1 Find the coefficient of x^2 in the expansion of

$$(2-5x)(1+3x)^{10}.$$

[4]

$(1+3x)^{10} = 1^{10} + 10 \times 1^9 \times (3x) + 45 \times 1^8 \times (3x)^2 + \ldots$

$= 1 + 30x + 45 \times 9x^2 + \ldots$

$= 1 + 30x + 405x^2 + \ldots$

$(2-5x)(1 + 30x + 405x^2 + \ldots)$

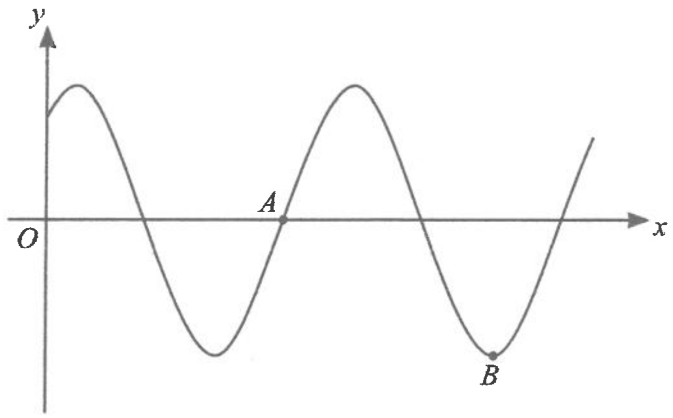
coeff of x^2: $2 \times 405x^2 + -5x \times 30x$

$= 810x^2 - 150x^2$

$= 660x^2$

coeff $= \underline{\underline{660}}$

2 **(a)**



The diagram shows the curve $y = k\cos(x - \frac{1}{6}\pi)$ where k is a positive constant and x is measured in radians. The curve crosses the x-axis at point A and B is a minimum point.

Find the coordinates of A and B. [3]

Cosine graph has been translated by $\begin{pmatrix} \pi/6 \\ 0 \end{pmatrix}$. On a normal cosine graph, A would be at $\frac{3\pi}{2}$: $\frac{3\pi}{2} + \frac{\pi}{6} = \frac{5\pi}{3}$

$\rightarrow$ $A = \left(\frac{5\pi}{3}, 0\right)$

x-coordinate of B on a normal graph is 3π: $3\pi + \frac{\pi}{6} = \frac{19\pi}{6}$

y-coordinates have been multiplied by k: $k \times -1 = -k$

$\rightarrow$ $B = \left(\frac{19\pi}{6}, -k\right)$

(b) Find the exact value of t that satisfies the equation

$$3\sin^{-1}(3t) + 2\cos^{-1}\left(\tfrac{1}{2}\sqrt{2}\right) = \pi.$$ [2]

$2\cos^{-1}\left(\frac{1}{2}\sqrt{2}\right) = \frac{\pi}{2}$:

$\rightarrow$ $3\sin^{-1}(3t) + \frac{\pi}{2} = \pi$

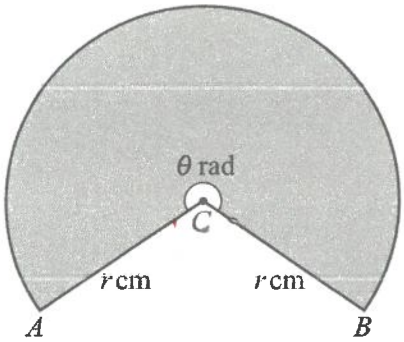
$3\sin^{-1}(3t) = \frac{\pi}{2}$

$\sin^{-1}(3t) = \pi/6$

$3t = \frac{1}{2}$

$t = 1/6$

3



The diagram shows a sector of a circle with centre C. The radii CA and CB each have length r cm and the size of the reflex angle ACB is θ radians. The sector, shaded in the diagram, has a perimeter of 65 cm and an area of 225 cm^2.

