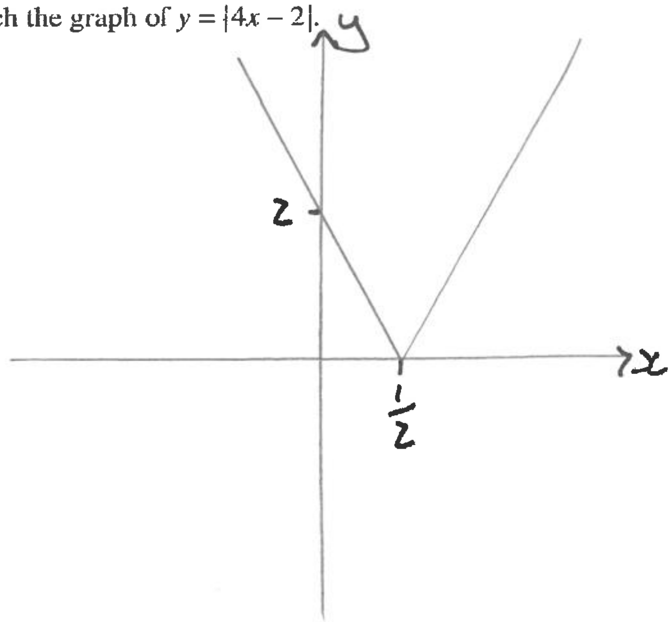
1 **(a)** Sketch the graph of $y = |4x - 2|$. [1]



(b) Solve the inequality $1 + 3x < |4x - 2|$. [4]

+/- Method: Critical values:

+: $1 + 3x = 4x - 2$ −: $1 + 3x = -(4x - 2)$

$1 = x - 2$ $1 + 3x = -4x + 2$

$x = 3$ $1 + 7x = 2$

$7x = 1$

$x = \frac{1}{7}$

$x < \frac{1}{7}$ e.g. $x = 0$	$\frac{1}{7} < x < 3$ e.g. $x = 1$	$x > 3$ e.g. $x = 4$
$1 + 3(0) < \|4(0) - 2\|$	$1 + 3(1) < \|4(1) - 2\|$	$1 + 3(4) < \|4(4) - 2\|$
$1 < \|-2\|$	$4 < \|2\|$	$13 < \|14\|$
$1 < 2$	$4 < 2$	$13 < 14$
✓	✗	✓

So $x < \frac{1}{7}$ or $x > 3$

2 The parametric equations of a curve are

$$x = (\ln t)^2, \qquad y = e^{2-t^2},$$

for $t > 0$.

Find the gradient of the curve at the point where $t = e$, simplifying your answer. [4]

$x = (\ln t)^2$ $\qquad$ $y = e^{2-t^2}$

$x = u^2 \quad u = \ln t$ $\qquad$ $y = e^v \quad v = 2-t^2$

$\frac{dx}{du} = 2u \quad \frac{du}{dt} = \frac{1}{t}$ $\qquad$ $\frac{dy}{dv} = e^v \quad \frac{dv}{dt} = -2t$

$\frac{dx}{dt} = 2u \times \frac{1}{t}$ $\qquad$ $\frac{dy}{dt} = -2te^v$

$= 2\ln t \times \frac{1}{t}$ $\qquad$ $= -2te^{2-t^2}$

$= \frac{2\ln t}{t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -2te^{2-t^2} \times \frac{t}{2\ln t}$

$= \frac{-2t^2e^{2-t^2}}{2\ln t}$

when $t = e$:

gradient $= \frac{-2e^2 \times e^{2-e^2}}{2\ln e}$

$= \frac{-2e^{4-e^2}}{2}$

$= -e^{4-e^2}$

3 The polynomial $2x^3 + ax^2 - 11x + b$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 12. ①

Find the values of a and b. [5]

