1 A curve has equation $y = 5 + 3x - 2x^2$ and a straight line has equation $y = kx + 13$, where k is a constant.

Find the set of values of k for which the curve and the line do **not** meet. [4]

Solve simultaneously:

$$-2x^2 + 3x + 5 = kx + 13$$

$$-2x^2 + 3x - kx - 8 = 0$$

$$\underset{a}{-2x^2} + \underset{b}{(3-k)}x \underset{c}{- 8} = 0$$

Curve and line do not intersect, so $b^2 - 4ac < 0$:

$$(3-k)^2 - 4(-2)(-8) < 0$$

$$9 - 6k + k^2 - 64 < 0$$

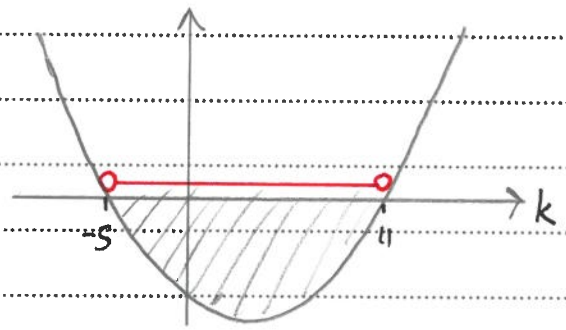
$$k^2 - 6k - 55 < 0$$

$$(k-11)(k+5) < 0$$

Critical values:

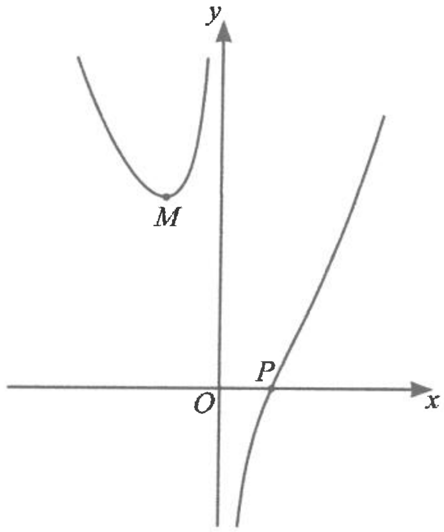
$$(k-11)(k+5) = 0$$

$$k = 11 \quad \text{or} \quad k = -5$$



$$\underline{\underline{-5 < k < 11}}$$

2



The diagram shows the curve with equation $y = 2x^2 - \frac{5}{x} + 3$. The curve crosses the x-axis at the point $P(1, 0)$ and M is a minimum point.

(a) Find the gradient of the curve at P. [2]

$y = 2x^2 - 5x^{-1} + 3$

$\frac{dy}{dx} = 4x + 5x^{-2}$

Sub. $x=1$:

gradient $= 4(1) + 5(1)^{-2}$

$= 9$

(b) Find the coordinates of M. Give each coordinate correct to 3 significant figures. [3]

Stationary point, so $\frac{dy}{dx} = 0$:

$4x + \frac{5}{x^2} = 0$

$4x = \frac{-5}{x^2}$ $\times x^2$

$4x^3 = -5$

$x^3 = -5/4$

$x = -1.08$ STO

$y = 2(-1.077..)^2 - \frac{5}{(-1.077)} + 3$

$y = 9.96$

$\rightarrow (-1.08, 9.96)$

3 **(a)** Find the complete expansion of $\left(2x - \frac{3}{x}\right)^4$. [4]

$$\left(2x - \frac{3}{x}\right)^4 = (2x)^4 + 4(2x)^3\left(\frac{-3}{x}\right) + 6(2x)^2\left(\frac{-3}{x}\right)^2 + 4(2x)\left(\frac{-3}{x}\right)^3 + \left(\frac{-3}{x}\right)^4$$

$$= 16x^4 + 4 \times 8x^3 \times \frac{-3}{x} + 6 \times 4x^2 \times \frac{9}{x^2} + 4 \times 2x \times \frac{-27}{x^3} + \frac{81}{x^4}$$

$$= \underline{16x^4 - 96x^2 + 216 - 216x^{-2} + 81x^{-4}}$$

(b) Hence determine the coefficient of x^2 in the expansion of $(x^2 + 5)\left(2x - \frac{3}{x}\right)^4$. [2]

