1 The graph of $y = f(x)$ is transformed to the graph of $y = 3 - f(x)$.

Describe fully, in the correct order, the two transformations that have been combined. [4]

① Reflection in the x-axis $\left(f(x) \to -f(x)\right)$

② Translation by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

2 **(a)** Find the first three terms, in ascending powers of x, in the expansion of $(1 + ax)^6$. [1]

$(1+ax)^6 = 1^6 + 6 \times 1^5 \times (ax) + 15 \times 1^4 \times (ax)^2 + \dots$

$= \underline{1 + 6ax + 15a^2x^2 + \dots}$

(b) Given that the coefficient of x^2 in the expansion of $(1 - 3x)(1 + ax)^6$ is -3, find the possible values of the constant a. [4]

$(1-3x)(1+6ax + 15a^2x^2 + \dots)$

coeff of x^2: $1 \times 15a^2x^2 - 3x \times 6ax$

$= 15a^2x^2 - 18ax^2$

coeff $= 15a^2 - 18a$

→ $15a^2 - 18a = -3$

$15a^2 - 18a + 3 = 0$

$5a^2 - 6a + 1 = 0$

$(5a - 1)(a - 1) = 0$

$\underline{a = \frac{1}{5}}$ or $\underline{a = 1}$

3 **(a)** Express $5y^2 - 30y + 50$ in the form $5(y + a)^2 + b$, where a and b are constants. [2]

$5[y^2 - 6y] + 50$

$5[(y-3)^2 - 9] + 50$

$5(y-3)^2 - 45 + 50$

$5(y-3)^2 + 5$

(b) The function f is defined by $f(x) = x^5 - 10x^3 + 50x$ for $x \in \mathbb{R}$.

Determine whether f is an increasing function, a decreasing function or neither. [3]

$f'(x) = 5x^4 - 30x^2 + 50$

using part (a), with $y = x^2$:

$f'(x) = \underbrace{5(x^2-3)^2}_{\text{always positive}} + 5$

positive

so $f'(x) > 0$

therefore increasing function

4 The first term of an arithmetic progression is 84 and the common difference is −3.

(a) Find the smallest value of n for which the nth term is negative. [2]

$u_n = a + (n-1)d$

$= 84 + (n-1)(-3)$

$= 84 - 3n + 3$

$= 87 - 3n$

$u_n < 0$:

$87 - 3n < 0$

$87 < 3n$

$3n > 87$

$n > 29$

$\underline{n = 30}$

It is given that the sum of the first $2k$ terms of this progression is equal to the sum of the first k terms.

(b) Find the value of k. [3]

