1 Solve the equation $8^{3-6x} = 4 \times 5^{-2x}$. Give your answer correct to 3 decimal places. [4]

$\ln 8^{3-6x} = \ln[4 \times 5^{-2x}]$

$(3-6x)\ln 8 = \ln 4 + \ln 5^{-2x}$

$3\ln 8 - 6x\ln 8 = \ln 4 - 2x\ln 5$

$2x\ln 5 - 6x\ln 8 = \ln 4 - 3\ln 8$

$x(2\ln 5 - 6\ln 8) = \ln 4 - 3\ln 8$

$x = \dfrac{\ln 4 - 3\ln 8}{2\ln 5 - 6\ln 8}$

$\underline{\underline{x = 0.524}}$

2 Find the exact coordinates of the stationary point of the curve $y = \overset{u}{e^{2x}}\, \overset{v}{\sin 2x}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. [5]

$u = e^{2x}$ $\qquad$ $v = \sin 2x$

$\frac{du}{dx} = 2e^{2x}$ $\qquad$ $\frac{dv}{dx} = 2\cos 2x$

$\frac{dy}{dx} = 2e^{2x}\cos 2x + 2e^{2x}\sin 2x$

$= 2e^{2x}(\cos 2x + \sin 2x)$

SP: $\frac{dy}{dx} = 0$: $2e^{2x}(\cos 2x + \sin 2x) = 0$

$2e^{2x} = 0$ or $\cos 2x + \sin 2x = 0$ ✓

X No sol^s

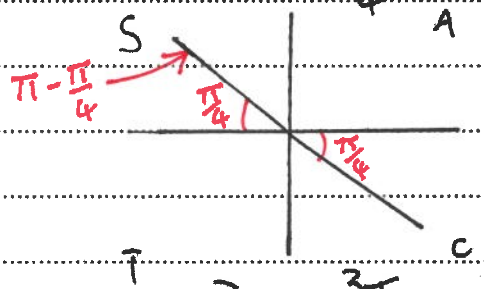
$\cos 2x + \sin 2x = 0$

$\sin 2x = -\cos 2x$

$\frac{\sin 2x}{\cos 2x} = -1$

$\tan 2x = -1$

$2x = \frac{-\pi}{4}$



$2x = \frac{3\pi}{4}$

$x = \frac{3\pi}{8}$

Sub into y: $y = e^{2\left(\frac{3\pi}{8}\right)} \sin\left(2 \times \frac{3\pi}{8}\right) = e^{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right)$

$= \frac{\sqrt{2}}{2} e^{\frac{3\pi}{4}}$

$\rightarrow \left(\frac{3\pi}{8}, \frac{\sqrt{2}}{2} e^{\frac{3\pi}{4}}\right)$

3 The square roots of $24-7\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $24-7\mathrm{i}$ in exact Cartesian form. [5]

if $\sqrt{24-7i} = x+iy$:

$24-7i = (x+iy)^2$

$24-7i = x^2 + 2xyi - y^2$

Real components:

$24 = x^2 - y^2$ ①

Imaginary components:

$-7 = 2xy$

$x = \dfrac{-7}{2y}$ ②

sub. ② → ①: $24 = \left(\dfrac{-7}{2y}\right)^2 - y^2$

$24 = \dfrac{49}{4y^2} - y^2$

$\times 4y^2 \quad \times 4y^2 \quad \times 4y^2$

$96y^2 = 49 - 4y^4$

$4y^4 + 96y^2 - 49 = 0$

let $X = y^2$:

$4X^2 + 96X - 49 = 0$

$X = \dfrac{-96 \pm \sqrt{(96)^2 - 4(4)(-49)}}{2(4)}$

$X = \dfrac{1}{2}$ or $X = \dfrac{-49}{2}$

$y^2 = \dfrac{1}{2}$ $\quad$ $y^2 = \dfrac{-49}{2}$

$y = \pm\dfrac{1}{\sqrt{2}}$ $\quad$ × y is real

sub. $y = \dfrac{1}{\sqrt{2}}$ into ②:

$x = \dfrac{-7}{2/\sqrt{2}}$

$= \dfrac{-7\sqrt{2}}{2}$

$\rightarrow \dfrac{-7\sqrt{2}}{2} + \dfrac{1}{\sqrt{2}}i$

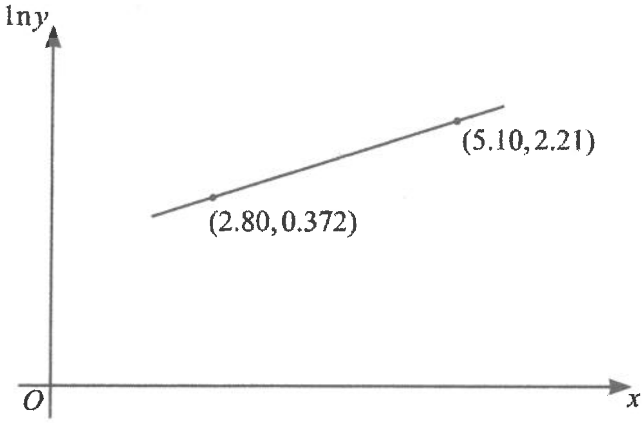
sub. $y = \dfrac{-1}{\sqrt{2}}$ into ②:

$x = \dfrac{-7}{-2/\sqrt{2}}$

$= \dfrac{7\sqrt{2}}{2}$

$\rightarrow \dfrac{7\sqrt{2}}{2} - \dfrac{1}{\sqrt{2}}i$

4



The variables x and y satisfy the equation $ky = e^{cx}$, where k and c are constants. The graph of $\ln y$ against x is a straight line passing through the points $(2.80, 0.372)$ and $(5.10, 2.21)$, as shown in the diagram.

Find the values of k and c. Give each value correct to 2 significant figures. [4]

$\ln(ky) = \ln(e^{cx})$

$\ln k + \ln y = cx$

$\ln y = cx - \ln k$

Sub. $(2.8, 0.372)$:

$0.372 = 2.8c - \ln k$ ①

Sub. $(5.1, 2.21)$:

$2.21 = 5.1c - \ln k$ ②

② − ①:

$1.838 = 2.3c$

$c = 0.7991..$ STO

→ ①: $0.372 = 2.8(0.7991..) - \ln k$

$\ln k = 2.2375... - 0.372$

$\ln k = 1.865...$

$k = e^{1.865...}$

$k = 6.5$ (2sf)

$c = 0.80$ (2sf)

5 Express $\frac{6x^2-2x+2}{(x-1)(2x+1)}$ in partial fractions. [5]

↑ improper fraction → algebraic division first

$(x-1)(2x+1) = 2x^2 - x - 1$

$$\begin{array}{r|l} & 3 \\ 2x^2 - x - 1 & 6x^2 - 2x + 2 \\ & 6x^2 - 3x - 3 \\ \hline & \quad\quad x + 5 \end{array}$$

$\rightarrow 3 + \frac{x+5}{(x-1)(2x+1)}$

$\frac{x+5}{(x-1)(2x+1)} \equiv \frac{A}{x-1} + \frac{B}{2x+1}$

$x+5 \equiv A(2x+1) + B(x-1)$

$x=1$:

$1+5 = 3A + 0$

$6 = 3A$

$A = 2$

$x = \frac{-1}{2}$:

$\frac{-1}{2} + 5 = 0 + B\left(\frac{-1}{2} - 1\right)$

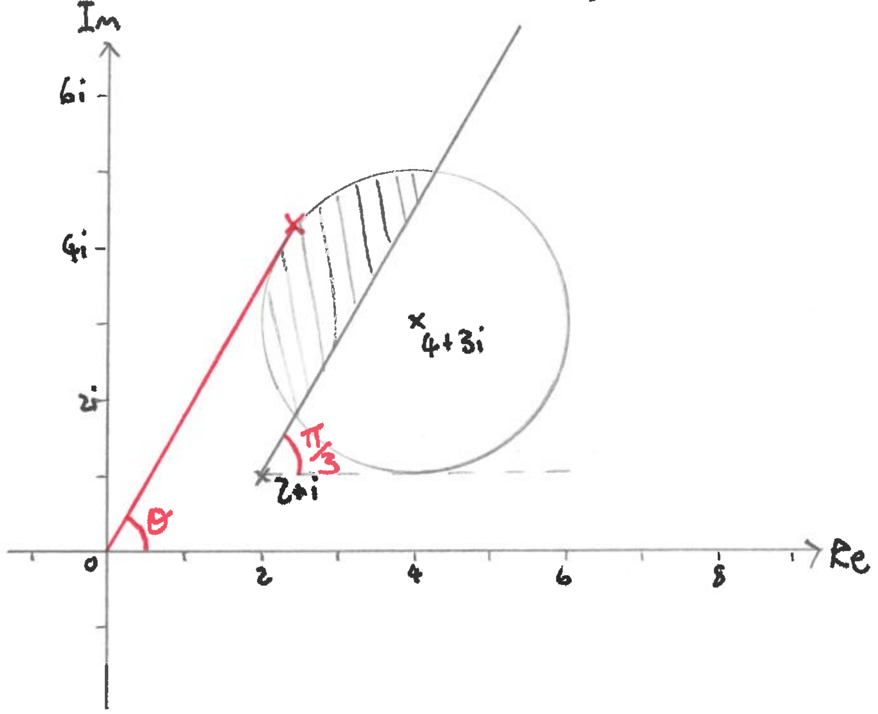
$\frac{9}{2} = \frac{-3}{2}B$

$9 = -3B$

$B = -3$

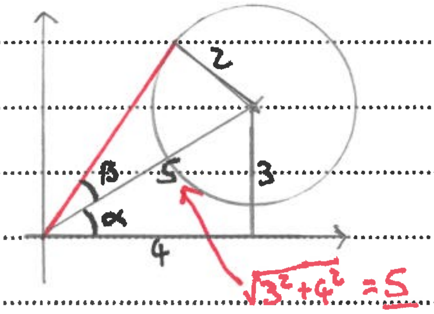
$\rightarrow 3 + \frac{2}{x-1} - \frac{3}{2x+1}$

6 **(a)** On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z-4-3\mathrm{i}| \leqslant 2$ and $\arg(z-2-\mathrm{i}) \geqslant \frac{1}{3}\pi$. [5]



(b) Calculate the greatest value of $\arg z$ for points in this region. [2]

Greatest value of arg(z) is θ on the diagram, where z is a tangent to the circle.



$\tan\alpha = \frac{3}{4}$

$\alpha = \tan^{-1}\left(\frac{3}{4}\right)$

$\sin\beta = \frac{2}{5}$

$\beta = \sin^{-1}\left(\frac{2}{5}\right)$

$\theta = \tan^{-1}\left(\frac{3}{4}\right) + \sin^{-1}\left(\frac{2}{5}\right)$

$= 1.06$

7 Let $f(x) = 8x^3 + 54x^2 - 17x - 21$.

(a) Show that $x + 7$ is a factor of $f(x)$. [1]

$f(-7) = 8(-7)^3 + 54(-7)^2 - 17(-7) - 21$

$= -2744 + 2646 + 119 - 21$

$= 0$

so $(x+7)$ is a factor

(b) Find the quotient when $f(x)$ is divided by $x + 7$. [2]

$$
\begin{array}{r|l}
 & 8x^2 - 2x - 3 \\
x + 7 & 8x^3 + 54x^2 - 17x - 21 \\
 & \underline{8x^3 + 56x^2} \\
 & \quad -2x^2 - 17x \\
 & \quad \underline{-2x^2 - 14x} \\
 & \qquad -3x - 21 \\
 & \qquad \underline{-3x - 21}
\end{array}
$$

quotient : $8x^2 - 2x - 3$

(c) Hence solve the equation

$$8\cos^3\theta + 54\cos^2\theta - 17\cos\theta - 21 = 0,$$

for $0° \leqslant \theta \leqslant 360°$. [3]

Let $x = \cos\theta$:

$$8x^3 + 54x^2 - 17x - 21 = 0$$

$$(x + 7)(8x^2 - 2x - 3) = 0$$

factorise: $ac = -24$; -6 and 4:

$8x^2 + 4x - 6x - 3$

$4x(2x + 1) - 3(2x + 1)$

$(4x - 3)(2x + 1)$

$\rightarrow (x + 7)(4x - 3)(2x + 1) = 0$

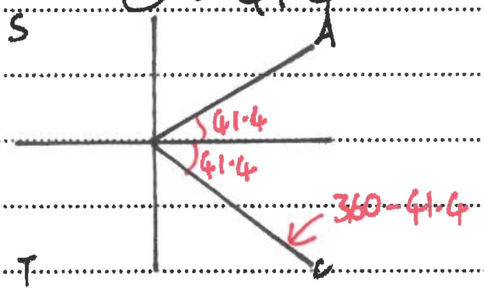
$x = -7$ $\quad x = \frac{3}{4}$ $\quad x = \frac{-1}{2}$

$\cos\theta = -7$ $\quad \cos\theta = \frac{3}{4}$ ① $\quad \cos\theta = \frac{-1}{2}$ ②

✗ no sol^{n}s

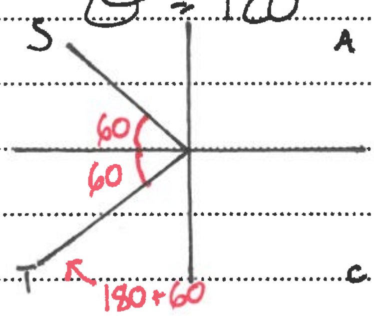
①: $\cos\theta = \frac{3}{4}$

$\theta = 41.4°$



②: $\cos\theta = \frac{-1}{2}$

$\theta = 120°$



$\theta = 41.4°, 318.6°$ $\qquad \theta = 120°, 240°$

$\theta = 41.4°, 120°, 240°, 318.6°$

8 **(a)** Express $3\cos 2x - \sqrt{3}\sin 2x$ in the form $R\cos(2x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and α. [3]

$3\cos 2x - \sqrt{3}\sin 2x = R\cos 2x\cos\alpha - R\sin 2x\sin\alpha$

$R\cos\alpha = 3$ ① $\quad R\sin\alpha = \sqrt{3}$ ②

② ÷ ①: $\tan\alpha = \frac{\sqrt{3}}{3}$

$\alpha = \frac{\pi}{6}$

①^2 + ②^2: $R^2 = (\sqrt{3})^2 + 3^2$
$= 3 + 9$
$= 12$
$R = \sqrt{12}$
$= \underline{2\sqrt{3}}$

$\rightarrow \underline{2\sqrt{3}\cos\left(2x + \frac{\pi}{6}\right)}$

(b) Hence find the exact value of $\int_0^{\frac{1}{12}\pi} \frac{3}{(3\cos 2x - \sqrt{3}\sin 2x)^2}\,dx$, simplifying your answer. [5]

$$\int_0^{\frac{\pi}{12}} \frac{3}{\left(2\sqrt{3}\cos\left(2x+\frac{\pi}{6}\right)\right)^2}\,dx \leftarrow \text{from part (a)}$$

$$= \int_0^{\frac{\pi}{12}} \frac{3}{12\cos^2\left(2x+\frac{\pi}{6}\right)}\,dx$$

$$= \int_0^{\frac{\pi}{12}} \frac{1}{4}\sec^2\left(2x+\frac{\pi}{6}\right)dx$$

$$= \left[\frac{1}{8}\tan\left(2x+\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{12}}$$

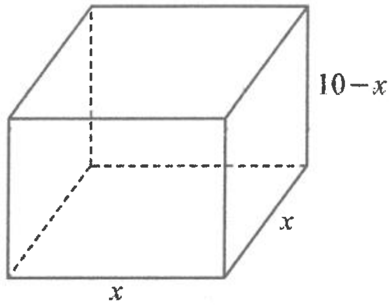
$$= \left[\frac{1}{8}\tan\left(\frac{\pi}{6}+\frac{\pi}{6}\right)\right] - \left[\frac{1}{8}\tan\left(\frac{\pi}{6}\right)\right]$$

$$= \left[\frac{1}{8}\times\sqrt{3}\right] - \left[\frac{1}{8}\times\frac{\sqrt{3}}{3}\right]$$

$$= \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{24}$$

$$= \frac{\sqrt{3}}{12}$$

9



A container in the shape of a cuboid has a square base of side x and a height of $(10-x)$. It is given that x varies with time, t, where $t > 0$. The container decreases in volume at a rate which is inversely proportional to t. *