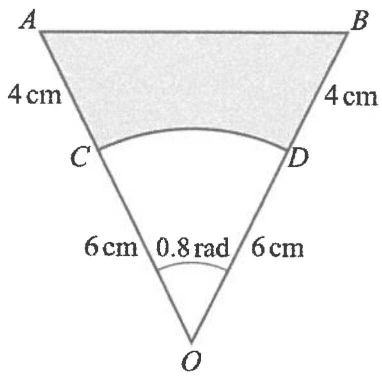
$$(x^2 + 5)(16x^4 - 96x^2 + 216 \ldots\ldots)$$

$$= 216x^2 - 480x^2$$

$$= -264x^2$$

coeff $= \underline{-264}$

4



The diagram shows a triangle OAB where $OA = OB = 10$ cm and angle $AOB = 0.8$ radians.
Points C and D on OA and OB respectively are such that the arc CD is part of a circle with centre O and radius 6 cm. The shaded region is bounded by the arc CD and the line segments CA, AB and BD.

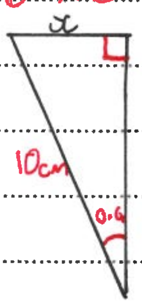
(a) Find the perimeter of the shaded region. [3]

Arc CD: Arc length $= r\theta$

$= 6 \times 0.8$

$= 4.8$ cm

Line AB:



← Triangle split in two

$\sin(0.4) = \dfrac{x}{10}$

$x = 10\sin(0.4)$

$= 3.894...$

$AB = 2 \times 3.894...$

$= 7.788...$ cm STO

Perimeter $= 4.8 + 4 + 4 + 7.788$

$= 20.6$ cm

(b) Find the area of the shaded region. [3]

Shaded region = triangle − Sector:

Triangle: Area $= \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 10 \times 10 \times \sin 0.8$

$= 35.867...$ STO

Sector: Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 6^2 \times 0.8$

$= 14.4$

→ $35.867 - 14.4 = 21.5\text{ cm}^2$

5 An arithmetic progression has first term 5 and common difference 6.

For this progression, find the sum of all the terms that lie between 150 and 400. [6]

$a = 5 \qquad d = 6$

Find first term greater than 150:

$a + (n-1)d > 150$

$5 + (n-1)6 > 150$

$5 + 6n - 6 > 150$

$6n > 151$

$n > 25.16\ldots$

$\rightarrow 26^{th}$ term

Find last term smaller than 400:

$5 + (n-1)6 < 400$

$5 + 6n - 6 < 400$

$6n < 401$

$n < 66.83\ldots$

$\rightarrow 66^{th}$ term

So we want the sum of the terms between u_{26} and u_{66} (inclusive)

all terms up to 66 all terms up to 25

$= S_{66} - S_{25}$

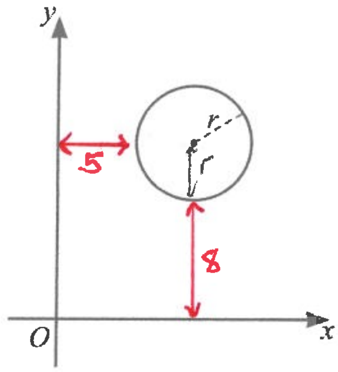
$= \frac{66}{2}[2(5) + 65(6)] - \frac{25}{2}[2(5) + 24(6)]$

$= 33 \times 400 - 12.5 \times 154$

$= 13\,200 - 1925$

$= 11\,275$

6

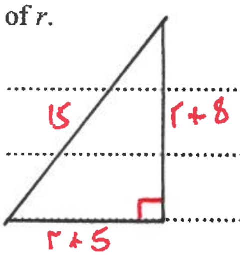


The diagram shows a circle C of radius r, where $x > 0$ and $y > 0$ for all points on C. The least distance between any point on C and the x-axis is 8 units, and the least distance between any point on C and the y-axis is 5 units.

(a) State the coordinates of the centre of the circle in terms of r. [1]

$(r+5\ ,\ r+8)$

(b) Given that the distance between the origin and the centre of the circle is 15 units, find the value of r. [3]



$(r+5)^2 + (r+8)^2 = 15^2$

$r^2 + 10r + 25 + r^2 + 16r + 64 = 225$

$2r^2 + 26r + 89 = 225$

$2r^2 + 26r - 136 = 0$

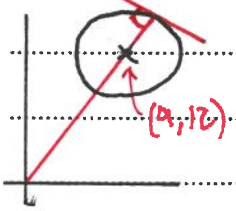
$r^2 + 13r - 68 = 0$

$(r+17)(r-4) = 0$

$r = -17$ X $\quad \underline{r = 4}$

(c) The point on the circle furthest from the origin is denoted by P.

