$

180−19.8



$x = 160.2°$

$\rightarrow x = 79.8°, 160.2°$

5 The square roots of $-4+6\sqrt{5}\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $-4+6\sqrt{5}\mathrm{i}$ in exact Cartesian form. [5]

$$\sqrt{-4+6\sqrt{5}i} = x + iy$$
$$-4+6\sqrt{5}i = (x+iy)^2$$
$$-4+6\sqrt{5}i = x^2 + 2xyi - y^2$$

Re:

$$x^2 - y^2 = -4 \quad ①$$

Im:

$$2xy = 6\sqrt{5}$$
$$y = \frac{3\sqrt{5}}{x} \quad ②$$

Sub. ② into ①: $x^2 - \left(\frac{3\sqrt{5}}{x}\right)^2 = -4$

$$x^2 - \frac{45}{x^2} = -4 \quad \times x^2$$
$$x^4 - 45 = -4x^2$$
$$x^4 + 4x^2 - 45 = 0$$

Let $Y = x^2$:

$$Y^2 + 4Y - 45 = 0$$
$$(Y+9)(Y-5) = 0$$
$$Y = -9 \quad \text{or} \quad Y = 5$$
$$x^2 = -9 \qquad x^2 = 5$$

$\times$ No real sol^{n}s $\qquad$ $x = \pm\sqrt{5}$

Sub. $x = \sqrt{5}$ into ②:

$$y = \frac{3\sqrt{5}}{\sqrt{5}} = 3$$

Sub. $x = -\sqrt{5}$ into ②:

$$y = \frac{3\sqrt{5}}{-\sqrt{5}} = -3$$

$\rightarrow z = \sqrt{5} + 3i$ or $z = -\sqrt{5} - 3i$

6 The variables x and θ satisfy the differential equation

$$\frac{dx}{d\theta} = \left(\frac{1}{5}x + 1\right)\sin^2 2\theta,$$

and $x = 5$ when $\theta = 0$.

Solve the differential equation and obtain an expression for x in terms of θ. [7]

$\frac{1 \times 5}{\frac{1}{5}x + 1 \times 5}\, dx = \sin^2 2\theta \, d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$\sin^2 2\theta = \frac{1}{2} - \frac{1}{2}\cos 4\theta$

$\int \frac{5}{x+5}\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$

$5\int \frac{1}{x+5}\, dx = \frac{\theta}{2} - \frac{1}{8}\sin 4\theta + C$

$5\ln|x+5| = \frac{\theta}{2} - \frac{1}{8}\sin 4\theta + C$

Sub. $x = 5$, $\theta = 0$:

$5\ln 10 = 0 - 0 + C$

$C = 5\ln 10$

$\rightarrow 5\ln|x+5| = \frac{\theta}{2} - \frac{1}{8}\sin 4\theta + 5\ln 10$

$\ln|x+5| = \frac{\theta}{10} - \frac{1}{40}\sin 4\theta + \ln 10$

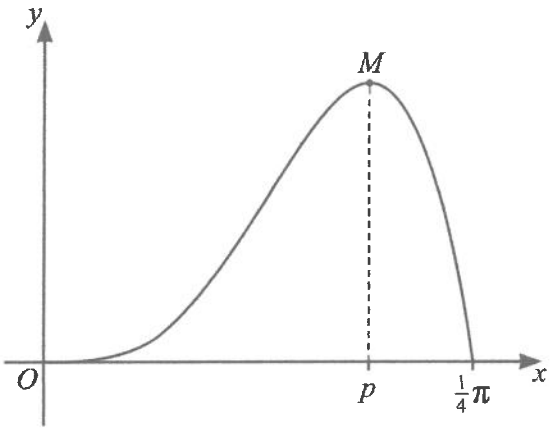
$x + 5 = e^{\frac{\theta}{10} - \frac{1}{40}\sin 4\theta + \ln 10}$

$x + 5 = e^{\ln 10} \times e^{\frac{\theta}{10} - \frac{1}{40}\sin 4\theta}$

$x + 5 = 10e^{\frac{\theta}{10} - \frac{1}{40}\sin 4\theta}$

$x = 10e^{\frac{\theta}{10} - \frac{1}{40}\sin 4\theta} - 5$

7



The diagram shows the curve $y = x^3 \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The curve has a maximum point at M, where $x = p$.

