

The diagram shows the curve $y = xe^{-\frac{1}{4}x^2}$, for $x \geq 0$, and its maximum point M .

- (a) Find the exact coordinates of M .

[4]

$$u = x$$

$$\frac{du}{dx} = 1$$

$$v = e^{-\frac{1}{4}x^2}$$

$$\frac{dv}{dx} = -\frac{1}{2}x e^{-\frac{1}{4}x^2}$$

$$\frac{dy}{dx} = \frac{1}{2}x^2 e^{-\frac{1}{4}x^2} + e^{-\frac{1}{4}x^2}$$

SP, so $\frac{dy}{dx} = 0$:

$$e^{-\frac{1}{4}x^2} \left(\frac{1}{2}x^2 + 1 \right) = 0$$

$$e^{-\frac{1}{4}x^2} = 0$$

no sol's

$$\text{or } \frac{1}{2}x^2 + 1 = 0$$

$$\frac{1}{2}x^2 = 1$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Point M is to the right of y -axis, so $x = +\sqrt{2}$:

$$y = \sqrt{2} e^{-\frac{1}{4}(\sqrt{2})^2}$$

$$= \sqrt{2} e^{-\frac{1}{2}}$$

$$\rightarrow \underline{(\sqrt{2}, \sqrt{2} e^{-\frac{1}{2}})}$$

- 1 Find the exact coordinates of the points on the curve $y = \frac{x^2}{1-3x}$ at which the gradient of the tangent is equal to 8. [5]

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$v = 1-3x$$

$$\frac{dv}{dx} = -3$$

$$\frac{dy}{dx} = \frac{(1-3x) \times 2x - x^2 \times (-3)}{(1-3x)^2}$$

$$= \frac{2x - 6x^2 + 3x^2}{(1-3x)^2}$$

$$= \frac{2x - 3x^2}{(1-3x)^2}$$

gradient = 8: $\frac{2x - 3x^2}{(1-3x)^2} = 8$

$$2x - 3x^2 = 8(1-3x)^2$$

$$2x - 3x^2 = 8(1 - 6x + 9x^2)$$

$$2x - 3x^2 = 72x^2 - 48x + 8$$

$$75x^2 - 50x + 8 = 0$$

$$x = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(75)(8)}}{2(75)}$$

$$x = \frac{2}{5} \quad \text{or} \quad x = \frac{4}{15}$$

$x = \frac{2}{5}$: $y = \frac{\left(\frac{2}{5}\right)^2}{1-3\left(\frac{2}{5}\right)}$

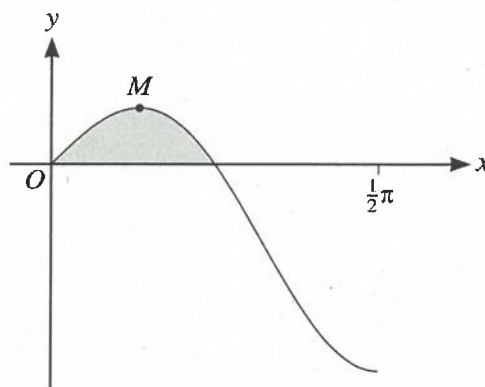
$$= \frac{\frac{4}{25}}{-\frac{1}{5}} = \frac{-4}{5}$$

$\left(\frac{2}{5}, -\frac{4}{5}\right)$

$x = \frac{4}{15}$: $y = \frac{\left(\frac{4}{15}\right)^2}{1-3\left(\frac{4}{15}\right)}$

$$= \frac{\frac{16}{225}}{\frac{3}{15}} = \frac{16}{45}$$

$\left(\frac{4}{15}, \frac{16}{45}\right)$



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leq x \leq \frac{1}{2}\pi$, and its maximum point M .

- (a) Find the x -coordinate of M , giving your answer correct to 3 significant figures.