Find the gradient of the tangent to the circle at P. [2]



Find gradient of line going through centre:

$m = \dfrac{12-0}{9-0} = \dfrac{4}{3}$

gradient of perpendicular $= \underline{\dfrac{-3}{4}}$

7 (a) Show that $3\tan^2\theta + 5\sin^2\theta \equiv \dfrac{8\sin^2\theta - 5\sin^4\theta}{1-\sin^2\theta}$. [3]

LHS:

$$\frac{3\sin^2\theta}{\cos^2\theta} + 5\sin^2\theta \times \frac{\cos^2\theta}{\cos^2\theta}$$

$$= \frac{3\sin^2\theta + 5\sin^2\theta\cos^2\theta}{\cos^2\theta} \quad 1-\sin^2\theta$$

$1-\sin^2\theta$

$$= \frac{3\sin^2\theta + 5\sin^2\theta(1-\sin^2\theta)}{1-\sin^2\theta}$$

$$= \frac{3\sin^2\theta + 5\sin^2\theta - 5\sin^4\theta}{1-\sin^2\theta}$$

$$= \frac{8\sin^2\theta - 5\sin^4\theta}{1-\sin^2\theta}$$

QED

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(b) Hence solve the equation $3\tan^2\theta + 5\sin^2\theta = 9$ for $0° < \theta < 270°$. [4]

from (a): $\dfrac{8\sin^2\theta - 5\sin^4\theta}{1-\sin^2\theta} = 9$

$8\sin^2\theta - 5\sin^4\theta = 9(1-\sin^2\theta)$

$8\sin^2\theta - 5\sin^4\theta = 9 - 9\sin^2\theta$

$17\sin^2\theta - 5\sin^4\theta - 9 = 0$

$-5\sin^4\theta + 17\sin^2\theta - 9 = 0$ $\times -1$

$5\sin^4\theta - 17\sin^2\theta + 9 = 0$

Let $Y = \sin^2\theta$:

$5Y^2 - 17Y + 9 = 0$

$Y = \dfrac{17 \pm \sqrt{(-17)^2 - 4(5)(9)}}{2(5)}$

$Y = 2.7440...$ or $Y = 0.6559...$

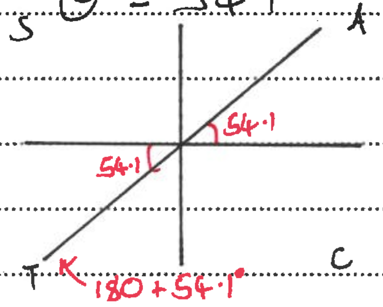
$\sin^2\theta = 2.7440...$ $\quad\quad$ $\sin^2\theta = 0.6559...$

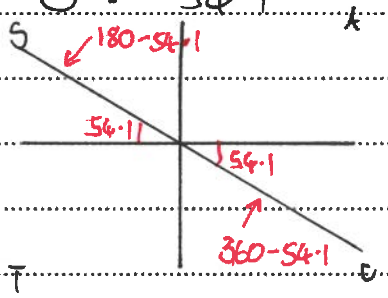
$\sin\theta = \pm 1.6565...$ $\quad\quad$ $\sin\theta = \pm 0.8099...$

✗ No sol^{n}s

$\sin\theta = 0.8099...$ $\quad\quad$ $\sin\theta = -0.8099$

$\theta = 54.1°$ $\quad\quad$ $\theta = -54.1°$





$\theta = 54.1°, 234.1°$ $\quad\quad$ $\theta = 125.9°, 305.9°$ outside range

$\underline{\theta = 54.1°, 125.9°, 234.1°}$

8 A geometric progression is such that its second term is -120 and its sum to infinity is 160.

(a) Find the common ratio. [4]

u_2: $ar = -120$ ①

S_∞: $\frac{a}{1-r} = 160$ $\times(1-r)$

$a = 160(1-r)$

$a = 160 - 160r$ ②

sub. ② into ①:

$(160-160r)r = -120$

$160r - 160r^2 = -120$

$-160r^2 + 160r + 120 = 0$ $\div -40$

$4r^2 - 4r - 3 = 0$

ac = −12
two numbers: −6, 2

$4r^2 + 2r - 6r - 3 = 0$

$2r(2r+1) - 3(2r+1) = 0$

$(2r-3)(2r+1) = 0$

$r = \frac{3}{2}$ or $r = \frac{-1}{2}$

Sum to infinity is only possible if sequence is converging, ie. $|r|<1$, so:

$\underline{r = \frac{-1}{2}}$ only

(b) The first nine terms of the progression are now removed.