①: $p\left(\frac{1}{2}\right) = 0$:

$$2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 - 11\left(\frac{1}{2}\right) + b = 0$$

$$2 \times \frac{1}{8} + a \times \frac{1}{4} - \frac{11}{2} + b = 0$$

$$\frac{1}{4} + \frac{a}{4} - \frac{11}{2} + b = 0$$

$$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \times 4$$

$$1 + a - 22 + 4b = 0$$

$$a + 4b = 21 \quad Ⓐ$$

②: $p(-1) = 12$:

$$2(-1)^3 + a(-1)^2 - 11(-1) + b = 12$$

$$2 \times -1 + a \times 1 + 11 + b = 12$$

$$-2 + a + 11 + b = 12$$

$$a + b = 3 \quad Ⓑ$$

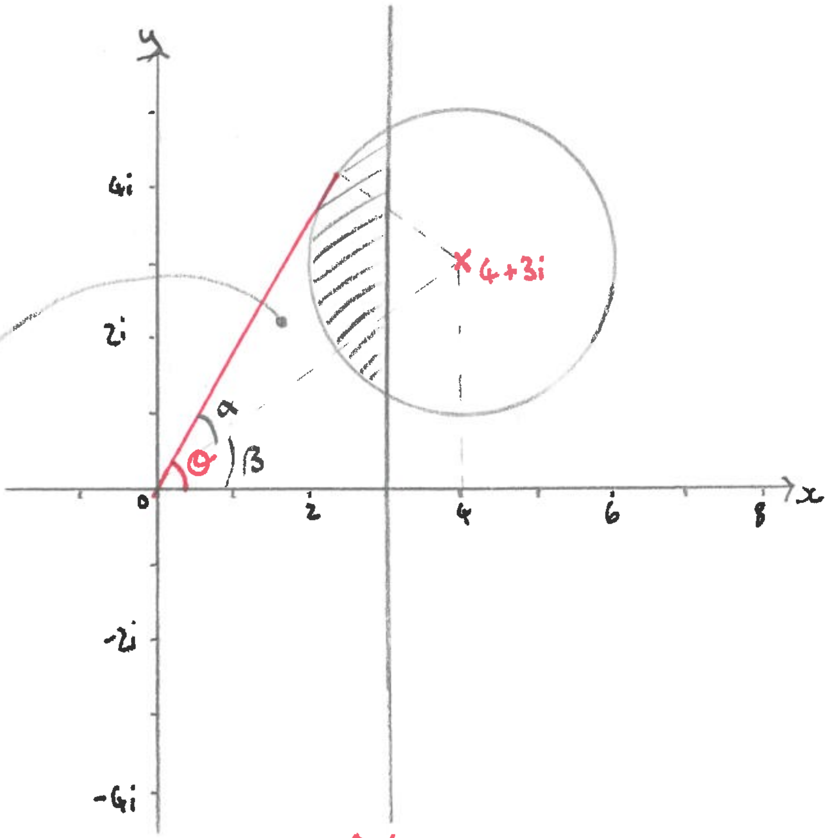
Ⓐ − Ⓑ: $3b = 18$

$$b = 6$$

→ Ⓑ: $a + 6 = 3$

$$a = -3$$

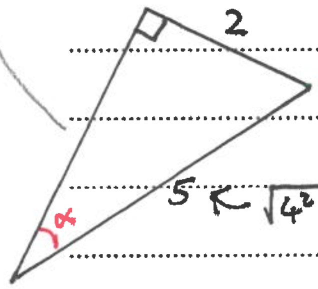
4 **(a)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z-4-3i| \leqslant 2$ and $\operatorname{Re} z \leqslant 3$. [4]



Re(z) ≤ 3

(b) Find the greatest value of arg z for points in this region. [2]

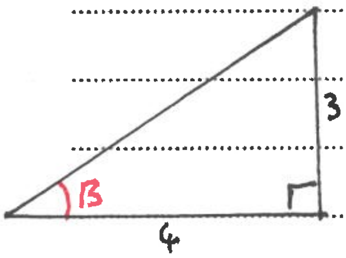
Greatest value of arg(z) is at tangent to circle as shown.



