1 (a) Express $3y^2 - 12y - 15$ in the form $3(y+a)^2 + b$, where a and b are constants. [2]

$3y^2 - 12y - 15 = 3[y^2 - 4y] - 15$

$= 3[(y-2)^2 - 4] - 15$

$= 3(y-2)^2 - 12 - 15$

$= \underline{\underline{3(y-2)^2 - 27}}$

(b) Hence find the exact solutions of the equation $3x^4 - 12x^2 - 15 = 0$. [3]

Let $y = x^2$:

$3y^2 - 12y - 15 = 0$

from (a): $3(y-2)^2 - 27 = 0$

$+27 \quad +27$

$3(y-2)^2 = 27$

$(y-2)^2 = 9$

$y - 2 = \pm 3$

$y - 2 = 3$ or $y - 2 = -3$

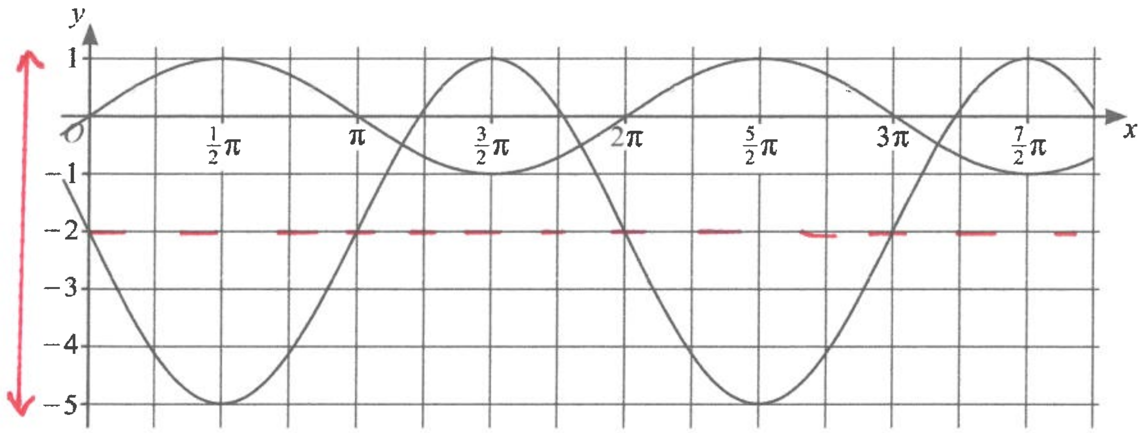
$y = 5$ or $y = -1$

$x^2 = 5$ or $x^2 = -1$

✗ no sol^s

$\underline{\underline{x = \sqrt{5} \text{ or } x = -\sqrt{5}}}$

2



The diagram shows two curves. One curve has equation $y = \sin x$ and the other curve has equation $y = f(x)$.

(a) In order to transform the curve $y = \sin x$ to the curve $y = f(x)$, the curve $y = \sin x$ is first reflected in the x-axis.

Describe fully a sequence of two further transformations which are required. [4]

Amplitude of $f(x)$ is 6, where it is 2 for $y = \sin x$

"middle" of graph vertically is at −2, where it is usually at 0.

So: ① Stretch parallel to y-axis, scale factor 3.

② Translation by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

(b) Find $f(x)$ in terms of $\sin x$. [2]

$\sin x \xrightarrow[\text{x-axis}]{\text{reflect in}} -\sin x \xrightarrow[\text{y}]{\text{stretch in}} -3\sin x \xrightarrow{\text{translate}} -3\sin x - 2$

$\rightarrow f(x) = -3\sin x - 2$

3 The coefficient of x^3 in the expansion of $(3+ax)^6$ is 160.

(a) Find the value of the constant a. [2]

$(3+ax)^6 = 3^6 + 6\times3^5\times ax + 15\times3^4\times(ax)^2 + 20\times3^3\times(ax)^3 + \dots$

$\rightarrow 20\times27\times a^3x^3$

$= 540a^3x^3$

coeff $= 540a^3$

$540a^3 = 160$

$a^3 = \frac{8}{27}$

$a = \frac{2}{3}$

(b) Hence find the coefficient of x^3 in the expansion of $(3+ax)^6(1-2x)$. [3]

$a=\frac{2}{3}$: $(3+\frac{2}{3}x)^6 = 3^6 + 6\times3^5\times\frac{2}{3}x + 15\times3^4\times(\frac{2}{3}x)^2 + 20\times3^3\times(\frac{2}{3}x)^3 + \dots$