When $t = \frac{1}{10}$, $x = \frac{1}{2}$ and the rate of decrease of x is $\frac{20}{37}$.

rate of decrease so $\frac{dx}{dt} = \frac{-20}{37}$

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{-1}{2t(20x-3x^2)}.$$

[5]

$$\text{Volume} = x \times x \times (10-x)$$
$$V = 10x^2 - x^3$$
$$\frac{dV}{dx} = 20x - 3x^2$$
$$*:\ \frac{dV}{dt} \propto \frac{-1}{t}$$
$$\frac{dV}{dt} = \frac{-k}{t}$$
$$\frac{dx}{dt} = \frac{dV}{dt} \times \frac{dx}{dV}$$
$$\frac{dx}{dt} = \frac{-k}{t} \times \frac{1}{20x-3x^2}$$

$\frac{dx}{dt} = \frac{-20}{37}$, $t = \frac{1}{10}$, $x = \frac{1}{2}$:

$$-\frac{20}{37} = -10k \times \frac{1}{10-\frac{3}{4}}$$
$$-\frac{20}{37} = \frac{-10k}{37/4}$$

$$-\frac{20}{37} = \frac{-40k}{37}$$
$$-20 = -40k$$
$$k = \frac{1}{2}$$

$$\frac{dx}{dt} = \frac{-(\frac{1}{2})}{t} \times \frac{1}{20x-3x^2}$$
$$= \frac{-1}{2t} \times \frac{1}{20x-3x^2}$$
$$= \frac{-1}{2t(20x-3x^2)}$$

QED

(b) Solve the differential equation, obtaining an expression for t in terms of x. [6]

$$\int(20x - 3x^2)\,dx = \int \frac{-1}{2t}\,dt$$

$$\int(20x - 3x^2)\,dx = \frac{-1}{2}\int \frac{1}{t}\,dt$$

$$10x^2 - x^3 = \frac{-1}{2}\ln t + C$$

$t = \frac{1}{10}, x = \frac{1}{2}$:

$$10\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 = \frac{-1}{2}\ln\left(\frac{1}{10}\right) + C$$

$$\frac{10}{4} - \frac{1}{8} = \frac{1}{2}\ln 10 + C$$

$$\frac{19}{8} - \frac{1}{2}\ln 10 = C$$

$$\rightarrow 10x^2 - x^3 = \frac{-1}{2}\ln t + \frac{19}{8} - \frac{1}{2}\ln 10$$

$\times 2$ $\times 2$ $\times 2$ $\times 2$ $\times 2$

$$20x^2 - 2x^3 = -\ln t + \frac{19}{4} - \ln 10$$

$$\ln t + \ln 10 = 2x^3 - 20x^2 + \frac{19}{4}$$

$$\ln(10t) = 2x^3 - 20x^2 + \frac{19}{4}$$

$$10t = e^{2x^3 - 20x^2 + \frac{19}{4}}$$

$$\underline{t = \frac{1}{10}e^{2x^3 - 20x^2 + \frac{19}{4}}}$$

10 The equations of two straight lines are

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + 2a\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} + a\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}),$$

where a is a constant.

(a) Given that the acute angle between the directions of these lines is $\frac{1}{4}\pi$, find the possible values of a. [6]