S_{2k}: $S_{2k} = \frac{2k}{2}[2(84) + (2k-1)(-3)]$

$= k(168 - 6k + 3)$

$= 171k - 6k^2$

S_k: $S_k = \frac{k}{2}[2(84) + (k-1)(-3)]$

$= \frac{k}{2}(168 - 3k + 3)$

$= \frac{171}{2}k - \frac{3}{2}k^2$

$S_{2k} = S_k$: $171k - 6k^2 = \frac{171}{2}k - \frac{3}{2}k^2$

$342k - 12k^2 = 171k - 3k^2$

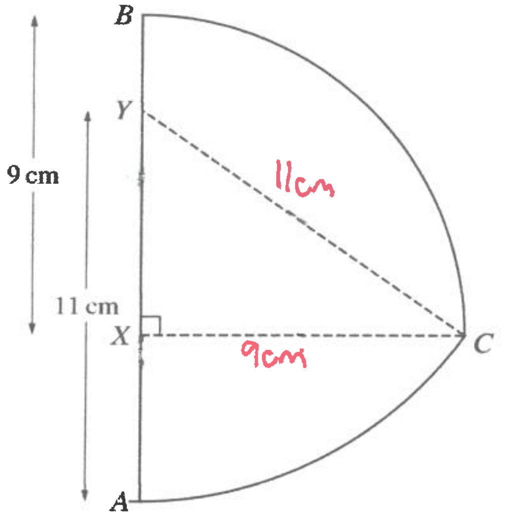
$9k^2 - 171k = 0$

$k^2 - 19k = 0$

$k(k-19) = 0$

$\cancel{k = 0}$ or $\underline{k = 19}$

5



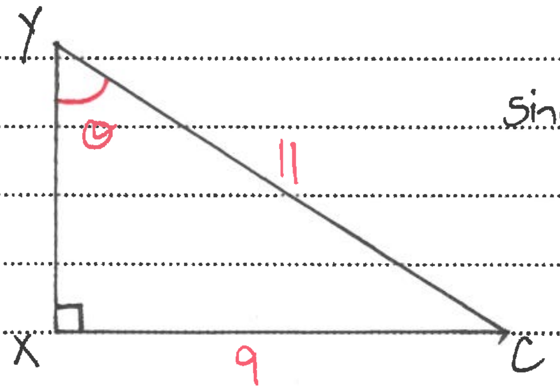
BX = CX

and

AY = CY

In the diagram, X and Y are points on the line AB such that $BX = 9$ cm and $AY = 11$ cm. Arc BC is part of a circle with centre X and radius 9 cm, where CX is perpendicular to AB. Arc AC is part of a circle with centre Y and radius 11 cm.

(a) Show that angle $XYC = 0.9582$ radians, correct to 4 significant figures. [1]



$\sin\theta = \frac{9}{11}$

$\theta = \underline{\underline{0.9582}}$ QED

(b) Find the perimeter of ABC. [6]

Length XY:

Pythagoras: $XY^2 + 9^2 = 11^2$

$XY^2 = 11^2 - 9^2$

$XY = \sqrt{11^2 - 9^2}$

$= \sqrt{40}$

Length BY: $BY = 9 - \sqrt{40}$

Length AB: $AY + BY = 11 + 9 - \sqrt{40}$

$= 20 - \sqrt{40}$

Arc BC: arc length $= 9 \times \frac{\pi}{2}$

$= \frac{9\pi}{2}$

Arc AC: arc length $= 11 \times 0.9582$

$= 10.5$ STO

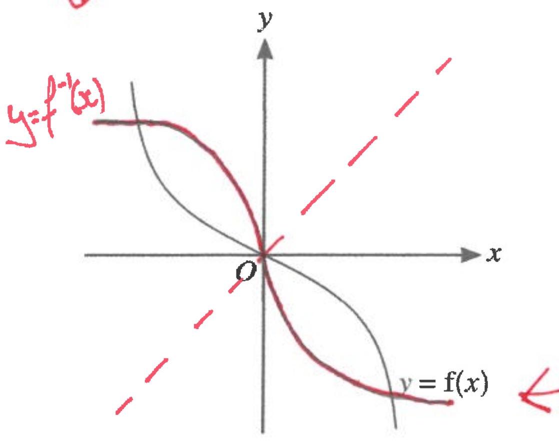
Perimeter = AB + Arc BC + Arc AC

$= 20 - \sqrt{40} + \frac{9\pi}{2} + 10.5$

$= 38.4$ cm

6

when x is negative, $f^{-1}(x)$ is positive



The diagram shows the graph of $y = f(x)$.

(a) On this diagram sketch the graph of $y = f^{-1}(x)$. - reflection in $y=x$ [1]

It is now given that $f(x) = -\dfrac{x}{\sqrt{4-x^2}}$ where $-2 < x < 2$.