$5 \leftarrow \sqrt{4^2+3^2}$

$\sin\alpha = \frac{2}{5}$

$\alpha = 0.4115$ STO



$\tan\beta = \frac{3}{4}$

$\beta = 0.6435$ STO

$\theta = \alpha + \beta$

$= \underline{\underline{1.06}}$

5 Find the exact value of $\int_0^6 \frac{x(x+1)}{x^2+4}\,dx$. [6]

$\frac{x(x+1)}{x^2+4} = \frac{x^2+x}{x^2+4}$ ← improper fraction, so use algebraic division:

$$\begin{array}{r} 1 \\ x^2+0x+4 \overline{) x^2+x+0} \\ x^2+0x+4 \\ \hline x-4 \end{array}$$

$\rightarrow 1 + \frac{x-4}{x^2+4}$

$\rightarrow \int_0^6 \left(1 + \frac{x}{x^2+4} - \frac{4}{x^2+4}\right) dx$

$\int \frac{1}{x^2+a^2}$ with $a^2=4$ so $a=2$

$= \int_0^6 1\,dx + \frac{1}{2}\int_0^6 \frac{2x}{x^2+4}\,dx - 4\int_0^6 \frac{1}{x^2+4}\,dx$

$= \left[x + \frac{1}{2}\ln|x^2+4| - 4 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^6$

$= \left[6 + \frac{1}{2}\ln(40) - 2\tan^{-1}(3)\right] - \left[0 + \frac{1}{2}\ln(4) - 2\tan^{-1}(0)\right]$

$= 6 + \frac{1}{2}\ln 40 - 2\tan^{-1}(3) - \frac{1}{2}\ln 4 - 0$

$= 6 + \ln 40^{\frac{1}{2}} - \ln 4^{\frac{1}{2}} - 2\tan^{-1}(3)$

$= 6 + \ln\left(\frac{40}{4}\right)^{\frac{1}{2}} - 2\tan^{-1}(3)$

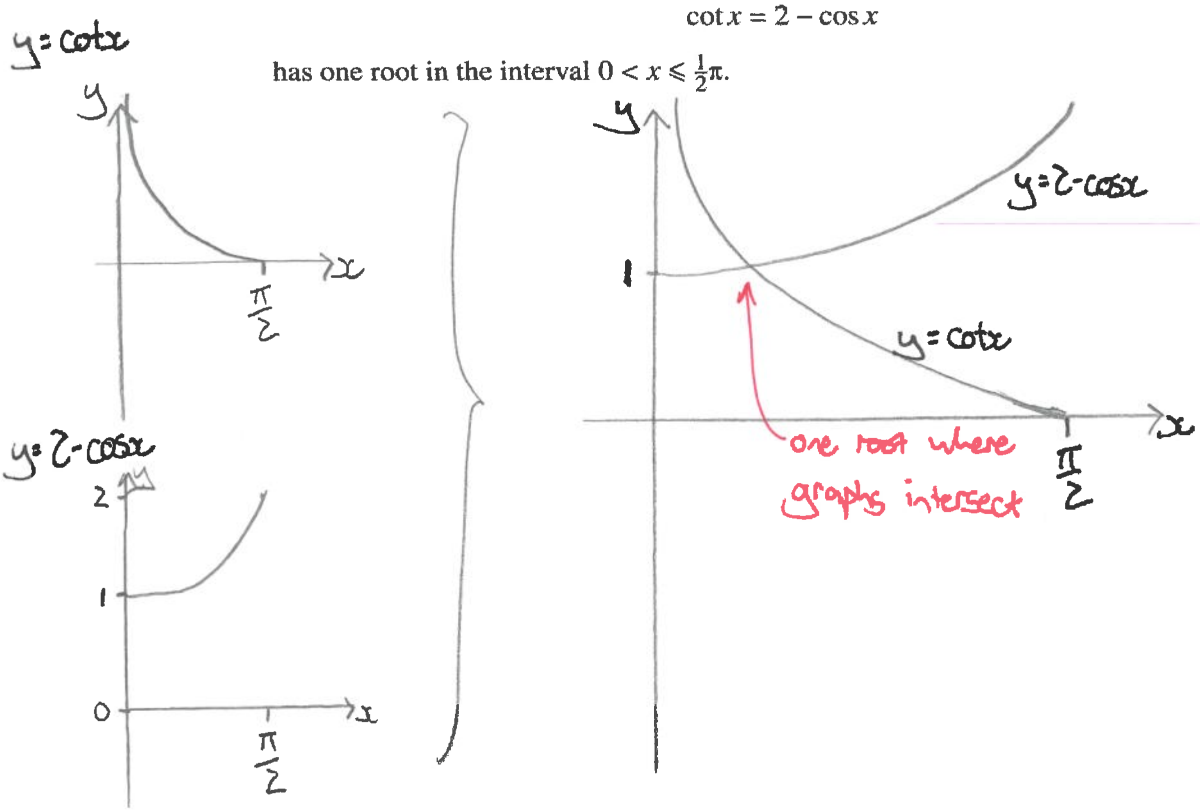
$= 6 + \ln 10^{\frac{1}{2}} - 2\tan^{-1}(3)$

