1 The coefficient of x^2 in the expansion of $(1-4x)^6$ is 12 times the coefficient of x^2 in the expansion of $(2+ax)^5$.

Find the value of the positive constant a. [3]

$(1-4x)^6 = 1^6 + 6\times1^5\times(-4x) + 15\times1^4\times(-4x)^2 + \dots$

$= 1 - 24x + \underline{240x^2} + \dots$

coeff of $x^2 = 240$

$(2+ax)^5 = 2^5 + 5\times2^4\times(ax) + 10\times2^3\times(ax)^2 + \dots$

$= 32 + 80ax + 80a^2x^2 + \dots$

coeff of $x^2 = 80a^2$

$240 = 12 \times 80a^2$

$240 = 960a^2$

$a^2 = \frac{1}{4}$

$a = \pm\frac{1}{2}$

a is positive : $\underline{\underline{a = \frac{1}{2}}}$

2 The curve $y = x^2$ is transformed to the curve $y = 4(x-3)^2 - 8$.

Describe fully a sequence of transformations that have been combined, making clear the order in which the transformations have been applied. [5]

$$x^2 \xrightarrow{①} (x-3)^2 \xrightarrow{②} 4(x-3)^2 \xrightarrow{③} 4(x-3)^2 - 8$$

$$f(x) \longrightarrow f(x-3) \longrightarrow 4f(x-3) \longrightarrow 4f(x-3) - 8$$

① Translation by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

② Stretch, parallel to y-axis, scale factor 4

③ Translation by $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$

$\frac{\sin\theta}{\cos\theta}$

3 **(a)** Show that the equation $\frac{7\tan\theta}{\cos\theta} + 12 = 0$ can be expressed as

$$12\sin^2\theta - 7\sin\theta - 12 = 0.$$

[3]

$\frac{7\frac{\sin\theta}{\cos\theta}}{\cos\theta} + 12 = 0$ $\times\cos\theta$ $\times\cos\theta$

$\frac{7\sin\theta}{\cos^2\theta} + 12 = 0$

$7\sin\theta + 12\cos^2\theta = 0$ $1-\sin^2\theta$

$7\sin\theta + 12(1-\sin^2\theta) = 0$

$7\sin\theta + 12 - 12\sin^2\theta = 0$

$-12\sin^2\theta + 7\sin\theta + 12 = 0$ $\times -1$

$\underline{12\sin^2\theta - 7\sin\theta - 12 = 0}$

(b) Hence solve the equation $\frac{7\tan\theta}{\cos\theta} + 12 = 0$ for $0° \leqslant \theta \leqslant 360°$. [3]

Let $Y = \sin\theta$:

$12Y^2 - 7Y - 12 = 0$

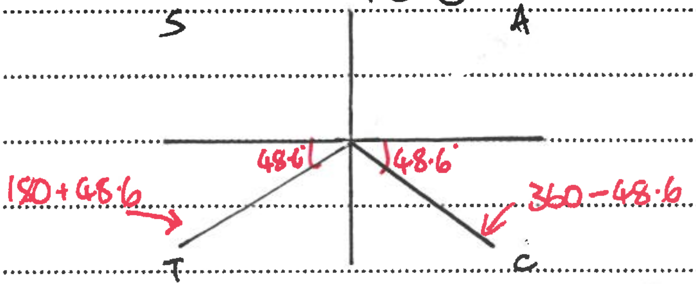
$Y = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(12)(-12)}}{2(12)}$

$Y = \frac{4}{3}$ or $Y = \frac{-3}{4}$

$\sin\theta = \frac{4}{3}$ $\sin\theta = \frac{-3}{4}$

no sol^{n}s

$\theta = -48.6°$



$\underline{\theta = 228.6°, 311.4°}$

4 The function f is defined as follows:

$$f(x) = \sqrt{x} - 1 \text{ for } x > 1.$$

(a) Find an expression for $f^{-1}(x)$. [1]