(b) Find an expression for $f^{-1}(x)$. [4]

$$y = \frac{-x}{\sqrt{4-x^2}}$$

Square both sides

$$y^2 = \frac{x^2}{4-x^2}$$

$$y^2(4-x^2) = x^2$$

$$4y^2 - x^2y^2 = x^2$$

$$4y^2 = x^2 + x^2y^2$$

$$4y^2 = x^2(1+y^2)$$

$$x^2 = \frac{4y^2}{1+y^2}$$

$$x = \pm\sqrt{\frac{4y^2}{1+y^2}}$$

$$x = \pm\frac{2y}{\sqrt{1+y^2}}$$

$$f^{-1}(x) = \pm\frac{2x}{\sqrt{1+x^2}}$$

To decide whether it is the positive or negative square root we need to use the graph (!)

$$f^{-1}(x) = \frac{\pm 2x}{\underbrace{\sqrt{1+x^2}}_{\text{this bit is always positive}}}$$

So for $f^{-1}(x)$ to be negative when x is positive, we want the negative root: $f^{-1}(x) = \dfrac{-2x}{\sqrt{1+x^2}}$

difficult!

The function g is defined by $g(x) = 2x$ for $-a < x < a$, where a is a constant.

(c) State the maximum possible value of a for which fg can be formed. [1]

The outputs of g become the inputs of f. The domain for f is $-2<x<2$, so this should be the range of g. Given $g(x)$ multiplies by 2, this means the domain must be $-1<x<1$ ($\rightarrow$ range: $-2<g(x)<2$)

(d) Assuming that fg can be formed, find and simplify an expression for $fg(x)$. [2]

$$fg(x) = -\frac{2x}{\sqrt{4-(2x)^2}}$$

$$= -\frac{2x}{\sqrt{4-4x^2}}$$

$$= -\frac{2x}{2\sqrt{1-x^2}}$$

$$= -\frac{x}{\sqrt{1-x^2}}$$

7 (a) Show that the equation $\frac{\tan x + \cos x}{\tan x - \cos x} = k$, where k is a constant, can be expressed as

$$(k+1)\sin^2 x + (k-1)\sin x - (k+1) = 0.$$ [4]

$$\frac{\tan x + \cos x}{\tan x - \cos x} = k$$

$$\tan x + \cos x = k\tan x - k\cos x$$

$$\frac{\sin x}{\cos x} + \cos x = k\frac{\sin x}{\cos x} - k\cos x$$

$\times\cos x$ $\times\cos x$ $\times\cos x$ $\times\cos x$

$$\sin x + \underset{\uparrow\, 1-\sin^2 x}{\cos^2 x} = k\sin x - k\underset{\uparrow\, 1-\sin^2 x}{\cos^2 x}$$

$$\sin x + 1 - \sin^2 x = k\sin x - k(1-\sin^2 x)$$

$$\sin x + 1 - \sin^2 x = k\sin x - k + k\sin^2 x$$

$$k\sin^2 x + \sin^2 x + k\sin x - \sin x - k - 1 = 0$$

$$(k+1)\sin^2 x + (k-1)\sin x - (k+1) = 0$$ QED

(b) Hence solve the equation $\dfrac{\tan x + \cos x}{\tan x - \cos x} = 4$ for $0° \leqslant x \leqslant 360°$. [4]

$k = 4$:

$$5\sin^2 x + 3\sin x - 5 = 0$$

Let $Y = \sin x$:

$$5Y^2 + 3Y - 5 = 0$$

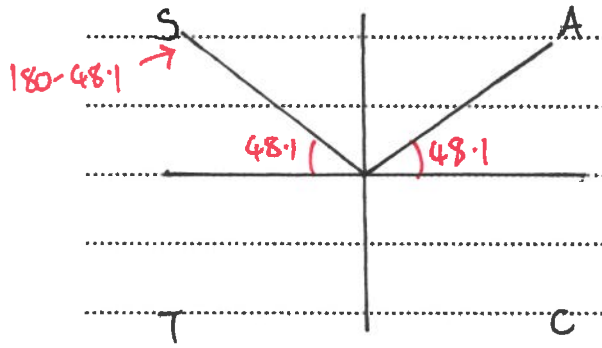
$$Y = \frac{-3 \pm \sqrt{3^2 - 4(5)(-5)}}{2(5)}$$

$$Y = \frac{-3 + \sqrt{109}}{10} \quad \text{or} \quad Y = \frac{-3 - \sqrt{109}}{10}$$

$$\sin x = \frac{-3 + \sqrt{109}}{10} \qquad \sin x = \frac{-3 - \sqrt{109}}{10}$$