Find the sum to infinity of the remaining terms of the progression. [3]

Sub. $r = \frac{-1}{2}$ into ①:

$a \times \frac{-1}{2} = -120$

$\underline{a = 240}$

Sum of first 9 terms:

$S_9 = \frac{240\left(1-\left(\frac{-1}{2}\right)^9\right)}{1 - \frac{-1}{2}}$

$\underline{S_9 = 160.3125}$

Sum of remaining terms $= S_\infty - S_9$

$= 160 - 160.3125$

$= \underline{-0.3125}$

9 A curve is such that $\frac{d^2y}{dx^2} = \frac{6}{x^4} - \frac{5}{x^3}$. It is given that the curve has a stationary point at $\left(\frac{1}{2}, 9\right)$.

(a) Use the expression for $\frac{d^2y}{dx^2}$ to determine whether the stationary point is a maximum or a minimum point. [2]

Sub. $x = \frac{1}{2}$ into $\frac{d^2y}{dx^2}$: $= \frac{6}{\left(\frac{1}{2}\right)^4} - \frac{5}{\left(\frac{1}{2}\right)^3}$

$= 96 - 40$

$= 56$ $\frac{d^2y}{dx^2} > 0$, so minimum

(b) Find the equation of the curve. [7]

$\frac{dy}{dx} = \int \left(6x^{-4} - 5x^{-3}\right) dx$

$= -2x^{-3} + \frac{5}{2}x^{-2} + c$

$\frac{dy}{dx} = 0$ at $x = \frac{1}{2}$:

$0 = -2\left(\frac{1}{2}\right)^{-3} + \frac{5}{2}\left(\frac{1}{2}\right)^{-2} + c$

$0 = -16 + 10 + c$

$c = 6$

$\rightarrow \frac{dy}{dx} = -2x^{-3} + \frac{5}{2}x^{-2} + 6$

$y = \int \left(-2x^{-3} + \frac{5}{2}x^{-2} + 6\right) dx$

$= x^{-2} - \frac{5}{2}x^{-1} + 6x + c$

Sub. $\left(\frac{1}{2}, 9\right)$:

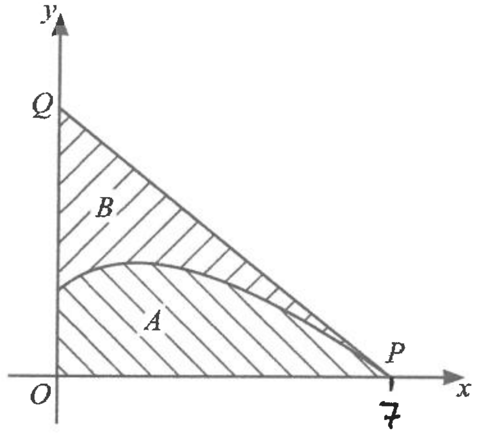
$9 = \left(\frac{1}{2}\right)^{-2} - \frac{5}{2}\left(\frac{1}{2}\right)^{-1} + 6\left(\frac{1}{2}\right) + c$

$9 = 4 - 5 + 3 + c$

$c = 7$

$\rightarrow y = x^{-2} - \frac{5}{2}x^{-1} + 6x + 7$

10



The diagram shows the curve with equation

$$y = 4(3x+4)^{\frac{1}{2}} - 2x - 6$$

for values of x such that $0 \leqslant x \leqslant 7$. The tangent to the curve at the point $P(7, 0)$ meets the y-axis at the point Q. Region A is bounded by the curve and the two axes. Region B is bounded by the curve, the line segment PQ and the y-axis.