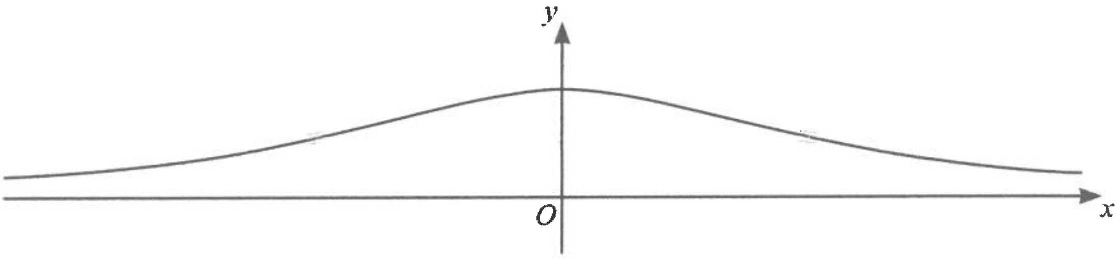
$y = \sqrt{x} - 1$

$y + 1 = \sqrt{x}$

$(y+1)^2 = x$

$x = (y+1)^2$

$f^{-1}(x) = (x+1)^2$



The diagram shows the graph of $y = g(x)$ where $g(x) = \frac{1}{x^2+2}$ for $x \in \mathbb{R}$.

(b) State the range of g and explain whether g^{-1} exists. [2]

Max. value of y is when $x = 0$:

$y = \frac{1}{0+2} = \frac{1}{2}$

Min. value of y is when $x \to \pm\infty$:

$y = \frac{1}{\infty + 2} = 0$ (but never actually reaches 0)

so $0 < g(x) \leq \frac{1}{2}$

g(x) is a many-to-one function. Only one-to-one functions have an inverse, so g^{-1} does not exist.

The function h is defined by $h(x) = \frac{1}{x^2+2}$ for $x \geqslant 0$.

(c) Solve the equation $hf(x) = f\left(\frac{25}{16}\right)$. Give your answer in the form $a + b\sqrt{c}$, where a, b and c are integers. [4]

$$f\left(\frac{25}{16}\right) = \sqrt{\frac{25}{16}} - 1$$

$$= \frac{5}{4} - 1$$

$$= \underline{\frac{1}{4}}$$

hf(x):

$$\frac{1}{(f(x))^2 + 2} = \frac{1}{(\sqrt{x}-1)^2 + 2}$$

$$= \frac{1}{x - 2\sqrt{x} + 1 + 2}$$

$$= \frac{1}{x - 2\sqrt{x} + 3}$$

$hf(x) = \frac{1}{4}$:

$$\frac{1}{x - 2\sqrt{x} + 3} \times \frac{1}{4}$$

$$4 = x - 2\sqrt{x} + 3$$

$$x - 2\sqrt{x} - 1 = 0$$

$$x - 2x^{\frac{1}{2}} - 1 = 0$$

let $Y = x^{\frac{1}{2}}$:

$$Y^2 - 2Y - 1 = 0$$

$$Y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$Y = 1 + \sqrt{2}$ or $Y = 1 - \sqrt{2}$

$\sqrt{x} = 1 + \sqrt{2}$ or $\sqrt{x} = 1 - \sqrt{2}$

$x = (1+\sqrt{2})^2$ $\quad$ $x = (1-\sqrt{2})^2$

$= 1 + 2\sqrt{2} + 2$ $\quad$ $= 1 - 2\sqrt{2} - 2$

$= \underline{3 + 2\sqrt{2}}$ $\quad$ $= \underline{-1 - 2\sqrt{2}}$

5 The first and second terms of an arithmetic progression are $\tan\theta$ and $\sin\theta$ respectively, where $0 < \theta < \frac{1}{2}\pi$.