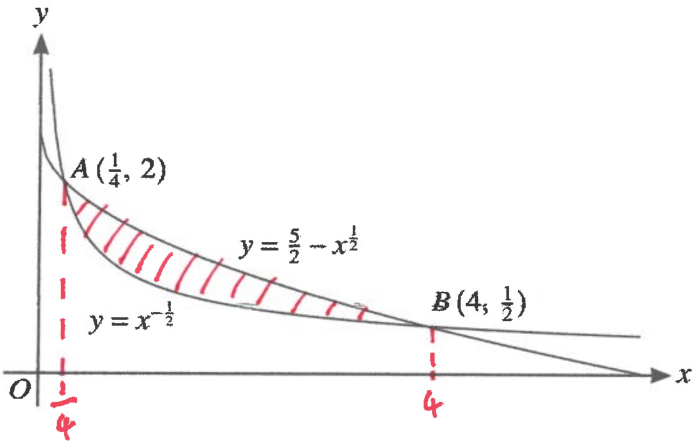
no solutions

$$x = 48.1°$$



$$x = 48.1°, 131.9°$$

8



The diagram shows the curves with equations $y = x^{-\frac{1}{2}}$ and $y = \frac{5}{2} - x^{\frac{1}{2}}$. The curves intersect at the points $A\left(\frac{1}{4}, 2\right)$ and $B\left(4, \frac{1}{2}\right)$.

(a) Find the area of the region between the two curves. [6]

$$\int_{\frac{1}{4}}^{4} \left[\left(\frac{5}{2} - x^{\frac{1}{2}}\right) - x^{-\frac{1}{2}}\right] dx$$

$$= \int_{\frac{1}{4}}^{4} \left(\frac{5}{2} - x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) dx$$

$$\left[\frac{5}{2}x - \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\right]_{\frac{1}{4}}^{4}$$

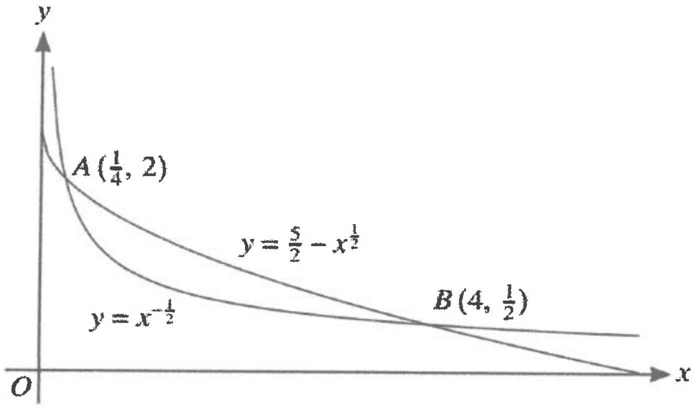
$$= \left[\frac{5}{2}(4) - \frac{2}{3}(4)^{\frac{3}{2}} - 2(4)^{\frac{1}{2}}\right] - \left[\frac{5}{2}\left(\frac{1}{4}\right) - \frac{2}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}} - 2\left(\frac{1}{4}\right)^{\frac{1}{2}}\right]$$

$$= \left[10 - \frac{2}{3} \times 8 - 2 \times 2\right] - \left[\frac{5}{8} - \frac{2}{3} \times \frac{1}{8} - 2 \times \frac{1}{2}\right]$$

$$= \left[10 - \frac{16}{3} - 4\right] - \left[\frac{5}{8} - \frac{1}{12} - 1\right]$$

$$= \left[\frac{2}{3}\right] - \left[\frac{-11}{24}\right]$$

$$= \frac{9}{8}$$



The diagram shows the curves with equations $y = x^{-\frac{1}{2}}$ and $y = \frac{5}{2} - x^{\frac{1}{2}}$. The curves intersect at the points $A\,(\frac{1}{4}, 2)$ and $B\,(4, \frac{1}{2})$.