$= 6 + \frac{1}{2}\ln 10 - 2\tan^{-1}(3)$

6 **(a)** By sketching a suitable pair of graphs, show that the equation

$$\cot x = 2 - \cos x$$

has one root in the interval $0 < x \leqslant \frac{1}{2}\pi$. [2]



(b) Show by calculation that this root lies between 0.6 and 0.8. [2]

Set = 0:

$2 - \cos x - \cot x = 0$

$x = 0.6$:

$2 - \cos(0.6) - \cot(0.6) = -0.287$

$x = 0.8$:

$2 - \cos(0.8) - \cot(0.8) = 0.332$

Sign change (and continuous function) between $x = 0.6$ and $x = 0.8$ shows there's a root between 0.6 and 0.8.

(c) Use the iterative formula $x_{n+1} = \tan^{-1}\left(\frac{1}{2 - \cos x_n}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 0.6000$

$x_2 = 0.7053$

$x_3 = 0.6792$

$x_4 = 0.6858$

$x_5 = 0.6842$

$x_6 = 0.6846$

$x_7 = 0.6845$

x_5, x_6 and x_7 all round to 0.68 to 2dp,

so $x = 0.68$

7 (a) By expressing 3θ as $2\theta + \theta$, prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [3]

$\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$2\cos^2\theta - 1$ $\quad$ $2\sin\theta\cos\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta \times \sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$1 - \cos^2\theta$

$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ QED

(b) Hence solve the equation

$$\cos 3\theta + \cos\theta\cos 2\theta = \cos^2\theta$$

for $0° \leqslant \theta \leqslant 180°$. [5]

from (a): $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\rightarrow 4\cos^3\theta - 3\cos\theta + \cos\theta\cos 2\theta = \cos^2\theta$

$2\cos^2\theta - 1$

$4\cos^3\theta - 3\cos\theta + \cos\theta(2\cos^2\theta - 1) = \cos^2\theta$

$4\cos^3\theta - 3\cos\theta + 2\cos^3\theta - \cos\theta = \cos^2\theta$

$6\cos^3\theta - 4\cos\theta = \cos^2\theta$

$6\cos^3\theta - \cos^2\theta - 4\cos\theta = 0$

$\cos\theta(6\cos^2\theta - \cos\theta - 4) = 0$

$\cos\theta = 0$

$\theta = 90°$

let $Y = \cos\theta$:

$6Y^2 - Y - 4 = 0$

$Y = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-4)}}{2(6)}$

$Y = \dfrac{1 + \sqrt{97}}{12}$ or $Y = \dfrac{1 - \sqrt{97}}{12}$

$\cos\theta = \dfrac{1 + \sqrt{97}}{12}$ $\cos\theta = \dfrac{1 - \sqrt{97}}{12}$