(a) Given that $\theta = \frac{1}{4}\pi$, find the exact sum of the first 40 terms of the progression. [4]

$\theta = \frac{\pi}{4}$:

$$u_1 = \tan\left(\frac{\pi}{4}\right)$$
$$= 1$$
$$u_2 = \sin\left(\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2}$$

$$d = u_2 - u_1$$
$$= \frac{\sqrt{2}}{2} - 1$$

$$S_{40} = \frac{40}{2}\left[2(1) + (39)\left(\frac{\sqrt{2}}{2} - 1\right)\right]$$
$$= 20\left[2 + \frac{39\sqrt{2}}{2} - 39\right]$$
$$= 20\left(\frac{39\sqrt{2}}{2} - 37\right)$$
$$= \underline{390\sqrt{2} - 740}$$

The first and second terms of a geometric progression are $\tan\theta$ and $\sin\theta$ respectively, where $0 < \theta < \frac{1}{2}\pi$.

(b) **(i)** Find the sum to infinity of the progression in terms of θ. [2]

$$r = \frac{u_2}{u_1} = \frac{\sin\theta}{\tan\theta} \leftarrow \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta \times \cos\theta}{\frac{\sin\theta}{\cos\theta} \times \cos\theta}$$

$$= \frac{\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}}$$

$$= \cos\theta$$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{\tan\theta}{1-\cos\theta}$$

(ii) Given that $\theta = \frac{1}{3}\pi$, find the sum of the first 10 terms of the progression. Give your answer correct to 3 significant figures. [3]

$$a = \tan\theta$$
$$= \tan\left(\frac{\pi}{3}\right)$$
$$= \sqrt{3}$$

$$r = \cos\theta$$
$$= \cos\left(\frac{\pi}{3}\right)$$
$$= \frac{1}{2}$$

$$S_{10} = \frac{\sqrt{3}\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$$

$$= 3.46 \ (3sf)$$

6 The curve with equation $y = 2x - 8x^{\frac{1}{2}}$ has a minimum point at A and intersects the positive x-axis at B.

(a) Find the coordinates of A and B. [4]

A: Min. pt: $\frac{dy}{dx} = 2 - 4x^{-\frac{1}{2}}$

$\frac{dy}{dx} = 0$: $2 - 4x^{-\frac{1}{2}} = 0$

$2 - \frac{4}{\sqrt{x}} = 0$

$2 = \frac{4}{\sqrt{x}}$

$2\sqrt{x} = 4$

$\sqrt{x} = 2$

$\underline{x = 4}$

y-coord: $y = 2(4) - 8(4)^{\frac{1}{2}}$

$= 8 - 8 \times 2$

$= \underline{-8}$

→ $\underline{A : (4, -8)}$

B: At x-axis, $y = 0$:

$2x - 8x^{\frac{1}{2}} = 0$

$2x^{\frac{1}{2}}(x^{\frac{1}{2}} - 4) = 0$

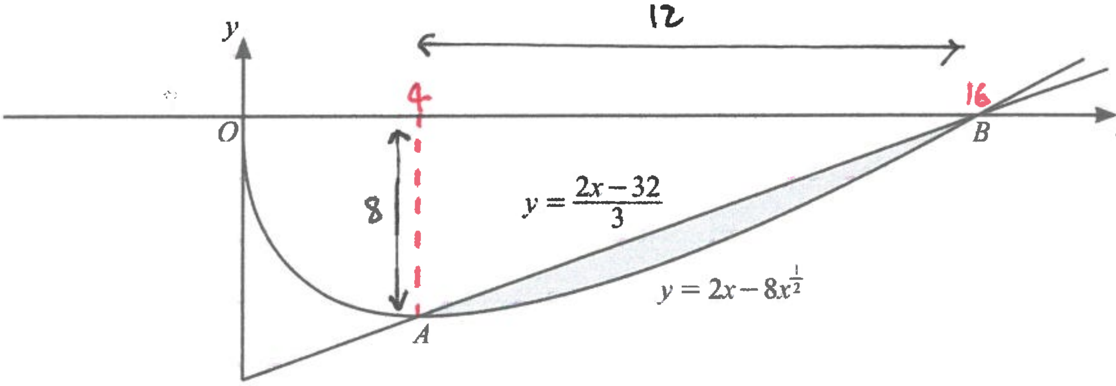
$2x^{\frac{1}{2}} = 0$ or $x^{\frac{1}{2}} - 4 = 0$

$x = 0$ ✗ $\quad x^{\frac{1}{2}} = 4$

$\underline{x = 16}$ ✓

→ $\underline{B : (16, 0)}$

(b)



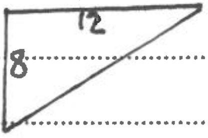
The diagram shows the curve with equation $y=2x-8x^{\frac{1}{2}}$ and the line AB. It is given that the equation of AB is $y=\frac{2x-32}{3}$.