(b) The normal to the curve $y = x^{-\frac{1}{2}}$ at the point $(1, 1)$ intersects the y-axis at the point $(0, p)$.

Find the value of p. [4]

$y = x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$

gradient at $x = 1$:

$-\frac{1}{2}(1)^{-\frac{3}{2}}$

$= -\frac{1}{2}$

gradient of normal:

$m = 2$

eqn of normal:

$y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$y = 2x - 1$

y-intercept

so $p = -1$

[Turn over

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9 The line $y = 2x + 5$ intersects the circle with equation $x^2 + y^2 = 20$ at A and B.

(a) Find the coordinates of A and B in surd form and hence find the exact length of the chord AB. [7]

Sub. $y = 2x + 5$ into eqn of circle:

$$x^2 + (2x+5)^2 = 20$$

$$x^2 + 4x^2 + 20x + 25 = 20$$

$$5x^2 + 20x + 5 = 0$$

$$x^2 + 4x + 1 = 0$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)}$$

$$x = -2 + \sqrt{3} \quad \text{or} \quad x = -2 - \sqrt{3}$$

$\underline{x = -2 + \sqrt{3}:}$

$$y = 2(-2+\sqrt{3}) + 5$$
$$= -4 + 2\sqrt{3} + 5$$
$$= 1 + 2\sqrt{3}$$

$\underline{x = -2 - \sqrt{3}:}$

$$y = 2(-2-\sqrt{3}) + 5$$
$$= -4 - 2\sqrt{3} + 5$$
$$= 1 - 2\sqrt{3}$$

$\underline{(-2+\sqrt{3}, 1+2\sqrt{3})}$ $\quad$ $\underline{(-2-\sqrt{3}, 1-2\sqrt{3})}$

Length of chord:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[(-2-\sqrt{3}) - (-2+\sqrt{3})]^2 + [(1-2\sqrt{3}) - (1+2\sqrt{3})]^2}$$
$$= \sqrt{[-2-\sqrt{3}+2-\sqrt{3}]^2 + [1-2\sqrt{3}-1-2\sqrt{3}]^2}$$
$$= \sqrt{(-2\sqrt{3})^2 + (-4\sqrt{3})^2}$$
$$= \sqrt{12 + 48} = \underline{2\sqrt{15}}$$

A straight line through the point (10, 0) with gradient m is a tangent to the circle.

(b) Find the two possible values of m. [5]

eqn of line: $y - y_1 = m(x - x_1)$

$y - 0 = m(x - 10)$

$y = mx - 10m$

sub. into eqn of circle:

$x^2 + (mx - 10m)^2 = 20$

$x^2 + m^2x^2 - 20m^2x + 100m^2 = 20$

$x^2 + m^2x^2 - 20m^2x + 100m^2 - 20 = 0$

$\underset{a}{(1 + m^2)}x^2 - \underset{b}{20m^2}x + \underset{c}{100m^2 - 20} = 0$

If line is tangent to circle: $b^2 - 4ac = 0$:

$(-20m^2)^2 - 4(1+m^2)(100m^2 - 20) = 0$

$400m^4 - 4(100m^2 - 20 + 100m^4 - 20m^2) = 0$

$400m^4 - 4(100m^4 + 80m^2 - 20) = 0$

$\cancel{400m^4} - \cancel{400m^4} - 320m^2 + 80 = 0$

$-320m^2 + 80 = 0$

$320m^2 = 80$

$m^2 = \frac{1}{4}$

$m = \pm\frac{1}{2}$

[Turn over

10 A curve has equation $y = f(x)$ and it is given that

$$f'(x) = \left(\tfrac{1}{2}x + k\right)^{-2} - (1 + k)^{-2},$$

this is just a constant (no x) so disappears when differentiated.

where k is a constant. The curve has a minimum point at $x = 2$.