$\theta = 25.3°$ $\theta = 137.5°$

$\theta = 25.3°, 90°, 137.5°$

8 It is given that $\dfrac{2+3a\mathrm{i}}{a+2\mathrm{i}} = \lambda(2-\mathrm{i})$, where a and λ are real constants.

(a) Show that $3a^2 + 4a - 4 = 0$. [4]

$2 + 3ai = \lambda(2-i)(a+2i)$

$2 + 3ai = \lambda(2a + 4i - ai + 2)$

$2 + 3ai = 2a\lambda + 4\lambda i - a\lambda i + 2\lambda$

$2 + 3ai = 2a\lambda + 2\lambda + (4\lambda - a\lambda)i$

Re: $2 = 2a\lambda + 2\lambda$ Im: $3a = 4\lambda - a\lambda$ (2)

$1 = a\lambda + \lambda$

$1 = \lambda(a+1)$

$\lambda = \dfrac{1}{a+1}$ (1)

Sub. (1) into (2):

$3a = 4\left(\dfrac{1}{a+1}\right) - a\left(\dfrac{1}{a+1}\right)$

$3a = \dfrac{4}{a+1} - \dfrac{a}{a+1}$

$\times(a+1)$ $\times(a+1)$ $\times(a+1)$

$3a(a+1) = 4 - a$

$3a^2 + 3a = 4 - a$

$3a^2 + 4a - 4 = 0$ QED

(b) Hence find the possible values of a and the corresponding values of λ. [3]

$3a^2 + 4a - 4 = 0$

$(3a - 2)(a + 2) = 0$

$3a - 2 = 0$ or $a + 2 = 0$

$a = \frac{2}{3}$ $\quad$ $a = -2$

→ ①: $a = \frac{2}{3}$: $\lambda = \frac{1}{\frac{2}{3} + 1}$

$= \frac{1}{\frac{5}{3}}$ $\times 3$ / $\times 3$

$= \frac{3}{5}$

→ $a = \frac{2}{3}$ and $\lambda = \frac{3}{5}$

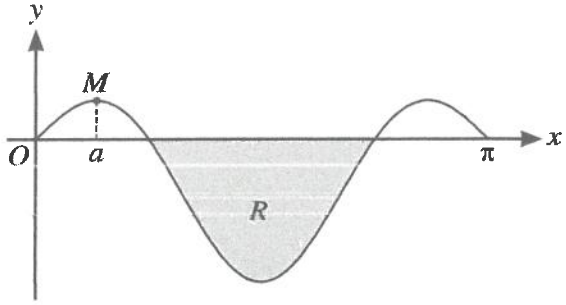
$a = -2$: $\lambda = \frac{1}{-2 + 1}$

$= \frac{1}{-1}$

$= -1$

→ $a = -2$ and $\lambda = -1$

9



The diagram shows the curve $y = \sin x \cos 2x$, for $0 \leqslant x \leqslant \pi$, and a maximum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

u v

(a) Find the value of a correct to 2 decimal places. [5]

$u = \sin x \qquad v = \cos 2x$

$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -2\sin 2x$

$\frac{dy}{dx} = -2\sin 2x \sin x + \cos 2x \cos x$

$2\sin x\cos x \qquad 2\cos^2 x - 1$

$= -4\sin x\cos x \times \sin x + (2\cos^2 x - 1)\cos x$

$= -4\sin^2 x\cos x + 2\cos^3 x - \cos x$

$1 - \cos^2 x$

$= -4(1-\cos^2 x)\cos x + 2\cos^3 x - \cos x$

$= -4\cos x + 4\cos^3 x + 2\cos^3 x - \cos x$

$= 6\cos^3 x - 5\cos x$

SP, so $\frac{dy}{dx} = 0$:

$6\cos^3 x - 5\cos x = 0$

$\cos x(6\cos^2 x - 5) = 0$

$\cos x = 0$ or $6\cos^2 x = 5$

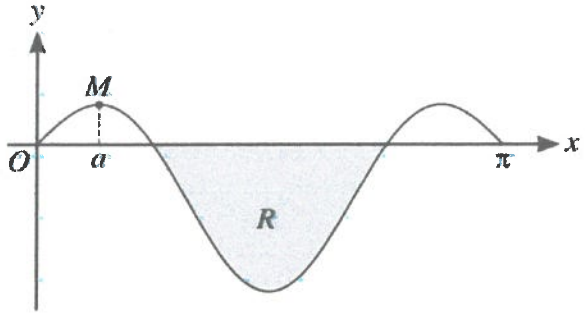
$x = \frac{\pi}{2} \qquad \cos^2 x = \frac{5}{6}$

$\cos x = \pm\sqrt{\frac{5}{6}}$

$\cos x = \sqrt{\frac{5}{6}}$ or $\cos x = -\sqrt{\frac{5}{6}}$

$x = 0.421$ or $x = 2.72$

Of the three stationary points, a is furthest to the left, so $x = 0.42$



The diagram shows the curve $y = \sin x \cos 2x$, for $0 \leqslant x \leqslant \pi$, and a maximum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(b) Find the exact area of the region R, giving your answer in simplified form. [4]

roots: $\sin x \cos 2x = 0$

$\sin x = 0$ or $\cos 2x = 0$

$x = 0, \pi$ $\quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \pi/4, \; 3\pi/4$

$$\int \sin x \cos 2x \, dx$$

$\cos 2x \to (2\cos^2 x - 1)$

$$= \int \sin x (2\cos^2 x - 1)\, dx$$

$$= \int 2\cos^2 x \sin x \, dx - \int \sin x \, dx$$

integrate normally

use substitution $u = \cos x$

$$= \int_{x=\frac{\pi}{4}}^{x=\frac{3\pi}{4}} 2\cos^2 x \sin x \, dx$$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\boxed{dx = \frac{du}{-\sin x}}$

$u = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ $\quad u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} 2u^2 \times \sin x \times \frac{du}{-\sin x} - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \, dx$$

$$= \int_{\sqrt{2}/2}^{-\frac{\sqrt{2}}{2}} -2u^2 \, du - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \, dx$$

$$= \left[-\frac{2}{3}u^3\right]_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} - \left[-\cos x\right]_{\pi/4}^{\frac{3\pi}{4}}$$

$$\left[-\frac{2}{3}\left(\frac{-\sqrt{2}}{2}\right)^3\right] - \left[-\frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] - \left[-\cos\left(\frac{3\pi}{4}\right)\right] + \left[-\cos\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$$

$$= -\frac{2\sqrt{2}}{3}$$

Area is positive,

so $= \frac{2\sqrt{2}}{3}$

10 The equations of the lines l and m are given by

$$l: \ \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad m: \ \mathbf{r} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix},$$

where c is a positive constant. It is given that the angle between l and m is 60°.

(a) Find the value of c. **[4]**

Find scalar product of direction vectors of l and m:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix} = \sqrt{1^2+1^2+2^2}\sqrt{(-2)^2+4^2+c^2}\cos(60)$$

$$-2+4+2c = \sqrt{6} \times \sqrt{20+c^2} \times \frac{1}{2}$$

$$2+2c = \sqrt{120+6c^2} \times \frac{1}{2}$$

$$4+4c = \sqrt{120+6c^2}$$

$$(4+4c)^2 = 120+6c^2$$

$$16+32c+16c^2 = 120+6c^2$$

$$10c^2+32c-104 = 0$$

$$5c^2+16c-52 = 0$$

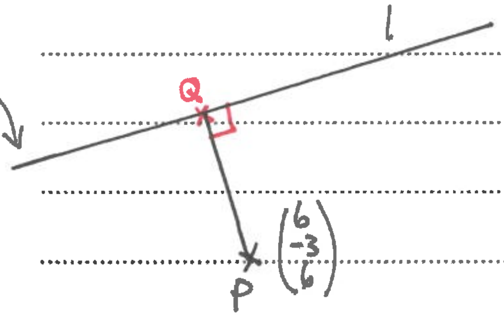
$$c = \frac{-16 \pm \sqrt{16^2-4(5)(-52)}}{2(5)}$$