Find the area of the shaded region between the curve and the line. [5]

from part (a): A = (4,-8) and B = (16,0)

Shaded Area = Area enclosed by $y=2x-8x^{\frac{1}{2}}$ − Area of triangle

Area of Triangle:



$$\frac{b\times h}{2}=\frac{8\times 12}{2}=\underline{48}$$

Area enclosed by curve:

$$\int_4^{16}\left(2x-8x^{\frac{1}{2}}\right)dx=\left[x^2-\frac{16}{3}x^{\frac{3}{2}}\right]_4^{16}$$

$$=\left[16^2-\frac{16}{3}(16)^{\frac{3}{2}}\right]-\left[4^2-\frac{16}{3}(4)^{\frac{3}{2}}\right]$$

$$=\left[256-\frac{1024}{3}\right]-\left[16-\frac{128}{3}\right]$$

$$=256-\frac{1024}{3}-16+\frac{128}{3}$$

$=-\frac{176}{3}$ ← negative because below x-axis. Area $=\frac{176}{3}$

Shaded Area $=\frac{176}{3}-48$

$$=\underline{\frac{32}{3}}$$

7 The equation of a circle is $(x-6)^2+(y+a)^2=18$. The line with equation $y=2a-x$ is a tangent to the circle.

(a) Find the two possible values of the constant a. [5]

Solve simultaneously:

$(x-6)^2 + (2a-x+a)^2 = 18$

$(x-6)^2 + (3a-x)^2 = 18$

$x^2 - 12x + 36 + 9a^2 - 6ax + x^2 = 18$

$2x^2 - 12x - 6ax + 18 + 9a^2 = 0$

line is a tangent to the circle, so $b^2-4ac=0$:

$2x^2 + (-12-6a)x + 18 + 9a^2 = 0$

$b^2-4ac=0$:

$(-12-6a)^2 - 4(2)(18+9a^2) = 0$

$144 + 144a + 36a^2 - 144 - 72a^2 = 0$

$-36a^2 + 144a = 0$

$-a^2 + 4a = 0$

$a^2 - 4a = 0$

$a(a-4) = 0$

$\underline{a=0}$ or $\underline{a=4}$

(b) For the greater value of a, find the equation of the diameter which is perpendicular to the given tangent. [3]

$a=4$: $(x-6)^2 + (y+4)^2 = 18$

$\rightarrow$ Centre $= (6,-4)$

gradient of tangent: $y = 8 - x$

$\uparrow$ gradient $= \underline{-1}$

gradient of perpendicular $= \frac{-1}{-1} = \underline{1}$

equation of diameter:

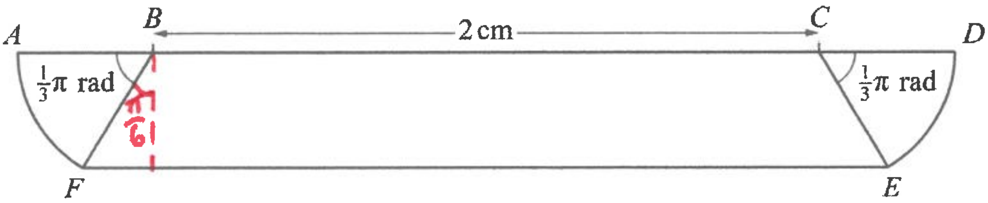
$y - y_1 = m(x - x_1)$

$y - {}^-4 = 1(x-6)$

$y + 4 = x - 6$

$\underline{y = x - 10}$

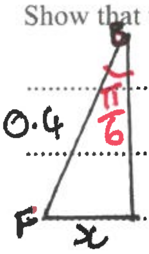
8



The diagram shows a symmetrical plate *ABCDEF*. The line *ABCD* is straight and the length of *BC* is 2 cm. Each of the two sectors *ABF* and *DCE* is of radius *r* cm and each of the angles *ABF* and *DCE* is equal to $\frac{1}{3}\pi$ radians.