[6]

$$y = \sin x \cos 2x$$

$$u = \sin x \quad v = \cos 2x$$

$$\frac{du}{dx} = \cos x \quad \frac{dv}{dx} = -2\sin 2x$$

$$\frac{dy}{dx} = \sin x \times -2\sin 2x + \cos 2x \times \cos x$$

$$\frac{dy}{dx} = 0$$

$$-2\sin x \cos 2x + \cos x \cos 2x = 0$$

$$-2\sin x \times 2\sin x \cos x + \cos x (2\cos^2 x - 1) = 0$$

$$-4\sin^2 x \cos x + 2\cos^3 x - \cos x = 0$$

$$-4(1 - \cos^2 x) \cos x + 2\cos^3 x - \cos x = 0$$

$$-4\cos x + 4\cos^3 x + 2\cos^3 x - \cos x = 0$$

$$6\cos^3 x - 5\cos x = 0$$

$$\cos x (6\cos^2 x - 5) = 0$$

$$\cos x = 0 \quad \text{or} \quad 6\cos^2 x - 5 = 0$$

$$x = \frac{\pi}{2} \quad \cos^2 x = \frac{5}{6}$$

$$\cos x = \sqrt{\frac{5}{6}} \quad \text{or} \quad \cos x = -\sqrt{\frac{5}{6}}$$

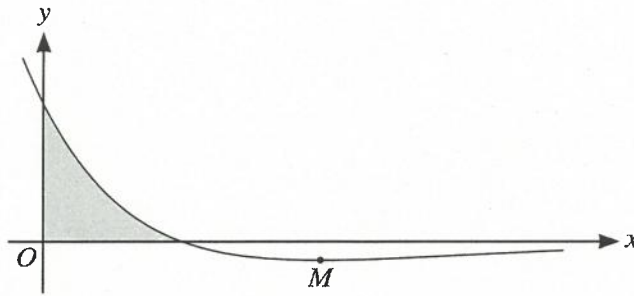
$$x = 0.421 \quad x = 2.72$$

X not the point we want on diagram

X outside domain

$$\underline{x = 0.421}$$

10



The diagram shows the curve $y = (2-x)e^{-\frac{1}{2}x}$, and its minimum point M .

- (a) Find the exact coordinates of M .

[5]

$$y = (2-x)e^{-\frac{1}{2}x}$$

$$u = 2-x \quad v = e^{-\frac{1}{2}x}$$

$$\frac{du}{dx} = -1 \quad \frac{dv}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x}$$

$$\frac{dy}{dx} = (2-x) \times -\frac{1}{2}e^{-\frac{1}{2}x} + e^{-\frac{1}{2}x} \times -1$$

$$\frac{dy}{dx} = 0: \quad -\frac{1}{2}(2-x)e^{-\frac{1}{2}x} - e^{-\frac{1}{2}x} = 0$$

$$\left(-1 + \frac{x}{2}\right)e^{-\frac{1}{2}x} - e^{-\frac{1}{2}x} = 0$$

$$e^{-\frac{1}{2}x} \left(-1 + \frac{x}{2} - 1\right) = 0$$

$$e^{-\frac{1}{2}x} \left(-2 + \frac{x}{2}\right) = 0$$

$$e^{-\frac{1}{2}x} = 0 \quad \text{or} \quad -2 + \frac{x}{2} = 0$$

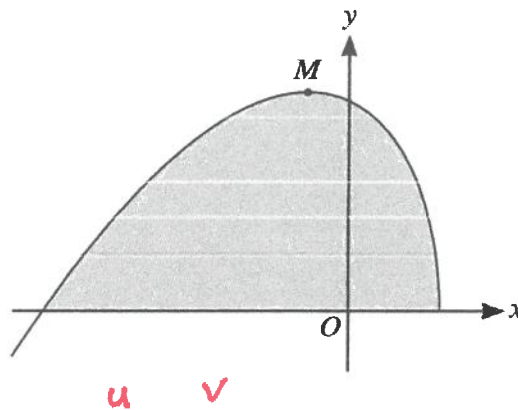
$\times$ not possible $\frac{x}{2} = 2$

$$x = 4$$

$\rightarrow$ original eqn: $y = (2-4)e^{-\frac{1}{2} \times 4}$

$$= -2e^{-2}$$

$\rightarrow (4, -2e^{-2})$



The diagram shows the curve $y = (x+5)\sqrt{3-2x}$ and its maximum point M .