$= 729 + 972x + 540x^2 + 160x^3 + \dots$

$\rightarrow (729 + 972x + 540x^2 + 160x^3 + \dots)(1-2x)$

coeff of x^3: $540x^2\times -2x + 160x^3\times1$

$= -1080x^3 + 160x^3$

$= -920x^3$

coeff $: -920$

4 The equation of a curve is $y = f(x)$, where $f(x) = (2x-1)\sqrt{3x-2} - 2$. The following points lie on the curve. Non-exact values have been given correct to 5 decimal places.

$A(2,4)$, $B(2.0001,k)$, $C(2.001,4.00625)$, $D(2.01,4.06261)$, $E(2.1,4.63566)$, $F(3,11.22876)$

(a) Find the value of k. Give your answer correct to 5 decimal places. [1]

Sub. $x = 2.0001$: $(2\times 2.0001 - 1)\sqrt{3(2.0001)-2} - 2$

$= \underline{4.00063}$

The table shows the gradients of the chords AB, AC, AD and AF.

Chord	AB	AC	AD	AE	AF
Gradient of chord	6.2501	6.2511	6.2608		7.2288

(b) Find the gradient of the chord AE. Give your answer correct to 4 decimal places. [1]

$m = \dfrac{4.63566 - 4}{2.1 - 2}$

$= \underline{6.3566}$

(c) Deduce the value of $f'(2)$ using the values in the table. [1]

What is the gradient becoming as the chords are progressively getting shorter?

AB is the shortest chord so the gradient appears to be getting closer to $\underline{6.25}$

(Check using calculator)

5 (a) Prove the identity $\frac{\sin^2 x - \cos x - 1}{1 + \cos x} \equiv -\cos x$. [3]

$1-\cos^2 x$

LHS:

$$\frac{1-\cos^2 x - \cos x - 1}{1+\cos x}$$

$$= \frac{-\cos^2 x - \cos x}{1 + \cos x}$$

$$= \frac{-\cos x \cancel{(\cos x + 1)}}{\cancel{1+\cos x}}$$

$$= \underline{-\cos x}$$ QED

(b) Hence solve the equation $\frac{\sin^2 x - \cos x - 1}{2 + 2\cos x} = \frac{1}{4}$ for $0° \leqslant x \leqslant 360°$. [3]

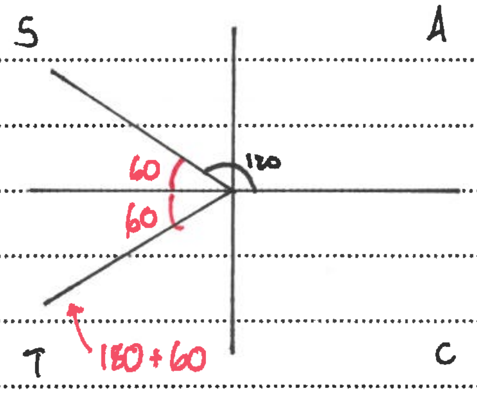
$$\frac{\sin^2 x - \cos x - 1}{1 + \cos x} \times \frac{1}{2} = \frac{\sin^2 x - \cos x - 1}{2 + 2\cos x}$$