(a) Show that p satisfies the equation $p = \frac{1}{2}\tan^{-1}\left(\frac{3}{2p}\right)$. [3]

$y = \overset{u}{x^3} \overset{v}{\cos 2x}$

$u = x^3 \qquad v = \cos 2x$

$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = -2\sin 2x$

$\frac{dy}{dx} = -2x^3 \sin 2x + 3x^2 \cos 2x$

SP, so $\frac{dy}{dx} = 0$: $-2x^3 \sin 2x + 3x^2 \cos 2x = 0$

$2x^3 \sin 2x = 3x^2 \cos 2x$

$\div \cos 2x \qquad \div \cos 2x$

$2x^3 \tan 2x = 3x^2$

$\div 2x^3 \qquad \div 2x^3$

$\tan 2x = \frac{3}{2x}$

$2x = \tan^{-1}\left(\frac{3}{2x}\right)$

$x = \frac{1}{2}\tan^{-1}\left(\frac{3}{2x}\right)$ QED

(b) Show by calculation that $0.5 < p < 0.7$. [2]

Set $=0$: $p - \frac{1}{2}\tan^{-1}\left(\frac{3}{2p}\right) = 0$

$x = 0.5$: $0.5 - \frac{1}{2}\tan^{-1}\left(\frac{3}{2\times0.5}\right) = -0.125$

$x = 0.7$: $0.7 - \frac{1}{2}\tan^{-1}\left(\frac{3}{2\times0.7}\right) = 0.133$

Sign change (and continuous function) between 0.5 and 0.7 shows there's a root between 0.5 and 0.7.

(c) Use an iterative formula based on the equation in part (a) to calculate p correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$p_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{3}{2p_n}\right)$

$p_1 = 0.50000$
$p_2 = 0.62452$
$p_3 = 0.58814$
$p_4 = 0.59856$
$p_5 = 0.59556$
$p_6 = 0.59642$
$p_7 = 0.59617$

p_5, p_6 and p_7 all round to 0.596 to 3dp

8 Two lines have equations $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

(a) Show that the lines are skew. [5]

① Show they're not parallel:

If parallel, $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = t \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

$\mathbf{i}$: $2 = -t$, $t = -2$ $\quad$ $\mathbf{j}$: $3 = -2t$, $t = -3/2$ $\quad$ $\mathbf{k}$: $-1 = t$

No common solution for t, so not parallel.

② Show they don't intersect:

$$\begin{pmatrix} -1 + 2\lambda \\ 3 + 3\lambda \\ -4 - \lambda \end{pmatrix} = \begin{pmatrix} 2 - \mu \\ -3 - 2\mu \\ -1 + \mu \end{pmatrix}$$

$\mathbf{i}$: $-1 + 2\lambda = 2 - \mu$ $\quad$ $\mathbf{j}$: $3 + 3\lambda = -3 - 2\mu$

$-3 + 2\lambda = -\mu$ $\quad$ $3\lambda = -6 - 2\mu$ ②

$\mu = 3 - 2\lambda$ ①

Sub. ① into ②:

$3\lambda = -6 - 2(3 - 2\lambda)$

$3\lambda = -6 - 6 + 4\lambda$

$-\lambda = -12$

$\lambda = 12$

Sub into ①:

$\mu = 3 - 2 \times 12$

$\mu = -21$

See if these values work for $\mathbf{k}$:

$\mathbf{k}$: $-4 - \lambda = -1 + \mu$

$-4 - 12 = -1 - 21$

$-16 \neq -22 \rightarrow$ don't intersect $\therefore$ skew

(b) Find the obtuse angle between the directions of the two lines. [3]

$$\begin{pmatrix}2\\3\\-1\end{pmatrix} \cdot \begin{pmatrix}-1\\-2\\1\end{pmatrix} = \sqrt{2^2+3^2+(-1)^2}\sqrt{(-1)^2+(-2)^2+1^2}\cos\theta$$

direction vector of l_1

direction vector of l_2.

