1 Find the set of values of x satisfying the inequality $|2^{x+1} - 2| < 0.5$, giving your answer to 3 significant figures. [4]

+/- Method:

+: $(2^{x+1} - 2) < 0.5$

$2^{x+1} - 2 < 0.5$

$2^{x+1} < 2.5$

$\log_2 2^{x+1} < \log_2 2.5$

$x+1 < \log_2 2.5$

$x < \log_2 2.5 - 1$

$x < 0.322$

−: $-(2^{x+1} - 2) < 0.5$

$-2^{x+1} + 2 < 0.5$

$-2^{x+1} < -1.5$

$2^{x+1} > 1.5$ (flip sign)

$\log_2 2^{x+1} > \log_2 1.5$

$x+1 > \log_2 1.5$

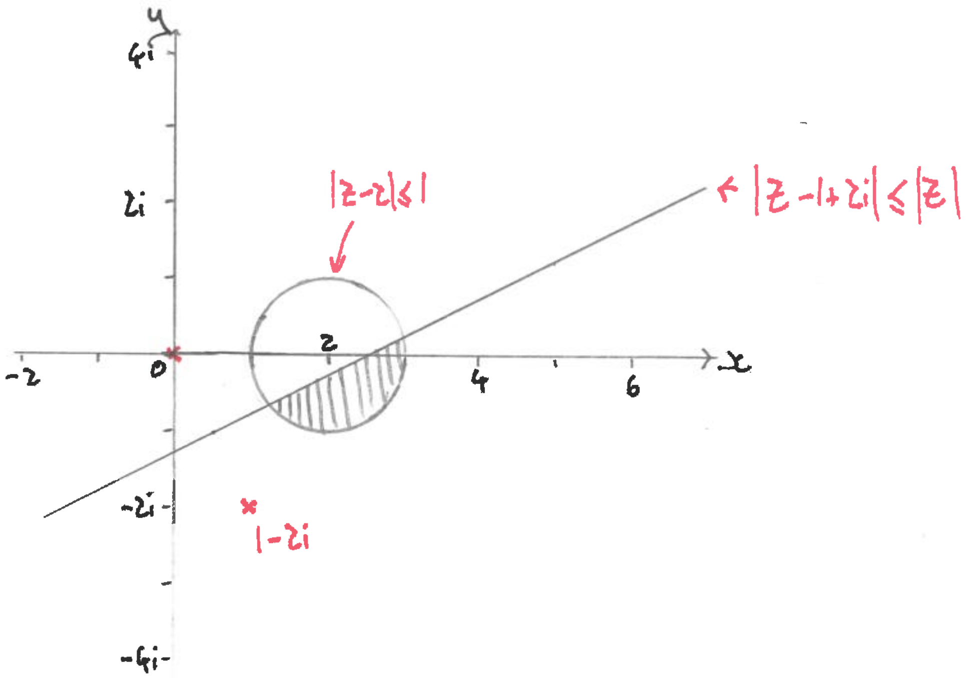
$x > \log_2 1.5 - 1$

$x > -0.415$

$-0.415 < x < 0.322$

2 On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z-1+2i| \leqslant |z|$ and $|z-2| \leqslant 1$. [5]

↑ perpendicular bisector of $(1,-2)$ and $(0,0)$



Optional: Find equation of perpendicular bisector to help plot it:

$$|z-1+2i| = |z|$$

$$|x+iy-1+2i| = |x+iy|$$

$$|(x-1)+i(y+2)| = |x+iy|$$

$$(x-1)^2 + (y+2)^2 = x^2 + y^2$$

$$x^2-2x+1 + y^2+4y+4 = x^2+y^2$$

$$-2x+4y+5 = 0$$

$$4y = 2x-5$$

$$\underline{y = \tfrac{1}{2}x - \tfrac{5}{4}}$$

3 The polynomial $2x^3 + ax^2 + bx + 6$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $(x+2)$ the remainder is -38 and when $p(x)$ is divided by $(2x-1)$ the remainder is $\frac{19}{2}$.

