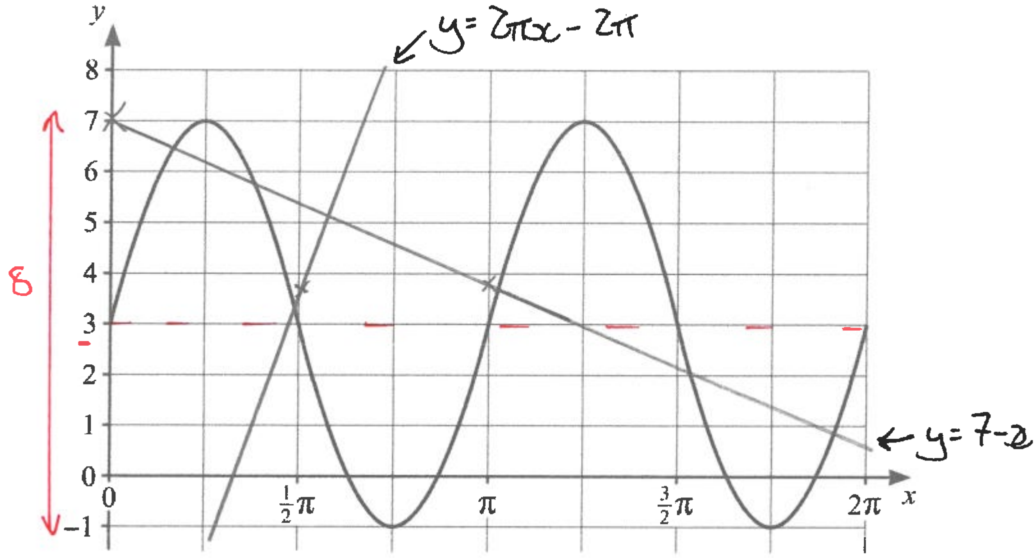
1



The diagram shows the curve with equation $y = a\sin(bx) + c$ for $0 \leqslant x \leqslant 2\pi$, where a, b and c are positive constants.

(a) State the values of a, b and c. [3]

Amplitude has been multiplied by 4, so $a = 4$

Period is π, where it is usually 2π, so $b = 2$

"middle" of graph vertically is 3, where it is usually 0, so $c = 3$

(b) For these values of a, b and c, determine the number of solutions in the interval $0 \leqslant x \leqslant 2\pi$ for each of the following equations:

(i) $a\sin(bx) + c = 7 - x$ [1]

Plot graph of $y = 7 - x$ (passes through $(0, 7)$ and $(\pi, 7 - \pi)$) $= 3.86$

Crosses in 5 places, so 5 solutions.

(ii) $a\sin(bx) + c = 2\pi(x - 1)$. [1]

-6.28

Plot $y = 2\pi x - 2\pi$ (passes through $(0, -2\pi)$, $(\frac{\pi}{2}, \pi^2 - 2\pi)$) $= 3.59$

Crosses in 1 place, so 1 solution.

Not many points fit on the axes

2 The first term of an arithmetic progression is −20 and the common difference is 5.

(a) Find the sum of the first 20 terms of the progression. [2]

Sequence: $-20, -15, -10, -5, \ldots$

$$S_{20} = \frac{20}{2}\left[2(-20) + 19(5)\right]$$
$$= 10\left[-40 + 95\right]$$
$$= 10 \times 55 = \underline{550}$$

It is given that the sum of the first $2k$ terms is 10 times the sum of the first k terms.

(b) Find the value of k. [3]

Sum of first 2k terms:

$$S_{2k} = \frac{2k}{2}\left[2(-20) + (2k-1)\times 5\right]$$
$$= k\left[-40 + 10k - 5\right]$$
$$= k(10k - 45)$$
$$= \underline{10k^2 - 45k}$$

Sum of first k terms:

$$S_k = \frac{k}{2}\left[2(-20) + (k-1)\times 5\right]$$
$$= \frac{k}{2}\left[-40 + 5k - 5\right]$$
$$= \frac{k}{2}(5k - 45)$$
$$= \underline{2.5k^2 - 22.5k}$$

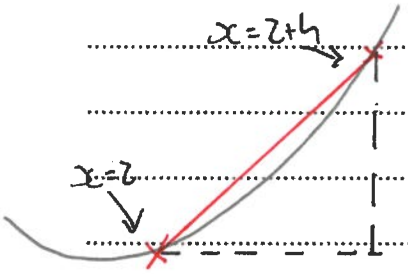
$S_{2k} = 10 \times S_k$:

$$10k^2 - 45k = 10(2.5k^2 - 22.5k)$$
$$10k^2 - 45k = 25k^2 - 225k$$
$$15k^2 - 180k = 0 \qquad k(k-12) = 0$$
$$k^2 - 12k = 0 \qquad k = 0 \text{ or } \underline{k = 12}$$

0 terms.