- (a) Find the exact coordinates of M .

[5]

$$u = x + 5 \quad v = (3 - 2x)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \frac{1}{2}x - 2(3 - 2x)^{-\frac{1}{2}}$$

$$= -(3 - 2x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -(x+5)(3-2x)^{-\frac{1}{2}} + (3-2x)^{\frac{1}{2}}$$

SP, so $\frac{dy}{dx} = 0$:

$$\frac{-(x+5)}{(3-2x)^{\frac{1}{2}}} + (3-2x)^{\frac{1}{2}} = 0$$

$$\frac{x+5}{(3-2x)^{\frac{1}{2}}} = (3-2x)^{\frac{1}{2}}$$

$$x+5 = 3-2x$$

$$3x = -2$$

$$x = -\frac{2}{3}$$

sub. in to find y :

$$y = \left(-\frac{2}{3} + 5\right)\sqrt{3 - 2\left(-\frac{2}{3}\right)}$$

$$= \frac{13}{3}\sqrt{\frac{13}{3}}$$

$$= \frac{13\sqrt{39}}{9}$$

$$\rightarrow \left(-\frac{2}{3}, \frac{13\sqrt{39}}{9}\right)$$

$$y = \cos^3 x \sqrt{\sin x}$$

$$u = \cos^3 x \quad v = (\sin x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 3\cos^2 x - \sin x \quad \frac{dv}{dx} = \frac{1}{2}(\sin x)^{-\frac{1}{2}} \times \cos x$$

$$\frac{dy}{dx} = \frac{1}{2} \cos^4 x (\sin x)^{-\frac{1}{2}} + (\sin x)^{\frac{1}{2}} \times 3 \cos^3 x \times -\sin x$$

$$\frac{dy}{dx} = 0: \quad \frac{1}{2} \cos^4 x (\sin x)^{-\frac{1}{2}} - 3 \cos^2 x (\sin x)^{\frac{3}{2}} = 0$$

$$\frac{1}{2} \frac{\cos^9 x}{\sqrt{\sin x}} - 3 \cos^2 x (\sin x)^{\frac{3}{2}} = 0$$

$$\frac{1}{2} \cos^4 x - 3 \cos^2 x \sin^2 x = 0$$

$$\frac{1}{2} \cos^4 x - 3 \cos^2 x (1 - \cos^2 x) = 0$$

$$\frac{1}{2} \cos^4 x - 3 \cos^2 x + 3 \cos^4 x = 0$$

$$\frac{7}{2} \cos^4 x - 3 \cos^2 x = 0$$

$$\cos^2 x \left(\frac{7}{2} \cos^2 x - 3 \right) = 0$$

$$\cos^2 x = 0 \quad \text{or} \quad \frac{7}{3} \cos^2 x - 3 = 0$$

$$\cos x = 0$$

$$\frac{7}{2} \cos^2 x = 3$$

$$x = \frac{\pi}{2}$$

$$\cos^2 x = \frac{6}{7}$$

X outside domain

$$\cos x = \sqrt{\frac{6}{7}} \quad \text{or} \quad \cos x = -\sqrt{\frac{6}{7}}$$

$$x = 0.388$$

$$x = 2.75$$

X outside domain

$$x = 0.388$$

- 9 The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

(a) Find the exact coordinates of the stationary point.