(a) Find the values of r and θ. [4]

Area: $\text{Area} = \frac{1}{2}r^2\theta$

$225 = \frac{1}{2}r^2\theta$

$450 = r^2\theta$

$\theta = \frac{450}{r^2}$ ①

Perimeter: $\text{Perimeter} = r + r + r\theta$

$65 = 2r + r\theta$ ②

① → ②: $65 = 2r + r\left(\frac{450}{r^2}\right)$

$65 = 2r + \frac{450}{r}$

$\times r \quad \times r \quad \times r$

$65r = 2r^2 + 450$

$2r^2 - 65r + 450 = 0$

$r = \frac{-(-65) \pm \sqrt{(-65)^2 - 4(2)(450)}}{2(2)}$

$r = 22.5$ or $r = 10$

$r = 22.5$: $\theta = \frac{450}{22.5^2}$

$= \frac{8}{9}$ ✗ (θ is a reflex angle)

$r = 10$: $\theta = \frac{450}{10^2}$

$= 4.5$

→ $r = 10$, $\theta = 4.5$

(b) Find the area of triangle ACB. [2]

Angle $ACB = 2\pi - 4.5$

$= 1.7832$ STO

Area of triangle $= \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 10 \times 10 \times \sin(1.7832)$

$= 48.9 \text{ cm}^2$

$\frac{\sin\theta}{\cos\theta}$

4 **(a)** Show that the equation $\cos\theta(7\tan\theta - 5\cos\theta) = 1$ can be written in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are integers to be found. [3]

$\cos\theta\left(7\frac{\sin\theta}{\cos\theta} - 5\cos\theta\right) = 1$

$7\sin\theta - 5\cos^2\theta = 1$

$7\sin\theta - 5(1-\sin^2\theta) = 1$

$7\sin\theta - 5 + 5\sin^2\theta = 1$

$5\sin^2\theta + 7\sin\theta - 6 = 0$ QED

(b) Hence solve the equation $\cos 2x(7\tan 2x - 5\cos 2x) = 1$ for $0° < x < 180°$. [3]

Replace θ with $2x$:

$0 < x < 180$

$0 < 2x < 360$

$5\sin^2(2x) + 7\sin(2x) - 6 = 0$

Let $Y = \sin 2x$:

$5Y^2 + 7Y - 6 = 0$

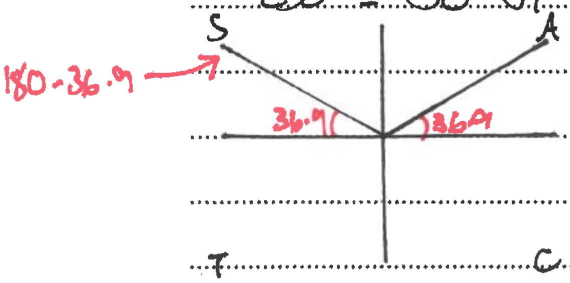
$(5Y-3)(Y+2) = 0$

$5Y - 3 = 0$ or $Y + 2 = 0$

$Y = \frac{3}{5}$ $\quad$ $Y = -2$

$\sin 2x = \frac{3}{5}$ $\quad$ $\sin 2x = -2$

$2x = 36.87°$ $\quad$ ✗ no sol^{n}s



$\rightarrow 2x = 36.87°, 143.13°$

$x = 18.4°, 71.6°$

5 The equation of a curve is $y = 2x^2 - \frac{1}{2x} + 3$.

(a) Find the coordinates of the stationary point. [3]