Scalar product with direction vectors:

$$\begin{pmatrix}3\\4\\a\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\2\end{pmatrix} = \sqrt{3^2+4^2+a^2} \times \sqrt{(-1)^2+2^2+2^2} \cos\theta$$

$$3\times -1 + 4\times 2 + 2a = \sqrt{25+a^2} \times \sqrt{9} \cos\theta$$

$$5 + 2a = 3\sqrt{25+a^2} \cos\theta$$

* Although the acute angle between the lines is $\frac{\pi}{4}$, the scalar product could give the answer $\frac{3\pi}{4}$. This doesn't matter because $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{3\pi}{4}\right) = \frac{-\sqrt{2}}{2}$ and we are going to square this:

$$5 + 2a = 3\sqrt{25+a^2} \times \left(\pm\frac{\sqrt{2}}{2}\right)$$

$$(5+2a)^2 = \left(3\sqrt{25+a^2} \times \left(\pm\frac{\sqrt{2}}{2}\right)\right)^2$$

$$25 + 20a + 4a^2 = 9(25+a^2)\left(\frac{2}{4}\right)$$

$$25 + 20a + 4a^2 = (225 + 9a^2) \times \frac{1}{2}$$

$$50 + 40a + 8a^2 = 225 + 9a^2$$

$$a^2 - 40a + 175 = 0$$

$$(a-5)(a-35) = 0$$

$$\underline{a = 5} \quad \text{or} \quad \underline{a = 35}$$

(b) Given instead that the lines intersect, find the value of a and the position vector of the point of intersection. [5]

$$\begin{pmatrix} 1 + 3\lambda \\ 1 + 4\lambda \\ 2a + a\lambda \end{pmatrix} = \begin{pmatrix} -3 - \mu \\ -1 + 2\mu \\ 4 + 2\mu \end{pmatrix}$$

$\underline{\underline{i}}$: $1 + 3\lambda = -3 - \mu$

$3\lambda + \mu = -4$ ①

$\underline{\underline{j}}$: $1 + 4\lambda = -1 + 2\mu$

$4\lambda - 2\mu = -2$ $\div 2$

$2\lambda - \mu = -1$ ②

① + ②: $5\lambda = -5$

$\lambda = -1$

→ ①: $3(-1) + \mu = -4$

$\mu = -1$

Sub. $\mu = 5$ to find point of intersection:

$$\underline{r} = \begin{pmatrix} -3 - -1 \\ -1 + -2 \\ 4 + -2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}}}$$

$\underline{\underline{k}}$: $2a + a\lambda = 4 + 2\mu$

$\lambda = -1, \mu = -1$:

$2a - a = 4 - 2$

$\underline{\underline{a = 2}}$

11 Use the substitution $2x = \tan\theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{12}{(1+4x^2)^2}\,dx .$$

Give your answer in the form $a + b\pi$, where a and b are rational numbers. [9]

$$\int_0^{\frac{1}{2}} \frac{12}{(1+(2x)^2)^2}\,dx$$

$2x = \tan\theta$

$x = \frac{1}{2}\tan\theta$

$\frac{dx}{d\theta} = \frac{1}{2}\sec^2\theta$

$\boxed{dx = \frac{1}{2}\sec^2\theta\, d\theta}$

$$\int_0^{\frac{1}{2}} \frac{12}{(1+(2x)^2)^2} \times \frac{1}{2}\sec^2\theta\, d\theta$$

$2(\frac{1}{2}) = \tan\theta$ $\quad$ $2(0) = \tan\theta$

$1 = \tan\theta$ $\quad$ $0 = \tan\theta$

$\theta = \frac{\pi}{4}$ $\quad$ $\theta = 0$

$$= \int_0^{\frac{\pi}{4}} \frac{12}{(1+\tan^2\theta)^2} \times \frac{1}{2}\sec^2\theta\, d\theta$$

$1+\tan^2\theta = \sec^2\theta$

$$= \int_0^{\frac{\pi}{4}} \frac{12}{(\sec^2\theta)^2} \times \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{6\sec^2\theta}{\sec^4\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{6}{\sec^2\theta}\, d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta + 1 = 2\cos^2\theta$

$6\cos^2\theta = 3\cos 2\theta + 3$

$$= \int_0^{\frac{\pi}{4}} 6\cos^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (3\cos 2\theta + 3)\, d\theta$$

$$= \left[\frac{3}{2}\sin 2\theta + 3\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2}\sin\left(\frac{\pi}{2}\right) + \frac{3\pi}{4}\right] - \left[0 + 0\right]$$

$$= \frac{3}{2} \times 1 + \frac{3\pi}{4}$$

$$= \underline{\underline{\frac{3}{2} + \frac{3\pi}{4}}}$$