1 2

Find the values of a and b. [5]

1: $p(-2) = -38$:

$2(-2)^3 + a(-2)^2 + b(-2) + 6 = -38$

$2\times -8 + a\times 4 - 2b + 6 = -38$

$-16 + 4a - 2b + 6 = -38$

$4a - 2b - 10 = -38$

$4a - 2b = -28$ A

2: $p\left(\frac{1}{2}\right) = \frac{19}{2}$:

$2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 6 = \frac{19}{2}$

$2\times\frac{1}{8} + a\times\frac{1}{4} + \frac{b}{2} + 6 = 19/2$

$\frac{1}{4} + \frac{a}{4} + b/2 + 6 = 19/2$

$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \times 4$

$1 + a + 2b + 24 = 38$

$a + 2b + 25 = 38$

$a + 2b = 13$ B

A + B: $5a = -15$

$\underline{a = -3}$

$\rightarrow$ B: $-3 + 2b = 13$

$2b = 16$

$\underline{b = 8}$

4 Solve the quadratic equation $(3+\mathrm{i})w^2 - 2w + 3 - \mathrm{i} = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

$$w = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3+i)(3-i)}}{2(3+i)}$$

$$= \frac{2 \pm \sqrt{4 - 4(9 - 3i + 3i + 1)}}{6 + 2i}$$

$$= \frac{2 \pm \sqrt{4 - 4(10)}}{6 + 2i}$$

$$= \frac{2 \pm \sqrt{-36}}{6 + 2i}$$

$$= \frac{2 \pm 6i}{6 + 2i}$$

① $\frac{2 + 6i}{6 + 2i} \times \frac{(6 - 2i)}{(6 - 2i)}$

$$= \frac{12 - 4i + 36i + 12}{36 - 12i + 12i + 4}$$

$$= \frac{24 + 32i}{40}$$

$$= \frac{3}{5} + \frac{4}{5}i$$

② $\frac{2 - 6i}{6 + 2i} \times \frac{(6 - 2i)}{(6 - 2i)}$

$$= \frac{12 - 4i - 36i - 12}{36 - 12i + 12i + 4}$$

$$= \frac{-40i}{40}$$

$$= -i$$

5 Find the exact coordinates of the stationary points of the curve $y = \frac{e^{3x^2-1}}{1-x^2}$. u v [6]

$u = e^{3x^2-1}$ $v = 1-x^2$

$\frac{du}{dx} = 6xe^{3x^2-1}$ $\frac{dv}{dx} = -2x$

$\frac{dy}{dx} = \frac{(1-x^2)\times 6xe^{3x^2-1} - -2xe^{3x^2-1}}{(1-x^2)^2}$

$= \frac{6xe^{3x^2-1} - 6x^3e^{3x^2-1} + 2xe^{3x^2-1}}{(1-x^2)^2}$

$= \frac{8xe^{3x^2-1} - 6x^3e^{3x^2-1}}{(1-x^2)^2}$

SP, so $\frac{dy}{dx} = 0$:

$\frac{8xe^{3x^2-1} - 6x^3e^{3x^2-1}}{(1-x^2)^2} = 0$

$8xe^{3x^2-1} - 6x^3e^{3x^2-1} = 0$

$2xe^{3x^2-1}(4 - 3x^2) = 0$

$2x = 0$ or $e^{3x^2-1} = 0$ or $4-3x^2=0$

$x = 0$ no sol^ns $3x^2 = 4$

$x^2 = \frac{4}{3}$

$x = \pm\frac{2\sqrt{3}}{3}$

Sub: $x=0$: $y = \frac{e^{0-1}}{1-0}$ $x=+\frac{2\sqrt{3}}{3}$: $y = \frac{e^{4-1}}{1-\frac{4}{3}}$ $x=-\frac{2\sqrt{3}}{3}$: $y = \frac{e^{4-1}}{1-\frac{4}{3}}$

$y = e^{-1}$ $y = -3e^3$ $y = -3e^3$

$(0, e^{-1})$, $\left(\frac{2\sqrt{3}}{3}, -3e^3\right)$, $\left(\frac{-2\sqrt{3}}{3}, -3e^3\right)$