[5]

$$y = x^{-\frac{2}{3}} \ln x$$

$$u = x^{-\frac{2}{3}}$$

$$v = \ln x$$

$$\frac{du}{dx} = -\frac{2}{3} x^{-\frac{5}{3}}$$

$$\frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = x^{-\frac{2}{3}} \times \frac{1}{x} + \ln x \times -\frac{2}{3} x^{-\frac{5}{3}}$$

$$\frac{dy}{dx} = 0:$$

$$x^{-\frac{2}{3}} \times x^{-1} - \frac{2}{3} x^{-\frac{5}{3}} \ln x = 0$$

$$x^{-\frac{5}{3}} - \frac{2}{3} x^{-\frac{5}{3}} \ln x = 0$$

$$x^{-\frac{5}{3}} (1 - \frac{2}{3} \ln x) = 0$$

$$x^{-\frac{5}{3}} = 0$$

$$\text{or } 1 - \frac{2}{3} \ln x = 0$$

$$x = 0$$

X outside domain

$$\frac{2}{3} \ln x = 1$$

$$\ln x = \frac{3}{2}$$

$$x = e^{\frac{3}{2}}$$

→ original eq:

$$y = (e^{\frac{3}{2}})^{-\frac{2}{3}} \ln(e^{\frac{3}{2}})$$

$$= e^{-1} \times \frac{3}{2}$$

$$= \frac{3}{2} e^{-1}$$

$$\rightarrow (e^{\frac{3}{2}}, \frac{3}{2} e^{-1})$$

9 Let $f(x) = \frac{1}{(9-x)\sqrt{x}}$.

(a) Find the x -coordinate of the stationary point of the curve with equation $y = f(x)$.

[4]

$$f(x) = (9-x)^{-1} x^{-\frac{1}{2}}$$

$$u = (9-x)^{-1}$$

$$v = x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = (9-x)^{-2}$$

$$\frac{dv}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$f'(x) = (9-x)^{-1} \times -\frac{1}{2}x^{-\frac{3}{2}} + x^{-\frac{1}{2}} \times (9-x)^{-2}$$

$$f'(x) = 0:$$

$$-\frac{1}{2}x^{-\frac{3}{2}}(9-x)^{-1} + x^{-\frac{1}{2}}(9-x)^{-2} = 0$$

$$\frac{-1}{2x^{\frac{3}{2}}(9-x)} + \frac{1}{x^{\frac{1}{2}}(9-x)^2} = 0$$

$$\frac{-1(9-x)}{2x^{\frac{3}{2}}(9-x)^2} + \frac{2x}{2x^{\frac{3}{2}}(9-x)^2} = 0$$

$$\frac{-1(9-x) + 2x}{2x^{\frac{3}{2}}(9-x)^2} = 0$$

$$-1(9-x) + 2x = 0$$

$$-9 + x + 2x = 0$$

$$-9 + 3x = 0$$

$$3x = 9$$

$$\underline{x = 3}$$

- 8 The equation of a curve is $y = e^{-5x} \tan^2 x$ for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

Find the x -coordinates of the stationary points of the curve. Give your answers correct to 3 decimal places where appropriate. [8]

$$y = e^{-5x} \tan^2 x$$

$$u = e^{-5x} \quad v = \tan^2 x$$

$$\frac{du}{dx} = -5e^{-5x} \quad \frac{dv}{dx} = 2 \tan x \times \sec^2 x$$

$$\frac{dy}{dx} = 2e^{-5x} \tan x \sec^2 x + \tan^2 x \times -5e^{-5x}$$

$$\frac{dy}{dx} = 0: \quad 2e^{-5x} \tan x \sec^2 x - 5e^{-5x} \tan^2 x = 0$$

$$2e^{-5x} \tan x (1 + \tan^2 x) - 5e^{-5x} \tan^2 x = 0$$

$$2e^{-5x} (\tan x + \tan^3 x) - 5e^{-5x} \tan^2 x = 0$$

$$2e^{-5x} \tan x + 2e^{-5x} \tan^3 x - 5e^{-5x} \tan^2 x = 0$$

$$e^{-5x} \tan x (2 + 2\tan^2 x - 5\tan x) = 0$$

$$e^{-5x} = 0 \quad \text{or} \quad \tan x = 0 \quad \text{or} \quad 2\tan^2 x - 5\tan x + 2 = 0$$

$\times$ not possible $x = 0$

$$\rightarrow \text{let } Y = \tan x: 2Y^2 - 5Y + 2 = 0$$

$$(2Y - 1)(Y - 2) = 0$$

$$Y = \frac{1}{2} \quad \text{or} \quad Y = 2$$

$$\tan x = \frac{1}{2} \quad \text{or} \quad \tan x = 2$$

$$x = 0.464$$

$$x = 1.107$$

$$x = 0, x = 0.464, x = 1.107$$

- 4 The curve with equation $y = e^{2x}(\sin x + 3 \cos x)$ has a stationary point in the interval $0 \leq x \leq \pi$.