$y = 2x^2 - \frac{1}{2}x^{-1} + 3$

$\frac{dy}{dx} = 4x + \frac{1}{2}x^{-2}$

SP: $\frac{dy}{dx} = 0$: $4x + \frac{1}{2}x^{-2} = 0$

$4x + \frac{1}{2x^2} = 0$

$\times 2x^2 \quad \times 2x^2 \quad \times 2x^2$

$8x^3 + 1 = 0$

$8x^3 = -1$

$x^3 = \frac{-1}{8}$

$x = \frac{-1}{2}$

$y = 2\left(\frac{-1}{2}\right)^2 - \frac{1}{2\left(\frac{-1}{2}\right)} + 3$

$= 2\left(\frac{1}{4}\right) - \frac{1}{-1} + 3$

$= \frac{1}{2} + 1 + 3$

$= \frac{9}{2}$

$\left(\frac{-1}{2}, \frac{9}{2}\right)$

(b) Determine the nature of the stationary point. [2]

$\frac{d^2y}{dx^2} = 4 - x^{-3}$

Sub. $x = \frac{-1}{2}$: $4 - \left(\frac{-1}{2}\right)^{-3}$

$= 4 - \frac{-1}{8}$

$= \frac{33}{8}$

$\frac{d^2y}{dx^2} > 0$, so minimum point

(c) For positive values of x, determine whether the curve shows a function that is increasing, decreasing or neither. Give a reason for your answer. [2]

$\frac{dy}{dx} = 4x + \frac{1}{2x^2}$

When x is positive, both terms in this expression are positive, so $\frac{dy}{dx}$ is always positive, so it's an increasing function.

6 A curve passes through the point $\left(\frac{4}{5}, -3\right)$ and is such that $\frac{dy}{dx} = \frac{-20}{(5x-3)^2}$.

(a) Find the equation of the curve. [4]

$y = \int \frac{-20}{(5x-3)^2}\,dx = \int -20(5x-3)^{-2}\,dx$

$= -20 \times -1 \times \frac{1}{5}(5x-3)^{-1} + C$

$y = 4(5x-3)^{-1} + C$

sub. $\left(\frac{4}{5}, -3\right)$: $-3 = 4\left(5\left(\frac{4}{5}\right) - 3\right)^{-1} + C$

$-3 = 4(4-3)^{-1} + C$

$-3 = 4 \times 1 + C$

$-3 = 4 + C$

$C = -7 \rightarrow y = 4(5x-3)^{-1} - 7$

(b) The curve is transformed by a stretch in the x-direction with scale factor $\frac{1}{2}$ followed by a translation of $\begin{pmatrix} 2 \\ 10 \end{pmatrix}$.

① ②

Find the equation of the new curve. [3]

①: Replace x with $2x$:

$y = 4(5(2x) - 3)^{-1} - 7$

$= 4(10x - 3)^{-1} - 7$

shift to right by 2.

②: (i) Replace x with $(x-2)$:

$y = 4(10(x-2) - 3)^{-1} - 7$

$y = 4(10x - 20 - 3)^{-1} - 7$

$= 4(10x - 23)^{-1} - 7$

(ii) add 10 to y-coords:

$y = 4(10x-23)^{-1} - 7 + 10$

$y = 4(10x-23)^{-1} + 3$

7 The first term of an arithmetic progression is 1.5 and the sum of the first ten terms is 127.5 .

(a) Find the common difference. [2]

$a = 1.5 \qquad S_{10} = \frac{10}{2}[2(1.5) + 9 \times d]$

$127.5 = 5[3 + 9d]$

$127.5 = 15 + 45d$

$45d = 112.5$

$d = 2.5$

(b) Find the sum of all the terms of the arithmetic progression whose values are between 25 and 100. [5]

$u_n = a + (n-1)d$

Find first term over 25:

$1.5 + (n-1) \times 2.5 > 25$

$(n-1) \times 2.5 > 23.5$

$n - 1 > 9.4$

$n > 10.4 \rightarrow 11^{th}$ term

Find last term less than 100:

$1.5 + (n-1) \times 2.5 < 100$

$(n-1) \times 2.5 < 98.5$

$n-1 < 39.4$

$n < 40.4 \rightarrow 40^{th}$ term

Sum between 11^{th} and 40^{th} terms (inclusive):

$\text{Sum} = S_{40} - S_{10}$

$= \frac{40}{2}[2(1.5) + 39(2.5)] - \frac{10}{2}[2(1.5) + 9(2.5)]$

$= 20(100.5) - 5(25.5)$

$= 2010 - 127.5$

$= 1882.5$

8 A circle with equation $x^2+y^2-6x+2y-15=0$ meets the y-axis at the points A and B. The tangents to the circle at A and B meet at the point P.