$c = 2$ or $c = \frac{-26}{5}$ ✗

P

(b) Show that the length of the perpendicular from (6, −3, 6) to l is $\sqrt{11}$. **[5]**

$$\begin{pmatrix} 3+\lambda \\ -2+\lambda \\ 1+2\lambda \end{pmatrix}$$



$$\vec{PQ} = \underline{q} - \underline{p}$$

$$= \begin{pmatrix} 3+\lambda \\ -2+\lambda \\ 1+2\lambda \end{pmatrix} - \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -3+\lambda \\ 1+\lambda \\ -5+2\lambda \end{pmatrix}$$

$$\begin{pmatrix} -3+\lambda \\ 1+\lambda \\ -5+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$$

$\vec{PQ}$ ↗ ↖ direction vector of l

$$-3+\lambda + 1+\lambda - 10 + 4\lambda = 0$$

$$6\lambda - 12 = 0$$

$$6\lambda = 12$$

$$\underline{\lambda = 2}$$

sub $\lambda = 2$ to find $\vec{PQ}$:

$$\vec{PQ} = \begin{pmatrix} -3+2 \\ 1+2 \\ -5+4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

length of $\vec{PQ}$:

$$|\vec{PQ}| = \sqrt{(-1)^2 + 3^2 + (-1)^2}$$

$$\underline{= \sqrt{11}} \quad \text{QED}$$

11 The variables x and y satisfy the differential equation

$$x^2\frac{dy}{dx} + y^2 + y = 0.$$

It is given that $x = 1$ when $y = 1$.

(a) Solve the differential equation to obtain an expression for y in terms of x. [8]

$$x^2\frac{dy}{dx} = -y^2 - y$$

$$\int \frac{1}{-y^2-y}\,dy = \int \frac{1}{x^2}\,dx$$

$$\int \frac{-1}{y^2+y}\,dy = \int x^{-2}\,dx$$

LHS:

$$\frac{-1}{y^2+y} \equiv \frac{-1}{y(y+1)} \equiv \frac{A}{y} + \frac{B}{y+1}$$

$$-1 \equiv A(y+1) + By$$

$y=-1$:

$$-1 = 0 - B$$

$$-1 = -B$$

$$B = 1$$

$y=0$:

$$-1 = A + 0$$

$$A = -1$$

$$\rightarrow \int \frac{-1}{y}\,dy + \int \frac{1}{y+1}\,dy = \int x^{-2}\,dx$$

$$-\ln|y| + \ln|y+1| = -x^{-1} + C$$

$$\ln\left|\frac{y+1}{y}\right| = -x^{-1} + C$$

$x=1,\ y=1:$

$$\ln\left|\frac{2}{1}\right| = -1 + C$$

$$\ln 2 + 1 = C$$

$$\rightarrow \ln\left|\frac{y+1}{y}\right| = \frac{-1}{x} + \ln 2 + 1$$

$$\frac{y+1}{y} = e^{-\frac{1}{x} + \ln 2 + 1}$$

$$\frac{y+1}{y} = e^{-\frac{1}{x}+1} \times e^{\ln 2}$$

$$\frac{y+1}{y} = 2e^{-\frac{1}{x}+1}$$

$$y+1 = 2ye^{-\frac{1}{x}+1}$$

$$y+1 = 2ye^{1-\frac{1}{x}}$$

$$y - 2ye^{1-\frac{1}{x}} = -1$$

$$y\left(1-2e^{1-\frac{1}{x}}\right) = -1$$

$$y = \frac{-1}{1-2e^{1-\frac{1}{x}}} \quad \begin{matrix}\times -1 \\ \times -1\end{matrix}$$

$$y = \frac{1}{2e^{1-\frac{1}{x}} - 1}$$

(b) State what happens to the value of y when x tends to infinity. Give your answer in an exact form. [1]

as $x \to \infty$, $e^{1-\frac{1}{x}} \to e^{1}$

so $y \to \dfrac{1}{2e^{1} - 1}$

$$= \frac{1}{2e-1}$$