So: $-\cos x \times \frac{1}{2} = \frac{1}{4}$

$-\cos x = \frac{1}{2}$

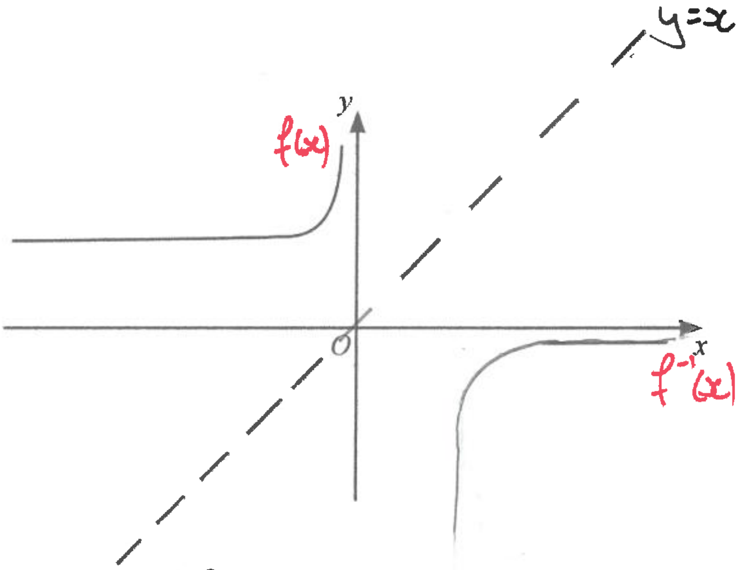
$\cos x = -\frac{1}{2}$

$x = 120°$



$\underline{x = 120°, 240°}$

6



The function f is defined by $f(x) = \frac{2}{x^2} + 4$ for $x < 0$. The diagram shows the graph of $y = f(x)$.

(a) On this diagram, sketch the graph of $y = f^{-1}(x)$. Show any relevant mirror line. [2]

(b) Find an expression for $f^{-1}(x)$. [3]

$y = \frac{2}{x^2} + 4$

$y - 4 = \frac{2}{x^2}$

$x^2(y-4) = 2$

$x^2 = \frac{2}{y-4}$

$x = \pm\sqrt{\frac{2}{y-4}}$

domain of f is $x < 0$, so x is always negative, meaning we want the negative root:

$x = -\sqrt{\frac{2}{y-4}}$

$f^{-1}(x) = -\sqrt{\frac{2}{x-4}}$

(c) Solve the equation $f(x) = 4.5$. [1]

$\frac{2}{x^2} + 4 = 4.5$

$\frac{2}{x^2} = 0.5$

$2 = 0.5x^2$

$x^2 = 4$

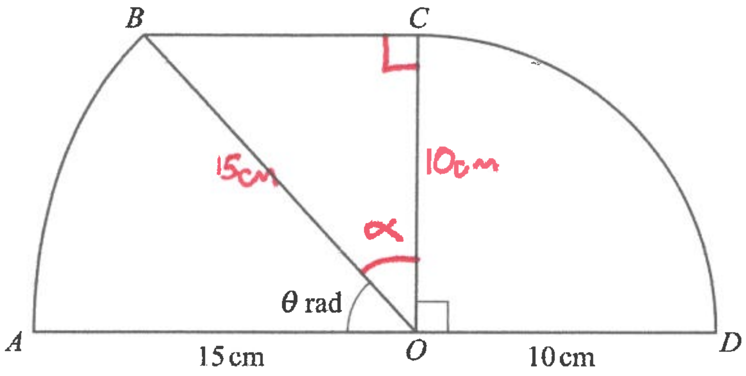
$x = \pm 2$

but $x < 0$: $x = -2$

(d) Explain why the equation $f^{-1}(x) = f(x)$ has no solution. [1]

For there to be a solution, f(x) and $f^{-1}(x)$ would have to intersect on the graph above.

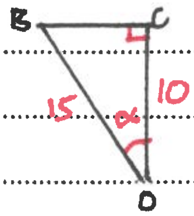
7



In the diagram, AOD and BC are two parallel straight lines. Arc AB is part of a circle with centre O and radius 15 cm. Angle $BOA = \theta$ radians. Arc CD is part of a circle with centre O and radius 10 cm. Angle $COD = \frac{1}{2}\pi$ radians.

(a) Show that $\theta = 0.7297$, correct to 4 decimal places. [1]

$OB = 15\text{cm}$ (radius, so = OA), $OC = 10\text{cm}$ (radius, so = OD)



$\cos\alpha = \frac{10}{15}$

$\alpha = 0.84106\ldots$

$\theta = \frac{\pi}{2} - 0.84106\ldots$

$= 0.7297$ QED

STO

(b) Find the perimeter and the area of the shape $ABCD$. Give your answers correct to 3 significant figures. [7]

Perimeter: $AB = r\theta$

$= 15 \times 0.7297$

$= 10.9459$

$BC = \sqrt{15^2 - 10^2}$

$= 5\sqrt{5}$

$CD = r\theta$

$= 10 \times \frac{\pi}{2}$

$= 5\pi$

Perimeter $= 10 + 15 + 10.9459 + 5\sqrt{5} + 5\pi$

$= 62.8\text{cm}$

Area: $OAB = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 15^2 \times 0.7297$

$= 82.0944$

$OBC = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 10 \times 5\sqrt{5}$

$= 25\sqrt{5}$

$OCD = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 10^2 \times \frac{\pi}{2}$

$= 25\pi$

Area $= 82.0944 + 25\sqrt{5} + 25\pi$

$= \underline{217 \text{ cm}^2}$ (3sf)

8 **(a)** The first three terms of an arithmetic progression are 25, $4p-1$ and $13-p$, where p is a constant.