(a) Find the area of region A. [4]

$\text{Area} = \int_0^7 \left[4(3x+4)^{\frac{1}{2}} - 2x - 6\right] dx$

$= \left[4 \times \frac{2}{3} \times \frac{1}{3}(3x+4)^{3/2} - x^2 - 6x\right]_0^7$

$= \left[\frac{8}{9}(3x+4)^{3/2} - x^2 - 6x\right]_0^7$

$= \left[\frac{8}{9}(21+4)^{3/2} - 7^2 - 42\right] - \left[\frac{8}{9}(4)^{3/2} - 0 - 0\right]$

$= \left[\frac{8}{9}(25)^{3/2} - 49 - 42\right] - \left[\frac{8}{9} \times 8\right]$

$= \left[\frac{8}{9} \times 125 - 91\right] - \left[\frac{64}{9}\right]$

$= \frac{181}{9} - \frac{64}{9}$

$= \underline{\underline{13}}$

(b) Find the area of region B. [5]

Find equation of tangent to curve to be able to find Q:

gradient: $\frac{dy}{dx} = 4 \times \frac{1}{2} \times 3(3x+4)^{-\frac{1}{2}} - 2$

$= 6(3x+4)^{-\frac{1}{2}} - 2$

Sub. $x = 7$:

gradient $= 6(21+4)^{-\frac{1}{2}} - 2$

$= 6(25)^{-\frac{1}{2}} - 2$

$= 6 \times \frac{1}{5} - 2$

$= \frac{-4}{5}$

eqn of line:

$y - y_1 = m(x - x_1)$

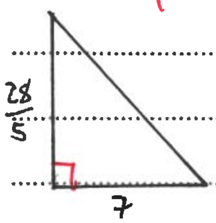
$y - 0 = \frac{-4}{5}(x - 7)$

$y = \frac{-4}{5}x + \frac{28}{5}$

y-intercept of this line is at Q, so:

$Q = (0, \frac{28}{5})$

Area of triangle OQP:



Area $= \frac{1}{2} \times 7 \times \frac{28}{5}$

$= 19.6$

Region B = Area of triangle − answer from (a)

$= 19.6 - 13$

$= 6.6$

11 Functions f and g are defined for all real values of x by

$$f(x) = 4x^2 - c \quad \text{and} \quad g(x) = 2x + k,$$

where c and k are positive constants. It is given that $g^{-1}(3k+1) = c$.

(a) Show that $gf(x) = 8x^2 - k - 1$. [4]

Use the info about g(x) and $g^{-1}(x)$ first:

find $g^{-1}(x)$: $g(x) = 2x + k$

$y = 2x + k$

$y - k = 2x$

$x = \frac{y-k}{2}$

$g^{-1}(x) = \frac{x-k}{2}$

$g^{-1}(3k+1) = c$:

$\frac{(3k+1) - k}{2} = c$

$\frac{2k+1}{2} = c$

Sub. into f(x):

$f(x) = 4x^2 - \frac{2k+1}{2}$

gf(x):

$gf(x) = 2\left[4x^2 - \frac{2k+1}{2}\right] + k$

$= 8x^2 - (2k+1) + k$

$= 8x^2 - 2k - 1 + k$

$= 8x^2 - k - 1$ QED

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(b) The curve with equation $y = 8x^2 - k - 1$ is transformed to the curve with equation $y = \mathrm{h}(x)$ by the following sequence of transformations.

1 Translation of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

2 Stretch in the y-direction by scale factor k

3 Reflection in the x-axis

Find an expression for $\mathrm{h}(x)$ in terms of x and k. [3]

1: $f(x) \to f(x-2)+3$:

$8(x-2)^2 - k - 1 + 3$

$8(x-2)^2 - k + 2$

2: $f(x) \to kf(x)$:

$8k(x-2)^2 - k^2 + 2k$

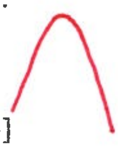
3: $f(x) \to -f(x)$:

$-8k(x-2)^2 + k^2 - 2k$

(c) The range of h is given by $\mathrm{h}(x) \leqslant 15$.

Find the values of c and k. [3]

Negative Quadratic



$\mathrm{h}(x) = -8k(x-2)^2 + k^2 - 2k$

If the range is $h(x) \leq 15$, then the maximum point is at $h(x) = 15$:

$h(x) = \underbrace{-8k(x-2)^2}_{} + k^2 - 2k$

When finding max. point, need to make this as big as possible. Biggest it can be is zero when $x = 2$.

At max point: $h(x) = 0 + k^2 - 2k$

$15 = k^2 - 2k$

$k^2 - 2k - 15 = 0$

$(k-5)(k+3) = 0$

$k = 5$ ✓ or $k = -3$ ✗ k is positive

Sub. into c:

$c = \dfrac{2(5)+1}{2}$

$c = 5.5$