(a) Find $f''(x)$ in terms of k and x, and hence find the set of possible values of k. [3]

$$f''(x) = -2 \times \tfrac{1}{2}\left(\tfrac{1}{2}x + k\right)^{-3}$$

$$= -\left(\tfrac{1}{2}x + k\right)^{-3}$$

Sub. $x = 2$:

$$f''(2) = -\left(\tfrac{1}{2}(2) + k\right)^{-3}$$

$$= -(1 + k)^{-3}$$

minimum point, so $f''(x) > 0$:

$$-(1 + k)^{-3} > 0$$

$$(1 + k)^{-3} < 0$$

$$\frac{1}{(1 + k)^3} < 0$$

reciprocal of both sides

$$(1 + k)^3 < 0$$

cube root:

$$1 + k < 0$$

$$k < -1$$

It is now given that $k = -3$ and the minimum point is at $(2, 3\tfrac{1}{2})$.

(b) Find $f(x)$. [4]

$k = -3$: $f'(x) = \left(\tfrac{1}{2}x - 3\right)^{-2} - (-2)^{-2}$

$$= \left(\tfrac{1}{2}x - 3\right)^{-2} - \tfrac{1}{4}$$

$$f(x) = \int \left[\left(\tfrac{1}{2}x - 3\right)^{-2} - \tfrac{1}{4}\right] dx$$

$$= -1 \times 2\left(\tfrac{1}{2}x - 3\right)^{-1} - \tfrac{1}{4}x + C$$

$$= -2\left(\tfrac{1}{2}x - 3\right)^{-1} - \tfrac{1}{4}x + C$$

continued...

Sub. $(2, \frac{7}{2})$: $\frac{7}{2} = -2\left(\frac{1}{2}(2)-3\right)^{-1} - \frac{1}{4}(2) + C$

$\frac{7}{2} = -2(-2)^{-1} - \frac{1}{2} + C$

$\frac{7}{2} = -2 \times \frac{-1}{2} - \frac{1}{2} + C$

$\frac{7}{2} = 1 - \frac{1}{2} + C$

$\frac{7}{2} = \frac{1}{2} + C$

$C = 3$

$\rightarrow f(x) = -2\left(\frac{1}{2}x-3\right)^{-1} - \frac{1}{4}x + 3$

(c) Find the coordinates of the other stationary point and determine its nature. **[4]**

SP: $f'(x) = 0$:

$\left(\frac{1}{2}x-3\right)^{-2} - (1-3)^{-2} = 0$

$\left(\frac{1}{2}x-3\right)^{-2} - \frac{1}{4} = 0$

$\frac{1}{\left(\frac{1}{2}x-3\right)^{2}} = \frac{1}{4}$

$4 = \left(\frac{1}{2}x-3\right)^{2}$

$\frac{1}{2}x - 3 = \pm 2$

$\frac{1}{2}x - 3 = -2$ or $\frac{1}{2}x - 3 = 2$

$\frac{1}{2}x = 1$ $\quad$ $\frac{1}{2}x = 5$

$x = 2$ $\quad$ $x = 10$ ✓

↑ from part (a) & (b)

Sub. $x = 10$ into $f(x)$:

$f(10) = -2\left(\frac{1}{2}(10)-3\right)^{-1} - \frac{1}{4}(10) + 3$

$= -2(2)^{-1} - \frac{10}{4} + 3$

$= \frac{-2}{2} - \frac{5}{2} + 3$

$= \frac{-1}{2}$

$\rightarrow \left(10, \frac{-1}{2}\right)$

Sub. $x = 10$ into $f''(x)$:

$-\left(\frac{1}{2}(10)-3\right)^{-3}$

$= -(2)^{-3}$

$= \frac{-1}{8}$ $\rightarrow f''(x) < 0$ so $\left(10, \frac{-1}{2}\right)$ is a maximum point