Find the value of the tenth term of the progression. [4]

$u_1 = 25$, $u_2 = 4p-1$, $u_3 = 13-p$

find d:

$u_2 - u_1$: $d = 4p - 1 - 25$
$= 4p - 26$ ①

$u_3 - u_2$: $d = 13 - p - (4p-1)$
$= 13 - p - 4p + 1$
$= 14 - 5p$ ②

① = ②: $4p - 26 = 14 - 5p$
$9p - 26 = 14$
$9p = 40$
$p = \frac{40}{9}$

$d = 4\left(\frac{40}{9}\right) - 26$
$= \frac{-74}{9}$

u_{10}: $u_{10} = a + 9d$
$= 25 + 9\left(\frac{-74}{9}\right)$
$= 25 - 74$
$= -49$

8. **(b)** The first three terms of a geometric progression are $\overset{u_1}{25}$, $\overset{u_2}{4q-1}$ and $\overset{u_3}{13-q}$, where q is a positive constant.

Find the sum to infinity of the progression. [4]

find r:

$\frac{u_2}{u_1}$: $r = \frac{4q-1}{25}$ ①

$\frac{u_3}{u_2}$: $r = \frac{13-q}{4q-1}$ ②

① = ②: $\frac{4q-1}{25} = \frac{13-q}{4q-1}$

$(4q-1)(4q-1) = 25(13-q)$

$16q^2 - 8q + 1 = 325 - 25q$

$16q^2 + 17q - 324 = 0$

$q = \frac{-17 \pm \sqrt{17^2 - 4(16)(-324)}}{2(16)}$

$q = 4$ ✓ or $q = \frac{-81}{16}$ ✗ q is positive

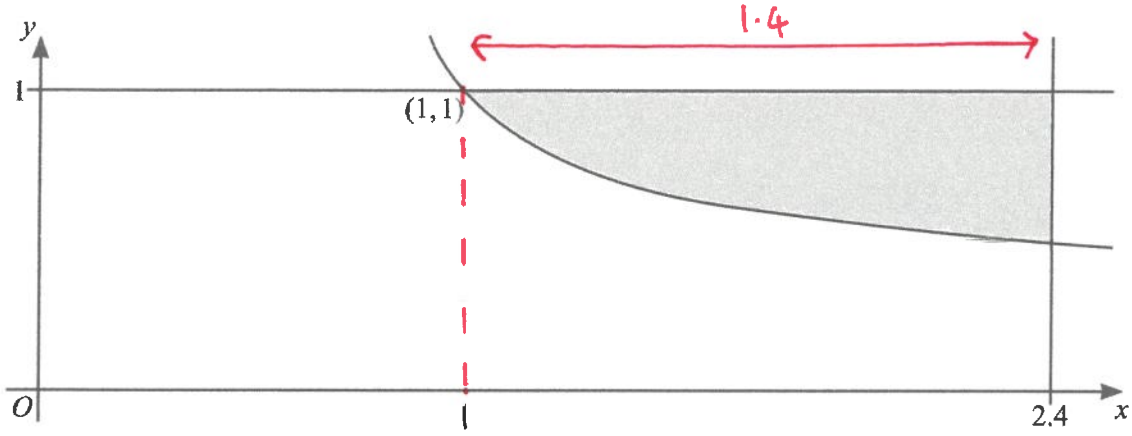
$\rightarrow r = \frac{4(4)-1}{25}$

$= \frac{3}{5}$

$S_\infty = \frac{a}{1-r} = \frac{25}{1-\frac{3}{5}}$

$= 62.5$

9



The diagram shows part of the curve with equation $y = \dfrac{1}{(5x-4)^{\frac{1}{3}}}$ and the lines $x = 2.4$ and $y = 1$. The curve intersects the line $y = 1$ at the point $(1, 1)$.