Find the coordinates of P. [8]

$x^2-6x+y^2+2y=15$

$(x-3)^2-9+(y+1)^2-1=15$

$(x-3)^2+(y+1)^2=25$

→ centre $=(3,-1)$, radius $=5$

Find points A & B: At y axis, $x=0$:

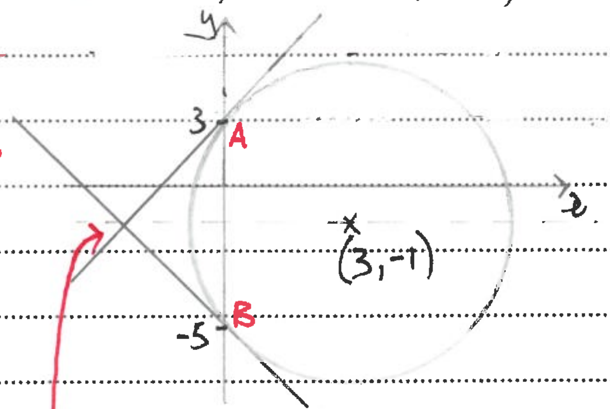
$0^2+y^2-6(0)+2y-15=0$

$y^2+2y-15=0$

$(y+5)(y-3)=0$

$y=-5$ or $y=3$

→ $(0,-5)$ and $(0,3)$



Through symmetry, y-coordinate of point of intersection must be $y=-1$.

Find gradient of radius between centre and A:

$m=\frac{-1-3}{3-0}=\frac{-4}{3}$

gradient of tangent through A:

$m_{perp}=\frac{3}{4}$

Eqn of line through A:

$$y - 3 = \frac{3}{4}(x - 0)$$

$$y - 3 = \frac{3}{4}x$$

$$y = \frac{3}{4}x + 3$$

Find x-coord of point of intersection by substituting $y = -1$:

$$-1 = \frac{3}{4}x + 3$$

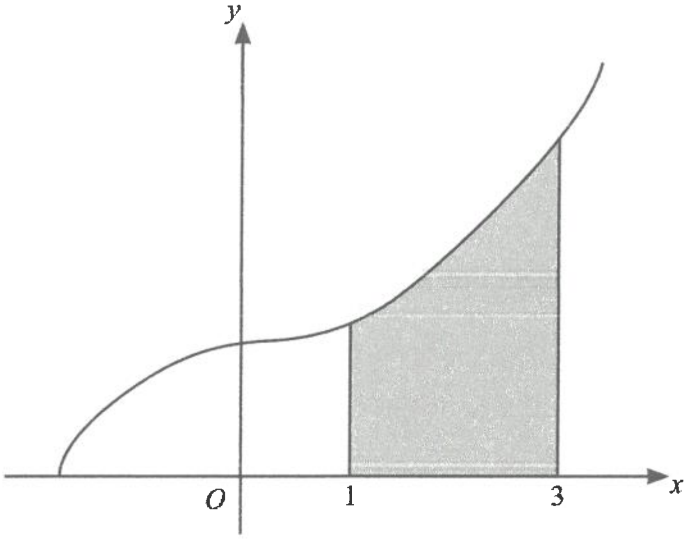
$$\frac{3}{4}x = -4$$

$$3x = -16$$

$$x = \frac{-16}{3}$$

$$\left(\frac{-16}{3}, -1\right)$$

9



The diagram shows the curve with equation $y = \sqrt{2x^3 + 10}$.