(a) It is given that $r = 0.4$ cm.

(i) Show that the length $EF = 2.4$ cm. [2]



$\sin\left(\frac{\pi}{6}\right) = \frac{x}{0.4}$

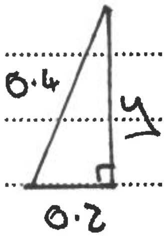
$x = 0.4\sin\left(\frac{\pi}{6}\right)$

$= \underline{0.2}$

$EF = 2 + 0.2 + 0.2$

$= \underline{\underline{2.4\text{cm}}}$

(ii) Find the area of the plate. Give your answer correct to 3 significant figures. [4]



$y^2 + 0.2^2 = 0.4^2$

$y^2 + 0.04 = 0.16$

$y^2 = 0.12$

$y = \frac{\sqrt{3}}{5}$

Area = 2x sectors + 2x triangles + rectangle

$= 2 \times \frac{1}{2}r^2\theta + 2 \times \frac{b \times h}{2} + 2 \times \frac{\sqrt{3}}{5}$

$= 2 \times \frac{1}{2} \times 0.4^2 \times \frac{\pi}{3} + 2 \times \frac{0.2 \times \frac{\sqrt{3}}{5}}{2} + \frac{2\sqrt{3}}{5}$

$= \frac{4\pi}{75} + \frac{\sqrt{3}}{25} + \frac{2\sqrt{3}}{5}$

$= \underline{\underline{0.930}}$ (3sf)

(b) It is given instead that the perimeter of the plate is 6 cm.

Find the value of r. Give your answer correct to 3 significant figures. [4]

$$\text{Perimeter} = \overset{\text{top}}{2} + \overset{AB + CD}{2r} + \overset{AF + DE}{2 \times \text{arc length}} + \overbrace{2 \times r\sin\left(\frac{\pi}{6}\right) + 2}^{\text{bottom}}$$

$$= 2 + 2r + 2 \times r\theta + 2r \times \frac{1}{2} + 2$$

$$= 2 + 2r + 2 \times r\left(\frac{\pi}{3}\right) + r + 2$$

$$= 4 + 3r + \frac{2\pi r}{3}$$

Perimeter $= 6$:

$$4 + 3r + \frac{2\pi r}{3} = 6$$

$$3r + \frac{2\pi r}{3} = 2$$

$$r\left(3 + \frac{2\pi}{3}\right) = 2$$

$$r = \frac{2}{3 + \frac{2\pi}{3}}$$

$$= \underline{0.393}$$

9 A function f is such that $f'(x) = 6(2x-3)^2 - 6x$ for $x \in \mathbb{R}$.

(a) Determine the set of values of x for which $f(x)$ is decreasing. [4]

decreasing when $f'(x) < 0$:

$$6(2x-3)^2 - 6x < 0$$

$$6(4x^2 - 12x + 9) - 6x < 0$$

$$24x^2 - 72x + 54 - 6x < 0$$

$$\underset{\div 6}{24x^2} - \underset{\div 6}{78x} + \underset{\div 6}{54} < \underset{\div 6}{0}$$

$$4x^2 - 13x + 9 < 0$$

find critical values:

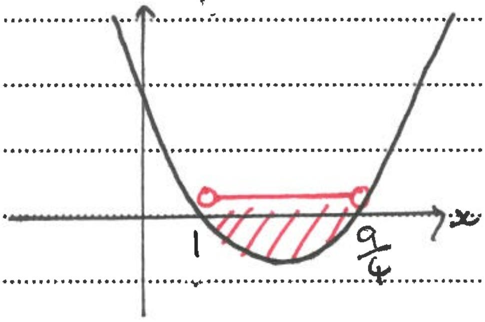
factorise: ac: 36 → -9, -4

$$4x^2 - 4x - 9x + 9 = 0$$

$$4x(x-1) - 9(x-1) = 0$$

$$(4x-9)(x-1) = 0$$

$$x = \frac{9}{4} \text{ or } x = 1$$



$$1 < x < \frac{9}{4}$$

9 A function f is such that $f'(x) = 6(2x-3)^2 - 6x$ for $x \in \mathbb{R}$.

(b) Given that $f(1) = -1$, find $f(x)$. [4]