3 The equation of a curve is $y = 2x^2 - 3$. Two points A and B with x-coordinates 2 and $(2+h)$ respectively lie on the curve.

(a) Find and simplify an expression for the gradient of the chord AB in terms of h. [3]



Find y-coords:

$x=2$: $y = 2(2)^2 - 3$
$= 8 - 3 = \underline{5} \rightarrow (2,5)$

$x = 2+h$: $y = 2(2+h)^2 - 3$
$= 2(4 + 4h + h^2) - 3$
$= 8 + 8h + 2h^2 - 3$
$= \underline{2h^2 + 8h + 5} \rightarrow (2+h, 2h^2+8h+5)$

gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2h^2 + 8h + 5 - 5}{2+h-2}$
$= \dfrac{2h^2 + 8h}{h} = 2h + 8$

(b) Explain how the gradient of the curve at the point A can be deduced from the answer to part **(a)**, and state the value of this gradient. [2]

To find the gradient at A, imagine making the chord as small as possible, so h becomes closer and closer to zero.

When $h \rightarrow 0$, gradient $= 2(0) + 8$
$= \underline{8}$

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4 Find the term independent of x in the expansion of each of the following:

(a) $\left(x+\dfrac{3}{x^2}\right)^6$ [2]

$\left(x+\dfrac{3}{x^2}\right)^6 = x^6 + 6x^5\left(\dfrac{3}{x^2}\right) + 15x^4\left(\dfrac{3}{x^2}\right)^2 * + 20x^3\left(\dfrac{3}{x^2}\right)^3 *$

$+ 15x^2\left(\dfrac{3}{x^2}\right)^4 + 6x\left(\dfrac{3}{x^2}\right)^5 + \left(\dfrac{3}{x^2}\right)^6$

$* \; 15x^4\left(\dfrac{3}{x^2}\right)^2 = 15\cancel{x^4} \times \dfrac{9}{\cancel{x^4}}$

$= 15 \times 9$

$= \underline{\underline{135}}$

(b) $(4x^3-5)\left(x+\dfrac{3}{x^2}\right)^6$. [4]

$* \; 20x^3\left(\dfrac{3}{x^2}\right)^3 = 20x^3 \times \dfrac{27}{x^6}$

$= \dfrac{540}{x^3}$

← from (a)

$\rightarrow (4x^3-5)\left(135 + \dfrac{540}{x^3}\right)$

$= 4\cancel{x^3} \times \dfrac{540}{\cancel{x^3}} - 5 \times 135$

$= 2160 - 675$

$= \underline{1485}$

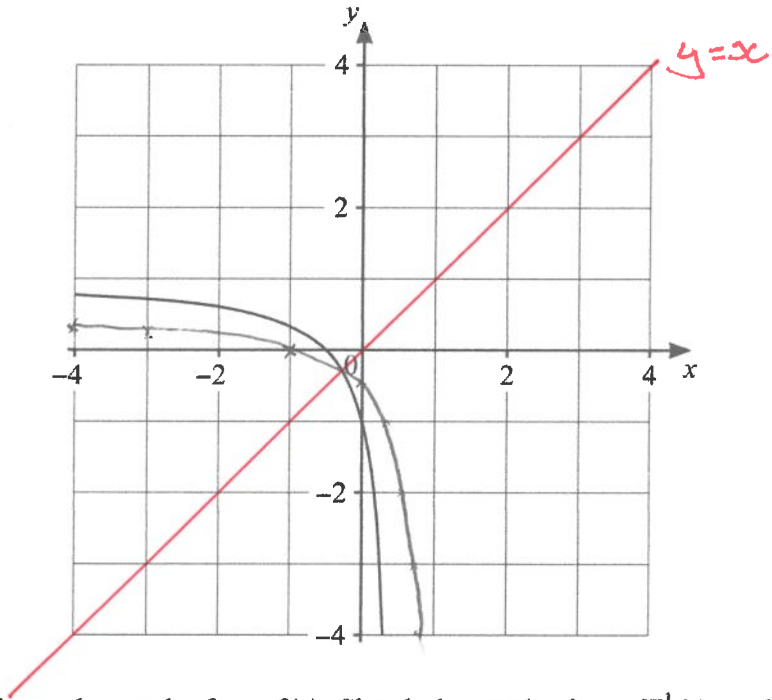
5 The function f is defined by $f(x) = \frac{2x+1}{2x-1}$ for $x < \frac{1}{2}$.