(a) Find the equation of the tangent to the curve at the point where $x = 3$. Give your answer in the form $ax + by + c = 0$ where a, b and c are integers. [5]

$y = (2x^3 + 10)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(2x^3 + 10)^{-\frac{1}{2}} \times 6x^2$

Sub. $x = 3$: $m = \frac{1}{2}(2(3)^3 + 10)^{-\frac{1}{2}} \times 6(3)^2$

$= \frac{1}{2}(64)^{-\frac{1}{2}} \times 54$

$= \frac{1}{2} \times \frac{1}{8} \times 54$

$= \frac{27}{8}$

Find y-coord when $x = 3$:

$y = \sqrt{2(3)^3 + 10}$

$= \sqrt{64} = 8$

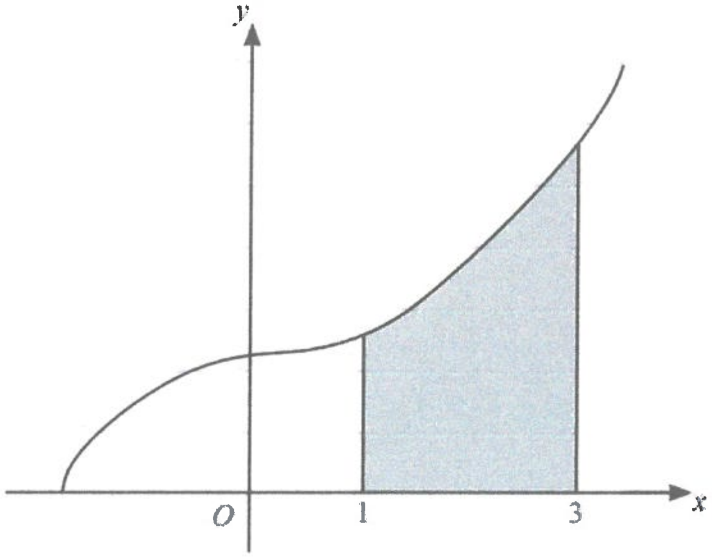
Eqn of line: $y - 8 = \frac{27}{8}(x - 3)$

$\times 8 \quad \times 8 \quad \times 8$

$8y - 64 = 27(x - 3)$

$8y - 64 = 27x - 81$

$27x - 8y - 17 = 0$



The diagram shows the curve with equation $y = \sqrt{2x^3 + 10}$.

(b) The region shaded in the diagram is enclosed by the curve and the straight lines $x = 1$, $x = 3$ and $y = 0$.

Find the volume of the solid obtained when the shaded region is rotated through 360° about the x-axis. [3]

$V = \pi \int y^2 \, dx$

$= \pi \int_1^3 (2x^3 + 10) \, dx$

$= \pi \left[\frac{1}{2}x^4 + 10x \right]_1^3$

$= \pi \left[\frac{1}{2} \times 3^4 + 10(3) \right] - \pi \left[\frac{1}{2} \times 1^4 + 10(1) \right]$

$= \pi \left[\frac{81}{2} + 30 \right] - \pi \left[\frac{1}{2} + 10 \right]$

$= 70.5\pi - 10.5\pi$

$= \underline{60\pi}$

10 The geometric progression $a_1, a_2, a_3, \ldots$ has first term 2 and common ratio r where $r > 0$. It is given that $\frac{9}{2}a_5 + 7a_3 = 8$.