Find the exact volume of the solid generated when the shaded region is rotated through 360° about the x-axis. [6]

Volume of shaded region = volume of cylinder − volume bounded by $y = \dfrac{1}{(5x-4)^{\frac{1}{3}}}$

Volume of cylinder : $V = \pi r^2 h$

$= \pi \times 1^2 \times 1.4$

$= \underline{1.4\pi}$

Volume bounded by $y = \dfrac{1}{(5x-4)^{\frac{1}{3}}}$:

$$V = \pi \int y^2 \, dx$$

$$= \pi \int_1^{2.4} \left(\frac{1}{(5x-4)^{\frac{1}{3}}} \right)^2 dx$$

$$= \pi \int_1^{2.4} \frac{1}{(5x-4)^{2/3}} \, dx$$

$$= \pi \int_1^{2.4} (5x-4)^{-2/3} \, dx$$

$$= \pi\left[3 \times \frac{1}{5}(5x-4)^{\frac{1}{3}}\right]_{1}^{2.4}$$

$$= \pi\left[\frac{3}{5}(5(2.4)-4)^{\frac{1}{3}}\right] - \pi\left[\frac{3}{5}(5-4)^{\frac{1}{3}}\right]$$

$$= \pi\left[\frac{3}{5} \times 8^{\frac{1}{3}}\right] - \pi\left[\frac{3}{5} \times 1^{\frac{1}{3}}\right]$$

$$= \pi\left[\frac{6}{5}\right] - \pi\left[\frac{3}{5}\right]$$

$$= \frac{3}{5}\pi$$

Volume $= 1.4\pi - \frac{3}{5}\pi$

$$= \frac{4}{5}\pi$$

10 The equation of a circle is $(x-3)^2+y^2=18$. The line with equation $y=mx+c$ passes through the point $(0,-9)$ and is a tangent to the circle.

Find the two possible values of m and, for each value of m, find the coordinates of the point at which the tangent touches the circle. [8]

Straight line: $y = mx + c$

intercept is $(0,-9)$: $y = mx - 9$

Solve simultaneously:

$(x-3)^2 + (mx-9)^2 = 18$

$x^2 - 6x + 9 + m^2x^2 - 18mx + 81 = 18$

$m^2x^2 + x^2 - 18mx - 6x + 72 = 0$

$(m^2+1)x^2 + (-18m-6)x + 72 = 0$ (1)

line is a tangent to the circle, so $b^2 - 4ac = 0$:

$(-18m-6)^2 - 4(m^2+1)(72) = 0$

$324m^2 + 216m + 36 - 288m^2 - 288 = 0$

$36m^2 + 216m - 252 = 0$

$m^2 + 6m - 7 = 0$

$(m+7)(m-1) = 0$

$m = -7$ or $m = 1$

Sub. $m=-7$ into (1):

$((-7)^2+1)x^2 + (-18(-7)-6)x + 72 = 0$

$50x^2 + 120x + 72 = 0$

$25x^2 + 60x + 36 = 0$

$x = \dfrac{-60 \pm \sqrt{60^2 - 4(25)(36)}}{2(25)}$

$x = \dfrac{-6}{5}$

$m=-7$

y-coord: $y = -7x - 9$

$= -7\left(\frac{-6}{5}\right) - 9$

$= \frac{-3}{5}$

So when $m=-7$, coordinate $= \left(\frac{-6}{5}, \frac{-3}{5}\right)$

Sub. $m=1$ into 1:

$(1^2+1)x^2 + (-18(1)-6)x + 72 = 0$

$2x^2 - 24x + 72 = 0$

$x^2 - 12x + 36 = 0$

$(x-6)^2 = 0$

$x = 6$

$m=1$

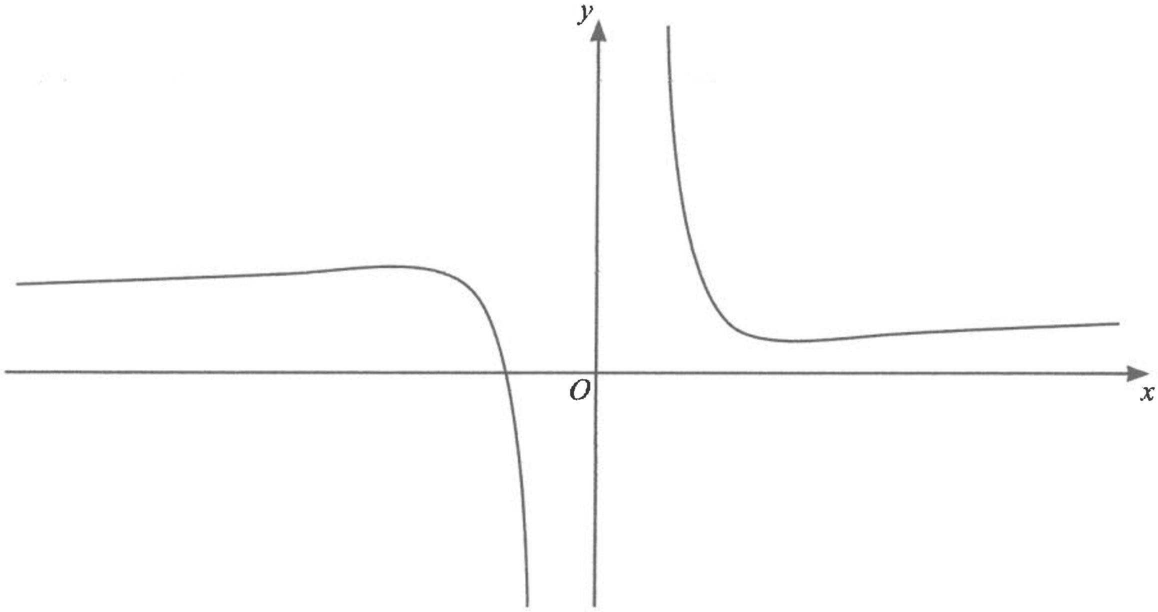
y-coord: $y = 1x - 9$

$= 6 - 9$

$= -3$

So when $m=1$, coordinate $= (6, -3)$

11



A function is defined by $f(x) = \frac{4}{x^3} - \frac{3}{x} + 2$ for $x \neq 0$. The graph of $y = f(x)$ is shown in the diagram.

(a) Find the set of values of x for which $f(x)$ is decreasing. [5]

$y = 4x^{-3} - 3x^{-1} + 2$

$\frac{dy}{dx} = -12x^{-4} + 3x^{-2}$

decreasing, so $\frac{dy}{dx} < 0$:

$\frac{-12}{x^4} + \frac{3}{x^2} < 0$

$\times x^4 \quad \times x^4 \quad \times x^4$

$-12 + 3x^2 < 0$

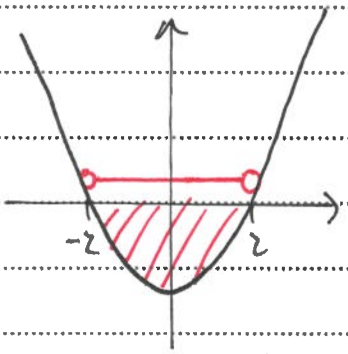
$3x^2 - 12 < 0$

$x^2 - 4 < 0$

Critical values: $x^2 - 4 = 0$

$(x-2)(x+2) = 0$

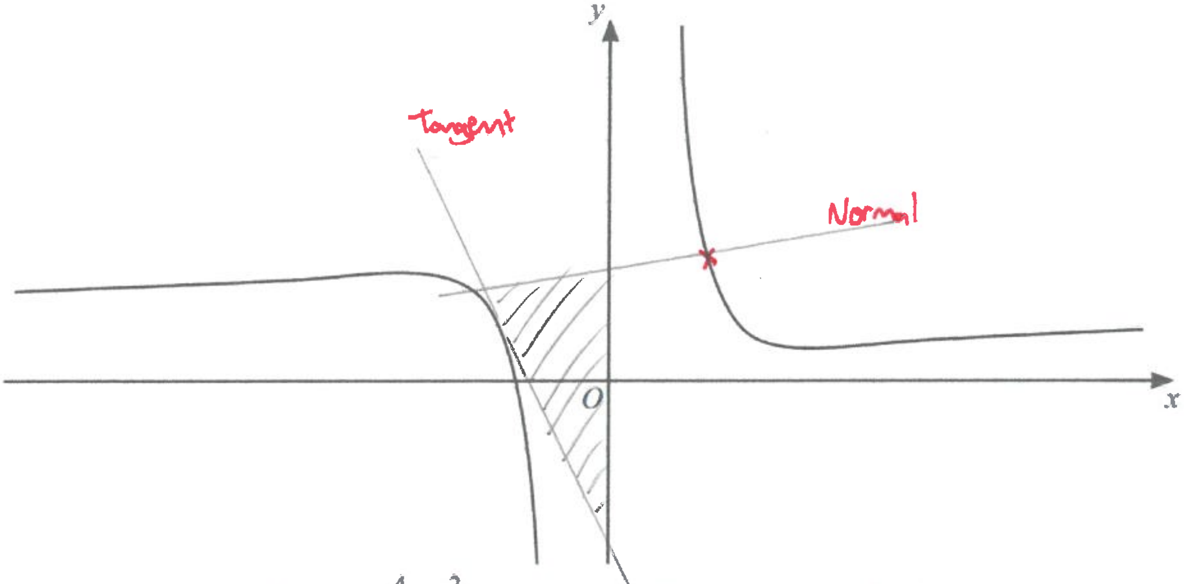
$x = 2$ or $x = -2$



so $-2 < x < 2$

but wait! $x \neq 0$ so:

$-2 < x < 2, x \neq 0$



A function is defined by $f(x) = \frac{4}{x^3} - \frac{3}{x} + 2$ for $x \neq 0$. The graph of $y = f(x)$ is shown in the diagram.

(b) A triangle is bounded by the y-axis, the normal to the curve at the point where $x = 1$ and the tangent to the curve at the point where $x = -1$.

Find the area of the triangle. Give your answer correct to 3 significant figures. [8]

Normal at $x=1$:

y-coord: $y = \frac{4}{1^3} - \frac{3}{1} + 2$

$y = 3$

gradient: $f(x) = 4x^{-3} - 3x^{-1} + 2$

$f'(x) = -12x^{-4} + 3x^{-2}$

$x=1$: $m = -12(1)^{-4} + 3(1)^{-2}$

$= -9$

gradient of normal $= \frac{-1}{-9}$

$= \frac{1}{9}$

eqn of normal: $y - 3 = \frac{1}{9}(x-1)$

$y - 3 = \frac{1}{9}x - \frac{1}{9}$

$y = \frac{1}{9}x + \frac{26}{9}$ 1

Tangent at $x = -1$:

y-coord: $y = \frac{4}{(-1)^3} - \frac{3}{(-1)} + 2$

$= -4 + 3 + 2$

$y = 1$

gradient: $f'(x) = -12x^{-4} + 3x^{-2}$

$x=-1$: $m = -12(-1)^{-4} + 3(-1)^{-2}$

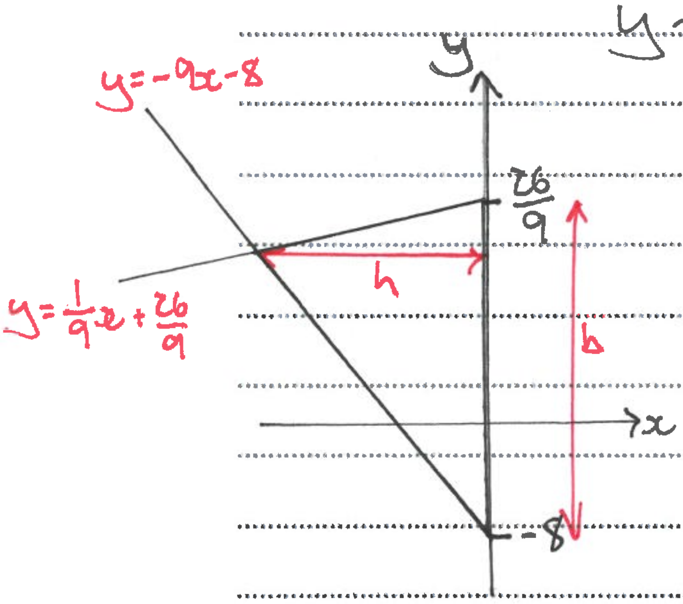
$= -9$

eqⁿ of tangent: $y - 1 = -9(x - -1)$

$y - 1 = -9(x+1)$

$y - 1 = -9x - 9$

$y = -9x - 8$ ②



base of triangle $= \frac{26}{9} - -8$

$= \frac{98}{9}$

for height, need to find x-coord of intersection of lines:

① = ②: $\frac{1}{9}x + \frac{26}{9} = -9x - 8$

$x + 26 = -81x - 72$

$82x = -98$

$x = \frac{-98}{82} = \frac{-49}{41}$ → height $= \frac{49}{41}$

Area $= \frac{1}{2} \times \frac{98}{9} \times \frac{49}{41}$

$= 6.51$ (3sf)