(a) **(i)** State the value of $f(-1)$. [1]

$f(-1) = \frac{2(-1)+1}{2(-1)-1}$

$= \frac{-1}{-3} = \frac{1}{3}$

(ii)



The diagram shows the graph of $y = f(x)$. Sketch the graph of $y = f^{-1}(x)$ on this diagram. Show any relevant mirror line. [2]

(iii) Find an expression for $f^{-1}(x)$ and state the domain of the function f^{-1}. [4]

$y = \frac{2x+1}{2x-1}$

$y(2x-1) = 2x+1$

$2xy - y = 2x+1$

$2xy - 2x = y+1$

$x(2y-2) = y+1$

$x = \frac{y+1}{2y-2}$

$f^{-1}(x) = \frac{x+1}{2x-2}$

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Domain of $f^{-1}(x)$ = Range of $f(x)$:

$f(x) = \frac{2x+1}{2x-1} \quad x < \frac{1}{2}$

Sub. something a tiny bit smaller than $\frac{1}{2}$, eg. $x = 0.49$:

$= \frac{2(0.49)+1}{2(0.49)-1} = \frac{1.98}{-0.02} = -99$ ← implies $f(x) \to -\infty$

Sub. something at other end of domain, e.g. $x = -1000$:

$= \frac{2(-1000)+1}{2(-1000)-1} = \frac{-1999}{-2001} = 0.999$ ← implies $f(x) \to 1$

So domain of $f^{-1}(x)$ is $x < 1$

The function g is defined by $g(x) = 3x + 2$ for $x \in \mathbb{R}$.

(b) Solve the equation $f(x) = gf\left(\frac{1}{4}\right)$. [3]

$f\left(\frac{1}{4}\right) = \frac{2\left(\frac{1}{4}\right)+1}{2\left(\frac{1}{4}\right)-1} = \frac{1.5}{-0.5}$

$= -3$

$gf\left(\frac{1}{4}\right) = g(-3)$

$= 3(-3) + 2$

$= -7$

$f(x) = gf\left(\frac{1}{4}\right)$:

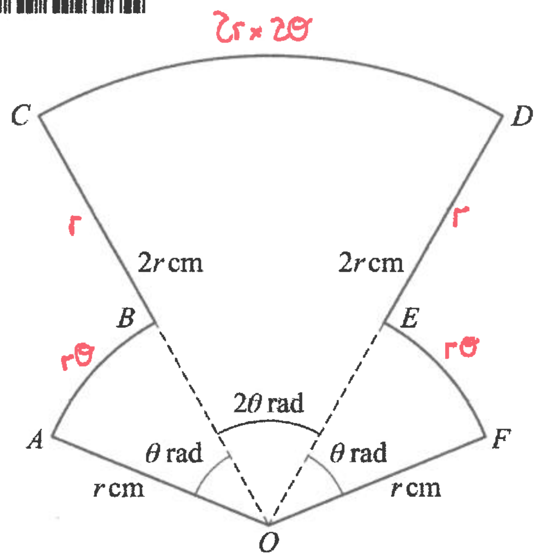
$\frac{2x+1}{2x-1} = -7$

$2x + 1 = -14x + 7$

$16x = 6$

$x = \frac{3}{8}$

6



The diagram shows a metal plate $OABCDEF$ consisting of sectors of two circles, each with centre O. The radii of sectors AOB and EOF are r cm and the radius of sector COD is $2r$ cm. Angle AOB = angle $EOF = \theta$ radians and angle $COD = 2\theta$ radians.

It is given that the perimeter of the plate is 14 cm and the area of the plate is 10 cm^2.