(a) Find the x -coordinate of this point, giving your answer correct to 2 decimal places. [4]

$$y = e^{2x}(\sin x + 3 \cos x)$$

$$u = e^{2x} \quad v = \sin x + 3 \cos x$$

$$\frac{du}{dx} = 2e^{2x} \quad \frac{dv}{dx} = \cos x - 3 \sin x$$

$$\frac{dy}{dx} = e^{2x}(\cos x - 3 \sin x) + 2e^{2x}(\sin x + 3 \cos x)$$

$$\frac{dy}{dx} = 0: e^{2x} \cos x - 3e^{2x} \sin x + 2e^{2x} \sin x + 6e^{2x} \cos x = 0$$

$$7e^{2x} \cos x - e^{2x} \sin x = 0$$

$$e^{2x}(7 \cos x - \sin x) = 0$$

$$e^{2x} = 0 \quad \text{or} \quad 7 \cos x - \sin x = 0$$

$\nearrow$ not possible

$$\sin x = 7 \cos x$$

$$\frac{\sin x}{\cos x} = 7$$

$$\tan x = 7$$

$$x = 1.43$$

STO

(b) Determine whether the stationary point is a maximum or a minimum. [2]

$$\frac{dy}{dx} = e^{2x}(7 \cos x - \sin x)$$

$$u = e^{2x} \quad v = 7 \cos x - \sin x$$

$$\frac{du}{dx} = 2e^{2x} \quad \frac{dv}{dx} = -7 \sin x + \cos x$$

$$\frac{d^2y}{dx^2} = e^{2x}(-7 \sin x + \cos x) + 2e^{2x}(7 \cos x - \sin x)$$

$$= -7e^{2x} \sin x + e^{2x} \cos x + 14e^{2x} \cos x - 2e^{2x} \sin x$$

$$= 15e^{2x} \cos x - 9e^{2x} \sin x$$

when $x = 1.43$, $\frac{d^2y}{dx^2} = -118.27 \dots \therefore$ it's a maximum.

- 4 The equation of a curve is $y = x \tan^{-1}(\frac{1}{2}x)$.

(a) Find $\frac{dy}{dx}$.

[3]

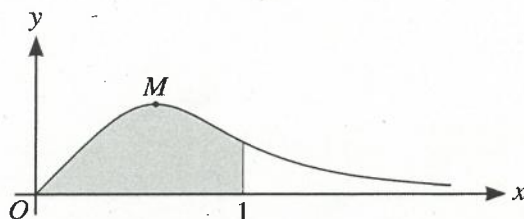
$$\begin{aligned}
 y &= u \tan^{-1}\left(\frac{1}{2}x\right) \\
 u &= x & v &= \tan^{-1}\left(\frac{1}{2}x\right) \\
 \frac{du}{dx} &= 1 & \frac{dv}{dx} &= \frac{1}{2} \times \frac{1}{1 + \left(\frac{x}{2}\right)^2} = \frac{1}{2} \times \frac{1}{1 + \frac{x^2}{4}} \\
 & & &= \frac{1}{2 + \frac{x^2}{2}} \\
 \frac{dy}{dx} &= x \times \frac{1}{2 + \frac{x^2}{2}} + \tan^{-1}\left(\frac{1}{2}x\right) \\
 &= \frac{x}{2 + \frac{x^2}{2}} + \tan^{-1}\left(\frac{1}{2}x\right) \\
 &= \frac{2x}{4 + x^2} + \tan^{-1}\left(\frac{1}{2}x\right)
 \end{aligned}$$

- (b) The tangent to the curve at the point where $x = 2$ meets the y-axis at the point with coordinates $(0, p)$.