6 (a) Show that the equation $\cot^2\theta + 2\cos 2\theta = 4$ can be written in the form

$$4\sin^4\theta + 3\sin^2\theta - 1 = 0.$$ [3]

$$\frac{\cos^2\theta}{\sin^2\theta} + 2\cos 2\theta = 4$$

$$\cos^2\theta + 2\cos 2\theta \times \sin^2\theta = 4\sin^2\theta$$

$1 - 2\sin^2\theta$

$$\cos^2\theta + 2(1-2\sin^2\theta)\sin^2\theta = 4\sin^2\theta$$

$1 - \sin^2\theta$

$$1 - \sin^2\theta + 2\sin^2\theta - 4\sin^4\theta = 4\sin^2\theta$$

$$1 + \sin^2\theta - 4\sin^4\theta = 4\sin^2\theta$$

$$1 - 3\sin^2\theta - 4\sin^4\theta = 0$$

$\times -1$ $\quad \times -1$ $\quad \times -1$ $\quad \times -1$

$$-1 + 3\sin^2\theta + 4\sin^4\theta = 0$$

$$4\sin^4\theta + 3\sin^2\theta - 1 = 0$$ QED

(b) Hence solve the equation $\cot^2\theta + 2\cos 2\theta = 4$, for $0° < \theta < 360°$. [3]

from (a): $4\sin^4\theta + 3\sin^2\theta - 1 = 0$

Let $y = \sin^2\theta$:

$4y^2 + 3y - 1 = 0$

$(4y - 1)(y + 1) = 0$

$4y - 1 = 0$ or $y + 1 = 0$

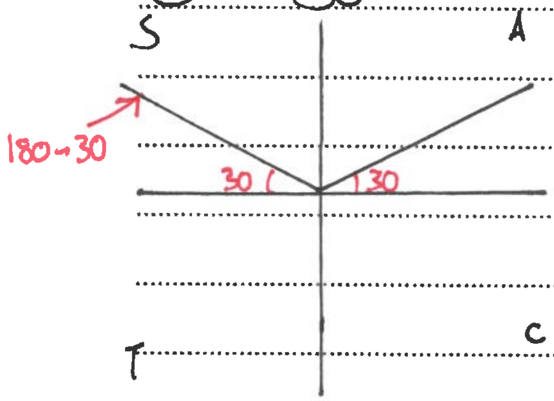
$y = \frac{1}{4}$ $\quad$ $y = -1$

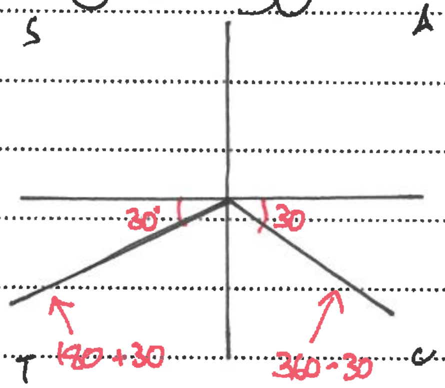
$\sin^2\theta = \frac{1}{4}$ $\quad$ $\sin^2\theta = -1$

X no sol^{n}s

$\sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{-1}{2}$

$\theta = 30°$ $\quad$ $\theta = -30°$





$\theta = 30°, 150°$ $\quad$ $\theta = 210°, 330°$

$\rightarrow \underline{\theta = 30°, 150°, 210°, 330°}$

7 The equation of a curve is $x^3 + y^2 + 3x^2 + 3y = 4$.

(a) Show that $\frac{dy}{dx} = -\frac{3x^2 + 6x}{2y + 3}$. [3]

differentiate:

$$3x^2 + 2y\frac{dy}{dx} + 6x + 3\frac{dy}{dx} = 0$$

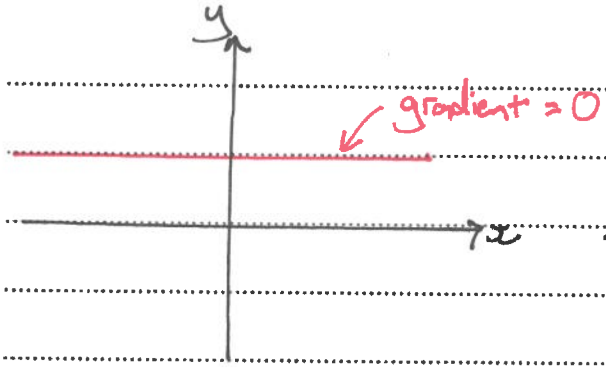
$$2y\frac{dy}{dx} + 3\frac{dy}{dx} = -3x^2 - 6x$$