$$2\times-1 + 3\times-2 + -1\times1 = \sqrt{14}\times\sqrt{6}\times\cos\theta$$

$$-2 + -6 + -1 = \sqrt{84}\cos\theta$$

$$-9 = \sqrt{84}\cos\theta$$

$$\cos\theta = \frac{-9}{\sqrt{84}}$$

$$\theta = 169.1^\circ$$

9 The polynomial $6x^3+ax^2+bx+9$ is denoted by $p(x)$, where a and b are constants. It is given that $(x-3)$ is a factor of $p(x)$, and when the first derivative $p'(x)$ is divided by $(x-3)$ the remainder is 72.

(a) Find the values of a and b. [5]

$(x-3)$ is a factor : $p(3) = 0$

$6(3)^3 + a(3)^2 + b(3) + 9 = 0$

$162 + 9a + 3b + 9 = 0$

$9a + 3b = -171$ ①

differentiate $p(x)$:

$p'(x) = 18x^2 + 2ax + b$

when $\div(x-3)$, remainder $= 72$: $p'(3) = 72$:

$18(3)^2 + 2a(3) + b = 72$

$162 + 6a + b = 72$

$6a + b = -90$ $\times 3$

$18a + 3b = -270$ ②

② − ①: $9a = -99$

$\underline{a = -11}$

Sub. into ①: $9(-11) + 3b = -171$

$-99 + 3b = -171$

$3b = -72$

$\underline{b = -24}$

(b) When a and b have the values found in part **(a)**, factorise $\mathrm{p}(x)$ completely. [3]

$p(x) = 6x^3 - 11x^2 - 24x + 9$

$$\begin{array}{r|l} & 6x^2 + 7x - 3 \\ \hline x-3 & 6x^3 - 11x^2 - 24x + 9 \\ & \underline{6x^3 - 18x^2} \\ & \quad 7x^2 - 24x \\ & \quad \underline{7x^2 - 21x} \\ & \qquad -3x + 9 \\ & \qquad \underline{-3x + 9} \end{array}$$

$\rightarrow (x-3)(\overset{a}{6}x^2 + \overset{b}{7}x \overset{c}{-3})$

$\rightarrow ac = -18$, two numbers: $9, -2$

$6x^2 + 9x - 2x - 3$

$3x(2x+3) - 1(2x+3)$

$(3x-1)(2x+3)$

$\rightarrow \underline{(x-3)(3x-1)(2x+3)}$

(c) Hence solve the inequality $\mathrm{p}(x) < 0$. [2]

Critical values: $(x-3)(3x-1)(2x+3) = 0$

$x = 3 \quad x = \frac{1}{3} \quad x = \frac{-3}{2}$

$x < \frac{-3}{2}$	$\frac{-3}{2} < x < \frac{1}{3}$	$\frac{1}{3} < x < 3$	$x > 3$
e.g. -2	e.g. 0	e.g. 2	e.g. 4
$p(-2) = -35$	$p(0) = 9$	$p(2) = -35$	$p(4) = 121$
$-35 < 0$	$9 \not< 0$	$-35 < 0$	$121 \not< 0$
✓	✗	✓	✗

$\underline{x < \frac{-3}{2}}$ or $\underline{\frac{1}{3} < x < 3}$

10 Let $f(x) = \dfrac{-7x^2+2x-6}{(1+x)(4+x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{-7x^2+2x-6}{(1+x)(4+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{4+x^2}$$

$$-7x^2+2x-6 \equiv A(4+x^2) + (Bx+C)(1+x)$$

$x=-1$:

$$-7(-1)^2+2(-1)-6 = A(4+(-1)^2) + 0$$
$$-15 = 5A$$
$$\underline{A = -3}$$

$x=0$:

$$0+0-6 = A(4) + C(1)$$
$$-6 = 4A_{=-3} + C$$
$$-6 = -12 + C$$
$$\underline{C = 6}$$