(a) Find the value of r. [3]

a_3: $a_3 = ar^2$ a_5: $a_5 = ar^4$

$= 2r^2$ $= 2r^4$

Sub. in: $\frac{9}{2} \times 2r^4 + 7 \times 2r^2 = 8$

$9r^4 + 14r^2 - 8 = 0$

Let $y = r^2$:

$9y^2 + 14y - 8 = 0$

$y = \frac{-14 \pm \sqrt{14^2 - 4(9)(-8)}}{2(9)}$

$y = \frac{4}{9}$ or $y = -2$

$r^2 = \frac{4}{9}$ $r^2 = -2$

× no sol^ns

$r = \frac{2}{3}$ or $r = \frac{-2}{3}$ × $r > 0$

(b) Find the sum of the first 20 terms of the geometric progression. Give your answer correct to 4 significant figures. [2]

$S_{20} = \frac{a(1 - r^n)}{1 - r}$

$= \frac{2\left(1 - \left(\frac{2}{3}\right)^{20}\right)}{1 - \frac{2}{3}}$

$= 5.998$ (4sf)

(c) Find the sum to infinity of the progression $a_2, a_5, a_8, \ldots$. [3]

a_2: $a_2 = 2 \times \frac{2}{3}$
$= \frac{4}{3}$

a_5: $a_5 = 2 \times \left(\frac{2}{3}\right)^4$
$= \frac{32}{81}$

a_8: $a_8 = 2 \times \left(\frac{2}{3}\right)^7$
$= \frac{256}{2187}$

$\rightarrow \frac{4}{3} \xrightarrow{\times \frac{8}{27}} \frac{32}{81} \xrightarrow{\times \frac{8}{27}} \frac{256}{2187}$

So:
$a = \frac{4}{3}$ and $r = \frac{8}{27}$

$S_\infty = \frac{a}{1-r}$
$= \frac{\frac{4}{3}}{1-\frac{8}{27}}$
$= \frac{36}{19}$

11 The function f is defined by $f(x) = 10 + 6x - x^2$ for $x \in \mathbb{R}$.

(a) By completing the square, find the range of f. [3]

$$f(x) = -x^2 + 6x + 10$$

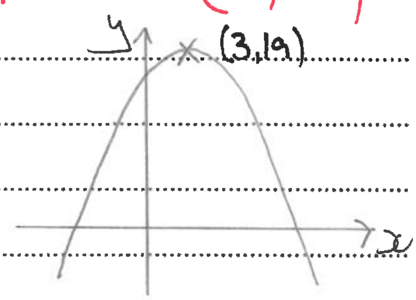
$$= -\left[x^2 - 6x\right] + 10$$

$$= -\left[(x-3)^2 - 9\right] + 10$$

$$= -(x-3)^2 + 9 + 10$$

$$f(x) = -(x-3)^2 + 19$$

Negative quadratic with max. point at (3, 19)



Range: $f(x) \leq 19$

The function g is defined by $g(x) = 4x + k$ for $x \in \mathbb{R}$ where k is a constant.

(b) It is given that the graph of $y = g^{-1}f(x)$ meets the graph of $y = g(x)$ at a single point P.

Determine the coordinates of P. [6]

$g^{-1}(x)$: $y = 4x + k$

$y - k = 4x$

$x = \dfrac{y-k}{4}$

$g^{-1}(x) = \dfrac{x-k}{4}$

$g^{-1}f(x)$: $g^{-1}f(x) = \dfrac{(f(x)) - k}{4}$

$= \dfrac{10 + 6x - x^2 - k}{4}$

$g^{-1}f(x) = g(x)$:

$\dfrac{10 + 6x - x^2 - k}{4} = 4x + k$

$10 + 6x - x^2 - k = 16x + 4k$

$x^2 + 10x + 5k - 10 = 0$ (1)

Only one point of intersection, so only one solution, so $b^2 - 4ac = 0$:

$10^2 - 4(1)(5k - 10) = 0$

$100 - 20k + 40 = 0$

$20k = 140$

$k = 7$

Sub into (1): $x^2 + 10x + 5(7) - 10 = 0$

$x^2 + 10x + 25 = 0$

$(x+5)^2 = 0$

$x = -5$

Sub. into $g(x)$:

$y = 4x + 7$

$= 4(-5) + 7$

$= -20 + 7$

$= -13$

$\rightarrow (-5, -13)$