$$\frac{dy}{dx}(2y + 3) = -3x^2 - 6x$$

$$\frac{dy}{dx} = \frac{-3x^2 - 6x}{2y + 3}$$

$$= -\frac{3x^2 + 6x}{2y + 3}$$ QED

(b) Hence find the coordinates of the points on the curve at which the tangent is parallel to the x-axis. [5]



numerator = 0

$$\frac{dy}{dx} = -\frac{3x^2+6x}{2y+3}$$

$$3x^2 + 6x = 0$$
$$x^2 + 2x = 0$$
$$x(x+2) = 0$$
$$\underline{x=0} \quad \text{or} \quad \underline{x=-2}$$

Sub. into equation:

$\underline{x=0:}$ $0^3 + y^2 + 3(0)^2 + 3y = 4$

$$y^2 + 3y = 4$$
$$y^2 + 3y - 4 = 0$$
$$(y+4)(y-1) = 0$$
$$\underline{y=-4}, \quad \underline{y=1}$$

$\underline{x=-2:}$ $(-2)^3 + y^2 + 3(-2)^2 + 3y = 4$

$$-8 + y^2 + 12 + 3y = 4$$
$$y^2 + 3y + 4 = 4$$
$$y^2 + 3y = 0$$
$$y(y+3) = 0$$
$$\underline{y=0}, \quad \underline{y=-3}$$

$\underline{(0,-4)}$, $\underline{(0,1)}$, $\underline{(-2,0)}$, $\underline{(-2,-3)}$

8 The variables x and y satisfy the differential equation

$$e^{4x}\frac{dy}{dx} = \cos^2 3y.$$

It is given that $y = 0$ when $x = 2$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

$$\int \frac{1}{\cos^2 3y}\, dy = \int \frac{1}{e^{4x}}\, dx$$

$$\int \sec^2 3y \, dy = \int e^{-4x}\, dx$$

$$\frac{1}{3}\tan 3y = -\frac{1}{4}e^{-4x} + C$$

$y=0,\ x=2$:

$$\frac{1}{3}\tan(0) = -\frac{1}{4}e^{-8} + C$$

$$0 = -\frac{1}{4}e^{-8} + C$$

$$C = \frac{1}{4}e^{-8}$$

→ $$\frac{1}{3}\tan 3y = -\frac{1}{4}e^{-4x} + \frac{1}{4}e^{-8}$$

$$\tan 3y = -\frac{3}{4}e^{-4x} + \frac{3}{4}e^{-8}$$

$$3y = \tan^{-1}\left(-\frac{3}{4}e^{-4x} + \frac{3}{4}e^{-8}\right)$$

$$y = \frac{1}{3}\tan^{-1}\left(-\frac{3}{4}e^{-4x} + \frac{3}{4}e^{-8}\right)$$

9 Let $f(x) = \dfrac{17x^2 - 7x + 16}{(2+3x^2)(2-x)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{17x^2 - 7x + 16}{(2+3x^2)(2-x)} \equiv \frac{Ax + B}{(2+3x^2)} + \frac{C}{2-x}$$

$$17x^2 - 7x + 16 \equiv (Ax+B)(2-x) + C(2+3x^2)$$

$\underline{x=2:}$

$17(2)^2 - 7(2) + 16 = 0 + C(2+3(2)^2)$

$68 - 14 + 16 = C(2+12)$

$70 = 14C$

$\underline{C = 5}$

$\underline{x=0:}$

$17(0)^2 - 7(0) + 16 = (0+B)(2) + C(2)$

$16 = 2B + 2C$

$16 = 2B + 10$

$2B = 6$

$\underline{B = 3}$

$\underline{x=1:}$

$17(1)^2 - 7(1) + 16 = (A+B)(1) + C(5)$

$17 - 7 + 16 = A + B + 5C$

$26 = A + 3 + 25$

$26 = A + 28$

$\underline{A = -2}$

$$\rightarrow \underline{\frac{-2x+3}{(2+3x^2)} + \frac{5}{2-x}}$$

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

A $(-2x+3)(2+3x^2)^{-1} = (-2x+3)\times 2^{-1}\left(1+\frac{3x^2}{2}\right)^{-1}$