Given that $r > \frac{3}{2}$ and $\theta < \frac{3}{4}$, find the values of r and θ. [6]

Perimeter:

$$r + r\theta + r + 2r\times 2\theta + r + r\theta + r = 14$$

$$4r + 2r\theta + 4r\theta = 14$$

$$4r + 6r\theta = 14$$

$$2r + 3r\theta = 7 \quad ①$$

2 small sectors — big sector

Area:

$$\frac{1}{2}r^2\theta \times 2 + \frac{1}{2}(2r)^2\times 2\theta = 10$$

$$r^2\theta + \frac{1}{2}\times 4r^2\times 2\theta = 10$$

$$r^2\theta + 4r^2\theta = 10$$

$$5r^2\theta = 10$$

$$r^2\theta = 2$$

$$\theta = \frac{2}{r^2} \quad ②$$

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Sub. ② into ①:

$$2r + 3r\left(\frac{2}{r^2}\right) = 7$$

$$2r + \frac{6}{r} = 7$$

$\times r \quad \times r \quad \times r$

$$2r^2 + 6 = 7r$$

$$2r^2 - 7r + 6 = 0$$

$$(2r - 3)(r - 2) = 0$$

$2r - 3 = 0$ or $\underline{r = 2}$

$2r = 3$

$r = 1.5$

$\times \quad r > \frac{3}{2}$

→② : $\underline{r = 2}$:

$$\theta = \frac{2}{2^2}$$

$$= \frac{2}{4}$$

$$\underline{\theta = \frac{1}{2}}$$

7 (a) By expressing $-2x^2+8x+11$ in the form $-a(x-b)^2+c$, where a, b and c are positive integers, find the coordinates of the vertex of the graph with equation $y=-2x^2+8x+11$. [3]

$-2[x^2-4x]+11$

$=-2[(x-2)^2-4]+11$

$=-2(x-2)^2+8+11$

$=-2(x-2)^2+19$

OR: $-a(x-b)^2+c$

$-a(x^2-2bx+b^2)+c$

$-ax^2+2abx-ab^2+c$

compare coeffs:

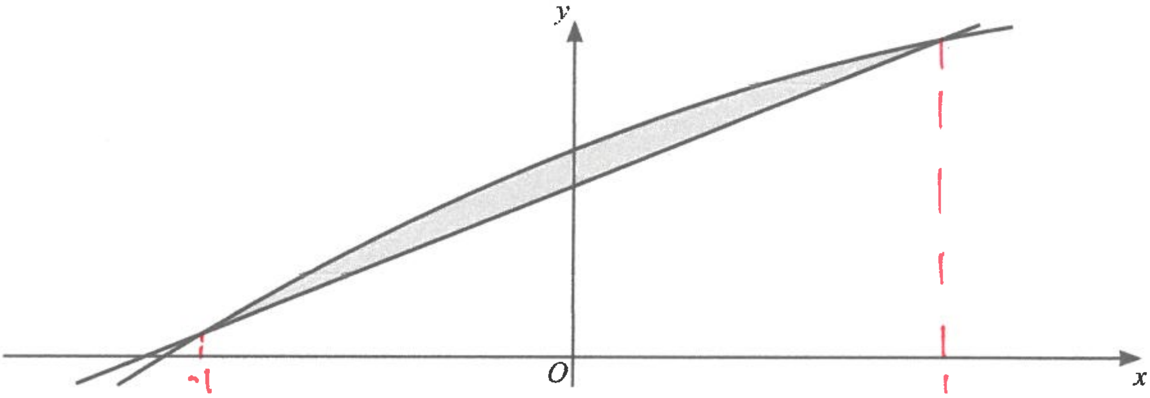
x^2: $-2=-a$, $a=2$

x: $8=2ab$, $8=4b$, $b=2$

const: $11=-ab^2+c$, $11=-8+c$, $c=19$

Vertex at $(2, 19)$

(b)



The diagram shows part of the curve with equation $y=-2x^2+8x+11$ and the line with equation $y=8x+9$.