Integrate:

$$f(x) = \int \left(6(2x-3)^2 - 6x\right) dx$$

$$= \frac{1}{3} \times \frac{1}{2} \times 6(2x-3)^3 - 3x^2 + C$$

$$= (2x-3)^3 - 3x^2 + C$$

$f(1) = -1$:

$$-1 = (2(1)-3)^3 - 3(1)^2 + C$$

$$-1 = (-1)^3 - 3 + C$$

$$-1 = -1 - 3 + C$$

$$-1 = -4 + C$$

$$C = 3$$

$$\rightarrow \underline{f(x) = (2x-3)^3 - 3x^2 + 3}$$

10 The equation of a curve is $y = (5-2x)^{\frac{3}{2}} + 5$ for $x < \frac{5}{2}$.

(a) A point P is moving along the curve in such a way that the y-coordinate of point P is decreasing at 5 units per second. ①

Find the rate at which the x-coordinate of point P is increasing when $y = 32$. [4]

$$\frac{dy}{dx} = \frac{3}{2} \times -2(5-2x)^{\frac{1}{2}}$$

$$= -3(5-2x)^{\frac{1}{2}}$$

Find x when $y=32$:

$$32 = (5-2x)^{3/2} + 5$$

$$27 = (5-2x)^{3/2}$$

$$3 = (5-2x)^{\frac{1}{2}}$$

$$9 = 5-2x$$

$$4 = -2x$$

$$\underline{x = -2}$$

at $x=-2$: $\frac{dy}{dx} = -3(5-2(-2))^{\frac{1}{2}}$

$$= -3(9)^{\frac{1}{2}}$$

$$= \underline{-9}$$

①: $\frac{dy}{dt} = -5$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$-9 = -5 \times \frac{dt}{dx}$$

$$\frac{dt}{dx} = \frac{9}{5} \longrightarrow \underline{\frac{dx}{dt} = \frac{5}{9}}$$

10 The equation of a curve is $y = (5-2x)^{\frac{3}{2}} + 5$ for $x < \frac{5}{2}$.

(b) Point A on the curve has y-coordinate 32. Point B on the curve is such that the gradient of the curve at B is -3.

Find the equation of the perpendicular bisector of AB. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. [6]

A: Find x when $y = 32$:

$$32 = (5-2x)^{3/2} + 5$$
$$27 = (5-2x)^{3/2}$$
$$3 = (5-2x)^{1/2}$$
$$9 = 5-2x$$
$$4 = -2x$$
$$\underline{x = -2} \quad A: (-2, 32)$$

B: $\frac{dy}{dx} = \frac{3}{2} \times -2 \times (5-2x)^{\frac{1}{2}}$

$$= -3(5-2x)^{\frac{1}{2}}$$

gradient = -3:

$$-3(5-2x)^{\frac{1}{2}} = -3$$
$$(5-2x)^{\frac{1}{2}} = 1$$
$$5-2x = 1$$
$$-2x = -4$$
$$\underline{x = 2}$$

y-coord: $y = (5-2(2))^{3/2} + 5$

$$= 1^{3/2} + 5$$
$$= \underline{6} \quad B: (2, 6)$$

gradient AB: $m = \frac{6-32}{2--2} = \frac{-26}{4}$

$$= \frac{-13}{2}$$

Mid-point of AB:

$$\left(\frac{-2+2}{2}, \frac{32+6}{2}\right)$$
$$\underline{\underline{(0, 19)}}$$

Gradient of perpendicular:

$$m_{perp} = \frac{2}{13}$$

Equation of perp. bisector:

$$y - 19 = \frac{2}{13}(x - 0)$$
$$y - 19 = \frac{2}{13}x$$
$$13y - 247 = 2x$$
$$\underline{\underline{2x - 13y + 247 = 0}}$$