$= \frac{1}{2}(-2x+3)\left(1+\frac{3x^2}{2}\right)^{-1}$

$= \frac{1}{2}(-2x+3)\left[1+(-1)\left(\frac{3x^2}{2}\right)+\ldots\right]$

$= \frac{1}{2}(-2x+3)\left(1-\frac{3x^2}{2}\right]$

$= \frac{1}{2}\left(-2x+3x^3+3-\frac{9}{2}x^2\right)$

$= \frac{3}{2}-x-\frac{9}{4}x^2+\frac{3}{2}x^3$

B $5(2-x)^{-1} = 5\times 2^{-1}\left(1-\frac{x}{2}\right)^{-1}$

$= \frac{5}{2}\left(1-\frac{x}{2}\right)^{-1}$

$= \frac{5}{2}\left[1+(-1)\left(\frac{-x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{-x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{-x}{2}\right)^3+\ldots\right]$

$= \frac{5}{2}\left[1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}\right]$

$= \frac{5}{2}+\frac{5}{4}x+\frac{5}{8}x^2+\frac{5}{16}x^3$

A + B: $\frac{3}{2}-x-\frac{9}{4}x^2+\frac{3}{2}x^3+\frac{5}{2}+\frac{5}{4}x+\frac{5}{8}x^2+\frac{5}{16}x^3$

$= 4+\frac{1}{4}x-\frac{13}{8}x^2+\frac{29}{16}x^3$

(c) State the set of values of x for which the expansion in **(b)** is valid. Give your answer in an exact form. [1]

$\left|\frac{3x^2}{2}\right| < 1$ and $\left|\frac{-x}{2}\right| < 1$

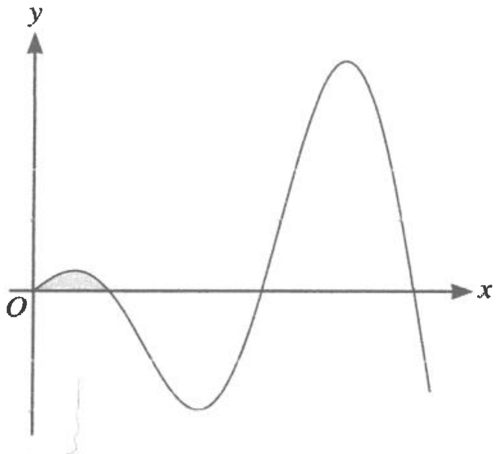
$|3x^2| < 2$ $\quad |x| < 2$

$|x^2| < \frac{2}{3}$

$|x| < \sqrt{\frac{2}{3}}$

so $|x| < \sqrt{\frac{2}{3}}$

10



The diagram shows the curve $y = \overset{u}{x} \overset{v}{\cos 2x}$, for $x \geqslant 0$.

(a) Find the equation of the tangent to the curve at the point where $x = \frac{1}{2}\pi$. [4]

$u = x$ $\quad\quad$ $v = \cos 2x$

$\frac{du}{dx} = 1$ $\quad\quad$ $\frac{dv}{dx} = -2\sin 2x$

$\frac{dy}{dx} = -2x\sin 2x + \cos 2x$

gradient at $x = \frac{\pi}{2}$:

$m = -2\left(\frac{\pi}{2}\right)\sin(\pi) + \cos(\pi)$

$\underline{m = -1}$

y-coordinate at $x = \frac{\pi}{2}$:

$y = \frac{\pi}{2} \times \cos(\pi)$

$\underline{y = \frac{-\pi}{2}}$

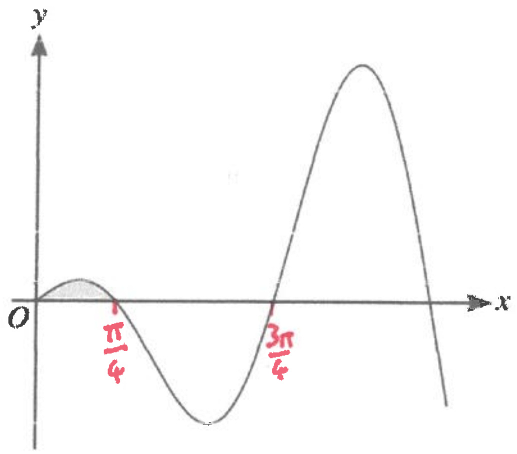
$\rightarrow y - y_1 = m(x - x_1)$

$y - -\frac{\pi}{2} = -1\left(x - \frac{\pi}{2}\right)$

$y + \frac{\pi}{2} = -x + \frac{\pi}{2}$

$\underline{y = -x}$

I L A T E



The diagram shows the curve $y = x\cos 2x$, for $x \geqslant 0$.

(b) Find the exact area of the shaded region shown in the diagram, bounded by the curve and the x-axis. [5]

roots: $x\cos 2x = 0$

$x = 0$ or $\cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$ ✓

$u = x \qquad v = \frac{1}{2}\sin 2x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \cos 2x$

$\rightarrow x \times \frac{1}{2}\sin 2x - \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin 2x \, dx$

$= \left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{1}{2}\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\cos\left(\frac{\pi}{2}\right)\right] - \left[0 + \frac{1}{4}\cos(0)\right]$

$= \left[\frac{\pi}{8} + 0\right] - \left[0 + \frac{1}{4}\right]$

$= \frac{\pi}{8} - \frac{1}{4}$

11 The line l has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The points A and B have position vectors $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively.

(a) Find a unit vector in the direction of l. [2]

direction vector of $l = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

Magnitude of direction vector $= \sqrt{(-1)^2 + 1^2 + 2^2}$
$= \sqrt{6}$

unit vector $= \dfrac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

The line m passes through the points A and B.

(b) Find a vector equation for m. [2]

$\overrightarrow{AB} = \underline{b} - \underline{a}$
$= \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$
$= \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$

point on line (A)

direction vector $\overrightarrow{AB}$

(c) Determine whether lines l and m are parallel, intersect or are skew. [5]

① Are they parallel?

If parallel, $\begin{pmatrix}-1\\1\\2\end{pmatrix} = t\begin{pmatrix}5\\-3\\2\end{pmatrix}$

$\underline{i}$: $-1 = 5t$, $t = \frac{-1}{5}$ $\quad$ $\underline{j}$: $1 = -3t$, $t = \frac{-1}{3}$ $\quad$ $\underline{k}$: $2 = 2t$, $t = 1$

No common solution for t, so not parallel.

② Do they intersect?

$$\begin{pmatrix}1-\lambda\\-2+\lambda\\-3+2\lambda\end{pmatrix} = \begin{pmatrix}-2+5\mu\\2-3\mu\\-1+2\mu\end{pmatrix}$$

$\underline{i}$: $1-\lambda = -2+5\mu$ $\quad$ $\underline{j}$: $-2+\lambda = 2-3\mu$

$3 = 5\mu + \lambda$ (A) $\quad$ $3\mu + \lambda = 4$ (B)

(A) − (B): $2\mu = -1$

$\mu = \frac{-1}{2}$

→ (B): $3\left(\frac{-1}{2}\right) + \lambda = 4$

$\frac{-3}{2} + \lambda = 4$

$\lambda = \frac{11}{2}$

Sub. into k:

$\underline{k}$: $-3 + 2\lambda = -1 + 2\mu$

$-3 + \frac{22}{2} = -1 + 2\left(\frac{-1}{2}\right)$

$-3 + 11 = -1 - 1$

$8 \neq -2$

Not equal, therefore lines don't intersect

Not parallel and don't intersect, so lines are skew.