$x=1$:

$$-7+2-6 = A(4+1) + (B+C)(1+1)$$
$$-11 = 5A + (B+C)\times 2$$
$$-11 = \underset{-3}{5A} + 2B + \underset{6}{2C}$$
$$-11 = -15 + 2B + 12$$
$$-8 = 2B$$
$$\underline{B = -4}$$

$$\rightarrow \underline{\underline{\frac{-3}{1+x} + \frac{-4x+6}{4+x^2}}}$$

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$. Give your answer in the form $a\pi - \ln b$, where a and b are constants. [6]

$$\int\left(\frac{-3}{1+x}+\frac{-4x+6}{4+x^2}\right)dx$$

$$=\int\left(\frac{-3}{1+x}+\frac{-4x}{4+x^2}+\frac{6}{4+x^2}\right)dx$$

$\int \frac{1}{x^2+a^2}$ with $a^2=4$ so $a=2$

$$=\int\left(-3\times\frac{1}{1+x}-2\times\frac{2x}{4+x^2}+6\times\frac{1}{4+x^2}\right)dx$$

$$=\left[-3\ln|1+x|-2\ln|4+x^2|+6\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$=\left[-3\ln(1+2)-2\ln(4+2^2)+3\tan^{-1}(1)\right]$$

$$-\left[-3\ln(1+0)-2\ln(4+0)+3\tan^{-1}(0)\right]$$

$$=\left[-3\ln 3-2\ln 8+\frac{3\pi}{4}\right]-\left[0-2\ln 4+0\right]$$

$$=-3\ln 3-2\ln 8+\frac{3\pi}{4}+2\ln 4$$

$$=-\ln 3^3-\ln 8^2+\ln 4^2+\frac{3\pi}{4}$$

$$=-\ln 27-\ln 64+\ln 16+\frac{3\pi}{4}$$

$$=\ln\left(\frac{16}{27\times 64}\right)+\frac{3\pi}{4}$$

$$=\ln\left(\frac{1}{108}\right)+\frac{3\pi}{4} \qquad =-\ln 108+\frac{3\pi}{4}$$

$$=\ln(108)^{-1}+\frac{3\pi}{4} \qquad =\frac{3\pi}{4}-\ln 108$$

11 Find the exact value of $\int_0^{\pi} \overset{u}{x^2} \overset{dv}{\cos\tfrac{1}{3}x}\,dx$. [6]

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$u = x^2 \qquad v = 3\sin\left(\tfrac{1}{3}x\right)$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \cos\left(\tfrac{1}{3}x\right)$

$\rightarrow 3x^2\sin\left(\tfrac{1}{3}x\right) - \int \overset{u}{6x}\,\overset{dv}{\sin\left(\tfrac{1}{3}x\right)}\,dx$

$u = 6x \qquad v = -3\cos\left(\tfrac{1}{3}x\right)$

$\frac{du}{dx} = 6 \qquad \frac{dv}{dx} = \sin\left(\tfrac{1}{3}x\right)$

$\rightarrow 3x^2\sin\left(\tfrac{1}{3}x\right) - \left[-18x\cos\left(\tfrac{1}{3}x\right) + \int 18\cos\left(\tfrac{1}{3}x\right)dx\right]$

$= \left[3x^2\sin\left(\tfrac{1}{3}x\right) + 18x\cos\left(\tfrac{1}{3}x\right) - 54\sin\left(\tfrac{1}{3}x\right)\right]_0^{\pi}$

$= \left[3\pi^2\sin\left(\tfrac{\pi}{3}\right) + 18\pi\cos\left(\tfrac{\pi}{3}\right) - 54\sin\left(\tfrac{\pi}{3}\right)\right] - [0]$

$= 3\pi^2 \times \frac{\sqrt{3}}{2} + 18\pi \times \frac{1}{2} - 54 \times \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}\pi^2 + 9\pi - 27\sqrt{3}$