Find p .

[3]

$$\begin{aligned}
 \text{when } x &= 2, \quad y = 2 \times \tan^{-1}\left(\frac{1}{2} \times 2\right) \\
 &= \frac{\pi}{2} \\
 \text{when } x &= 2, \quad \frac{dy}{dx} = \frac{2(2)}{4 + 2^2} + \tan^{-1}\left(\frac{1}{2} \times 2\right) \\
 &= \frac{1}{2} + \frac{\pi}{4} \\
 y - y_1 &= m(x - x_1) \\
 y - \frac{\pi}{2} &= \left(\frac{1}{2} + \frac{\pi}{4}\right)(x - 2) \\
 \text{when } x &= 0: \quad y - \frac{\pi}{2} = \left(\frac{1}{2} + \frac{\pi}{4}\right)(-2) \\
 y - \frac{\pi}{2} &= -1 - \frac{\pi}{2} \\
 y &= -1
 \end{aligned}$$



The diagram shows the curve $y = \frac{x}{1+3x^4}$, for $x \geq 0$, and its maximum point M .

- (a) Find the x -coordinate of M , giving your answer correct to 3 decimal places.

[4]

$$y = \frac{x}{1+3x^4}$$

$$u = x \quad v = 1 + 3x^4$$

$$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = 12x^3$$

$$\frac{dy}{dx} = \frac{1(1+3x^4) - 12x^4}{(1+3x^4)^2}$$

$$\frac{dy}{dx} = 0: \quad \frac{1+3x^4-12x^4}{(1+3x^4)^2} = 0$$

$$1+3x^4-12x^4 = 0$$

$$1-9x^4 = 0$$

$$9x^4 = 1$$

$$x^4 = \frac{1}{9}$$

$$x = \sqrt[4]{\frac{1}{9}}$$

$$\underline{\underline{x = 0.577}}$$

- 5 Find the exact coordinates of the stationary points of the curve $y = \frac{e^{3x^2-1}}{1-x^2}$ [6]

$$u = e^{3x^2-1}$$

$$v = 1-x^2$$

$$\frac{du}{dx} = 6xe^{3x^2-1}$$

$$\frac{dv}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{(1-x^2) \times 6xe^{3x^2-1} - 2xe^{3x^2-1}}{(1-x^2)^2}$$

$$= \frac{6xe^{3x^2-1} - 2xe^{3x^2-1}}{(1-x^2)^2}$$

$$= \frac{4xe^{3x^2-1}}{(1-x^2)^2}$$

SP, so $\frac{dy}{dx} = 0$:

$$\frac{4xe^{3x^2-1}}{(1-x^2)^2} = 0$$

$$4xe^{3x^2-1} = 0$$

$$2xe^{3x^2-1}(4-3x^2) = 0$$

$$\rightarrow 2x = 0 \quad \text{or} \quad e^{3x^2-1} = 0 \quad \text{or} \quad 4-3x^2 = 0$$

$$\underline{x = 0}$$

no solⁿs

$$3x^2 = 4$$

$$x^2 = \frac{4}{3}$$

$$\underline{x = \pm \frac{2\sqrt{3}}{3}}$$

Sub: $x=0$:

$$y = \frac{e^{0-1}}{1-0}$$

$$y = e^{-1}$$

$x = +\frac{2\sqrt{3}}{3}$:

$$y = \frac{e^{4-1}}{1-\frac{4}{3}}$$

$$y = -3e^3$$

$x = -\frac{2\sqrt{3}}{3}$:

$$y = \frac{e^{4-1}}{1-\frac{4}{3}}$$

$$y = -3e^3$$

$$\underline{(0, e^{-1})}$$

$$\underline{(\frac{2\sqrt{3}}{3}, -3e^3)}$$

$$\underline{(-\frac{2\sqrt{3}}{3}, -3e^3)}$$