Find the area of the shaded region. [5]

Find points of intersection:

$-2x^2+8x+11 = 8x+9$

$-2x^2+2=0$

$2x^2=2$

$x^2=1$

$x=\pm 1$

area under curve

area under straight line

$$\text{Area} = \int_{-1}^{1} (-2x^2 + 8x + 11)\,dx - \int_{-1}^{1} (8x + 9)\,dx$$

$$= \left[-\frac{2x^3}{3} + 4x^2 + 11x\right]_{-1}^{1} - \left[4x^2 + 9x\right]_{-1}^{1}$$

$$= \left[\left(\frac{-2}{3} + 4 + 11\right) - \left(\frac{2}{3} + 4 - 11\right)\right] - \left[(4+9) - (4-9)\right]$$

$$= \left[\left(\frac{43}{3}\right) - \left(\frac{-19}{3}\right)\right] - \left[(13) - (-5)\right]$$

$$= \frac{62}{3} - 18$$

$$= \frac{8}{3}$$

8 The equation of a circle is $x^2+y^2+px+2y+q=0$, where p and q are constants.

(a) Express the equation in the form $(x-a)^2+(y-b)^2=r^2$, where a is to be given in terms of p and r^2 is to be given in terms of p and q. [2]

$x^2+px+y^2+2y+q=0$

$\left(x+\frac{p}{2}\right)^2-\frac{p^2}{4}+(y+1)^2-1+q=0$

$\left(x+\frac{p}{2}\right)^2+(y+1)^2=1+\frac{p^2}{4}-q$

$\left(x--\frac{p}{2}\right)^2+(y--1)^2=1+\frac{p^2}{4}-q$

The line with equation $x+2y=10$ is the tangent to the circle at the point $A(4, 3)$.

(b) (i) Find the equation of the normal to the circle at the point A. [3]

Tangent: $2y=-x+10$

$y=\frac{-1}{2}x+5$

gradient of tangent $=\frac{-1}{2}$ gradient of normal $=2$

Sub. (4,3): $y-y_1=m(x-x_1)$

$y-3=2(x-4)$

$y-3=2x-8$

$y=2x-5$

(ii) Find the values of p and q. [5]

Normal to the circle goes through centre:

$y = 2x - 5$ passes through $\left(\frac{-p}{2}, -1\right)$

sub. $\left(\frac{-p}{2}, -1\right)$: $-1 = 2\left(\frac{-p}{2}\right) - 5$

$-1 = -p - 5$

$4 = -p$

$p = -4$

→ Eqn of circle: $\left(x + \frac{p}{2}\right)^2 + (y+1)^2 = 1 + \frac{p^2}{4} - q$

→ $(x-2)^2 + (y+1)^2 = 1 + 4 - q$

$(x-2)^2 + (y+1)^2 = 5 - q$

Circle passes through $(4,3)$ so sub this point:

$(4-2)^2 + (3+1)^2 = 5 - q$

$2^2 + 4^2 = 5 - q$

$4 + 16 = 5 - q$

$20 = 5 - q$

$15 = -q$

$q = -15$

9 The equation of a curve is $y = \frac{1}{2}k^2x^2 - 2kx + 2$ ① and the equation of a line is $y = kx + p$ ②, where k and p are constants with $0 < k < 1$.

(a) It is given that one of the points of intersection of the curve and the line has coordinates $\left(\frac{5}{2}, \frac{1}{2}\right)$.

Find the values of k and p, and find the coordinates of the other point of intersection. [7]

Sub. $\left(\frac{5}{2}, \frac{1}{2}\right)$ into ①: $\frac{1}{2} = \frac{1}{2}k^2 \times \left(\frac{5}{2}\right)^2 - 2k\left(\frac{5}{2}\right) + 2$

$\frac{1}{2} = \frac{1}{2}k^2 \times \frac{25}{4} - 5k + 2$

$\frac{1}{2} = \frac{25}{8}k^2 - 5k + 2$

×8 ×8 ×8 ×8

$4 = 25k^2 - 40k + 16$

$25k^2 - 40k + 12 = 0$ (a, b, c)

ac = 300 two numbers with product 300 and sum −40 : −30, −10

$25k^2 - 30k - 10k + 12 = 0$

$5k(5k-6) - 2(5k-6) = 0$

$(5k-2)(5k-6) = 0$

$k = \frac{2}{5}$ ✓ or $k = \frac{6}{5}$ ✗ $0<k<1$

Sub. $\left(\frac{5}{2}, \frac{1}{2}\right)$ into ②: $\frac{1}{2} = k\left(\frac{5}{2}\right) + p$

$\leftarrow k = \frac{2}{5}$

$\frac{1}{2} = \frac{2}{5} \times \frac{5}{2} + p$

$\frac{1}{2} = 1 + p$

$p = -\frac{1}{2}$

eqⁿ ①: $y = \frac{1}{2}\left(\frac{2}{5}\right)^2x^2 - 2\left(\frac{2}{5}\right)x + 2$ eqⁿ ②: $y = \frac{2}{5}x - \frac{1}{2}$

$= \frac{1}{2} \times \frac{4}{25}x^2 - \frac{4}{5}x + 2$

$y = \frac{2}{25}x^2 - \frac{4}{5}x + 2$

Solve simultaneously: $\frac{2}{25}x^2 - \frac{4}{5}x + 2 = \frac{2}{5}x - \frac{1}{2}$

×50 ×50 ×50 ×50 ×50

$4x^2 - 40x + 100 = 20x - 25$

$4x^2 - 60x + 125 = 0$ (a, b, c)

$ac = 500$. → two numbers w/ product 500 and sum -60 : $-50, -10$

$4x^2 - 10x - 50x + 125 = 0$

$2x(2x - 5) - 25(2x - 5) = 0$

$(2x - 25)(2x - 5) = 0$

$x = \frac{25}{2}$ ✓ $x = \frac{5}{2}$ × (already used)

Sub. $x = \frac{25}{2}$ into ②:

$y = \frac{2}{5} \times \frac{25}{2} - \frac{1}{2}$

$= 5 - \frac{1}{2} = \frac{9}{2}$ → $\left(\frac{25}{2}, \frac{9}{2}\right)$

(b) It is given instead that the line and the curve do **not** intersect.

Find the set of possible values of p. [3]

Solve simultaneously:

$\frac{1}{2}k^2x^2 - 2kx + 2 = kx + p$

$\frac{1}{2}k^2x^2 - 3kx + 2 - p = 0$

$a = \frac{1}{2}k^2$ $b = -3k$ $c = 2 - p$

No intersection, so $b^2 - 4ac < 0$:

$(-3k)^2 - 4\left(\frac{1}{2}k^2\right)(2 - p) < 0$

$9k^2 - 2k^2(2 - p) < 0$

$9k^2 - 4k^2 + 2k^2p < 0$

$5k^2 + 2k^2p < 0$

$2k^2p < -5k^2$

$p < \frac{-5k^2}{2k^2}$

$p < \frac{-5}{2}$

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10 A function f with domain $x > 0$ is such that $f'(x) = 8(2x-3)^{\frac{1}{3}} - 10x^{\frac{2}{3}}$. It is given that the curve with equation $y = f(x)$ passes through the point $(1, 0)$.

(a) Find the equation of the normal to the curve at the point $(1, 0)$. [3]

gradient of f at (1,0): $f'(1) = 8(2-3)^{\frac{1}{3}} - 10\times(1)^{\frac{2}{3}}$

$= 8\times -1 - 10 = -18$

gradient of normal: $m = \frac{1}{18} \rightarrow y - y_1 = m(x - x_1)$

$y - 0 = \frac{1}{18}(x-1)$

$y = \frac{1}{18}x - \frac{1}{18}$

(b) Find f(x). [4]

Integrate:

$f(x) = \frac{3}{4} \times \frac{1}{2} \times 8(2x-3)^{\frac{4}{3}} - \frac{3}{5} \times 10x^{\frac{5}{3}}$

$= 3(2x-3)^{4/3} - 6x^{5/3} + C$

Sub. (1,0):

$0 = 3(2(1)-3)^{4/3} - 6(1)^{5/3} + C$

$0 = 3(-1)^{4/3} - 6(1)^{5/3} + C$

$0 = 3\times 1 - 6\times 1 + C$

$0 = -3 + C$

$C = 3$

$f(x) = 3(2x-3)^{4/3} - 6x^{5/3} + 3$

It is given that the equation $f'(x) = 0$ can be expressed in the form

$$125x^2 - 128x + 192 = 0.$$

(c) Determine, making your reasoning clear, whether f is an increasing function, a decreasing function or neither. [3]

Complete the Square:

$$f'(x) = 125\left[x^2 - \frac{128}{125}x\right] + 192$$

$$= 125\left[\left(x - \frac{64}{125}\right)^2 - \frac{16\,384}{15\,625}\right] + 192$$

$$= 125\left(x - \frac{64}{125}\right)^2 - \frac{16\,384}{125} + 192$$

$$= \underbrace{125\left(x - \frac{64}{125}\right)^2}_{\text{always positive}} + \frac{40\,384}{125}$$

positive

$f'(x)$ is > 0 for all values of x

so $f(x)$ is an increasing function