1 A curve is such that its gradient at a point (x, y) is given by $\frac{dy}{dx} = x - 3x^{-\frac{1}{2}}$. It is given that the curve passes through the point $(4, 1)$.

Find the equation of the curve. [4]

Integrate: $y = \frac{1}{2}x^2 - 6x^{\frac{1}{2}} + c$

Sub. (4, 1): $1 = \frac{1}{2}(4)^2 - 6(4)^{\frac{1}{2}} + c$

$1 = \frac{1}{2} \times 16 - 6 \times 2 + c$

$1 = 8 - 12 + c$

$1 = -4 + c$

$c = 5$

→ $y = \frac{1}{2}x^2 - 6x^{\frac{1}{2}} + 5$

2 The circle with equation $(x-3)^2+(y-5)^2=40$ intersects the y-axis at points A and B.

(a) Find the y-coordinates of A and B, expressing your answers in terms of surds. [2]

At y-axis, $x=0$:

$$(0-3)^2+(y-5)^2=40$$
$$9+y^2-10y+25=40$$
$$y^2-10y+34=40$$
$$y^2-10y-6=0$$
$$y=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(-6)}}{2(1)}$$

$y=5+\sqrt{31}$ or $y=5-\sqrt{31}$

(b) Find the equation of the circle which has AB as its diameter. [2]

$A:(0,5+\sqrt{31})$ $B:(0,5-\sqrt{31})$

Centre of circle = mid-point of AB: $\left(\frac{0+0}{2},\frac{5+\sqrt{31}+5-\sqrt{31}}{2}\right)$

$=(0,5)$

$\rightarrow (x-0)^2+(y-5)^2=r^2$

Sub. A:

$$(0)^2+(5+\sqrt{31}-5)^2=r^2$$
$$(\sqrt{31})^2=r^2$$
$$31=r^2$$

$\rightarrow x^2+(y-5)^2=31$

3 (a) Show that the equation

$$5\cos\theta - \sin\theta\tan\theta + 1 = 0$$

may be expressed in the form $a\cos^2\theta + b\cos\theta + c = 0$, where a, b and c are constants to be found. [3]

$5\cos\theta - \sin\theta \times \dfrac{\sin\theta}{\cos\theta} + 1 = 0$

$5\cos\theta - \dfrac{\sin^2\theta}{\cos\theta} + 1 = 0 \quad \times\cos\theta$

$5\cos^2\theta - \sin^2\theta + \cos\theta = 0$

$(1-\cos^2\theta)$

$5\cos^2\theta - (1-\cos^2\theta) + \cos\theta = 0$

$5\cos^2\theta - 1 + \cos^2\theta + \cos\theta = 0$

$\underline{6\cos^2\theta + \cos\theta - 1 = 0}$

(b) Hence solve the equation $5\cos\theta - \sin\theta\tan\theta + 1 = 0$ for $0 < \theta < 2\pi$. [4]

$6\cos^2\theta + \cos\theta - 1 = 0$

Let $Y = \cos\theta$:

$6Y^2 + Y - 1 = 0$

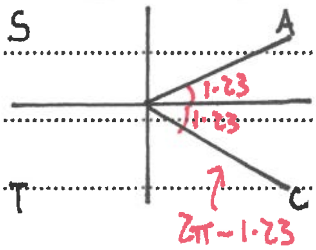
$(3Y-1)(2Y+1) = 0$

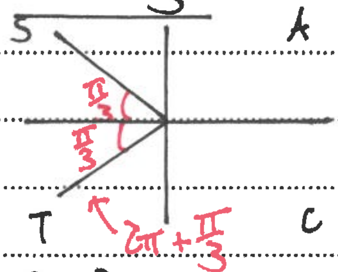
$3Y - 1 = 0$ or $2Y + 1 = 0$

$Y = \frac{1}{3}$ $\qquad$ $Y = -\frac{1}{2}$

$\cos\theta = \frac{1}{3}$ $\qquad$ $\cos\theta = -\frac{1}{2}$

$\theta = 1.23$ STO $\qquad$ $\theta = \frac{2\pi}{3}$





$\theta = 1.23, 5.05$ $\qquad$ $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\underline{\theta = 1.23, \frac{2\pi}{3}, \frac{4\pi}{3}, 5.05}$

4 (a) Expand the following in ascending powers of x up to and including the term in x^2.

(i) $(1+2x)^5$. [1]

$(1+2x)^5 = 1^5 + 5\times1^4\times(2x) + 10\times1^3\times(2x)^2 + \ldots$

$= 1 + 5\times2x + 10\times4x^2 + \ldots$

$= 1 + 10x + 40x^2 + \ldots$

(ii) $(1-ax)^6$, where a is a constant. [2]

$(1-ax)^6 = 1^6 + 6\times1^5\times(-ax) + 15\times1^4\times(-ax)^2 + \ldots$

$= 1 + 6\times-ax + 15\times a^2x^2 + \ldots$

$= 1 - 6ax + 15a^2x^2 + \ldots$

In the expansion of $(1+2x)^5(1-ax)^6$, the coefficient of x^2 is -5.

(b) Find the possible values of a. [4]

$(1 + 10x + 40x^2)(1 - 6ax + 15a^2x^2)$

x^2 term: $15a^2x^2 - 60ax^2 + 40x^2$

coefficient: $15a^2 - 60a + 40$

coeff $= -5$: $15a^2 - 60a + 40 = -5$

$15a^2 - 60a + 45 = 0$

$a^2 - 4a + 3 = 0$

$(a-3)(a-1) = 0$

$a = 3$ or $a = 1$

5 The first, second and third terms of a geometric progression are $2p + 6$, $5p$ and $8p + 2$ respectively.

(a) Find the possible values of the constant p. [3]

$$\underset{①}{2p+6} \xrightarrow{\times r} \underset{②}{5p} \xrightarrow{\times r} \underset{③}{8p+2}$$

Find r: $\frac{②}{①} = \frac{5p}{2p+6}$ $\quad \frac{③}{②} = \frac{8p+2}{5p}$

$$\frac{5p}{2p+6} = \frac{8p+2}{5p}$$

$$25p^2 = (8p+2)(2p+6)$$
$$25p^2 = 16p^2 + 52p + 12$$
$$9p^2 - 52p - 12 = 0$$
$$9p^2 - 54p + 2p - 12 = 0$$
$$9p(p-6) + 2(p-6) = 0$$

$$(9p+2)(p-6) = 0$$
$$9p+2=0 \text{ or } p-6=0$$
$$p = \frac{-2}{9} \text{ or } p = 6$$

(b) One of the values of p found in (a) is a negative fraction.

Use this value of p to find the sum to infinity of this progression. [4]

Sub. $p = \frac{-2}{9}$ into progression:

$$2\left(\frac{-2}{9}\right) + 6,\ 5\left(\frac{-2}{9}\right),\ 8\left(\frac{-2}{9}\right) + 2$$

$$\frac{50}{9},\ -\frac{10}{9},\ \frac{2}{9}$$

$$a = \frac{50}{9},\quad r = \frac{-\frac{10}{9}}{50/9} = \frac{-1}{5}$$

$$S_\infty = \frac{\frac{50}{9}}{1 - -\frac{1}{5}}$$

$$= \frac{\frac{50}{9}}{6/5} = \frac{125}{27}$$

6 A line has equation $y = 6x - c$ and a curve has equation $y = cx^2 + 2x - 3$, where c is a constant. The line is a tangent to the curve at point P.

Find the possible values of c and the corresponding coordinates of P. [7]

Solve simultaneously:

$$cx^2 + 2x - 3 = 6x - c$$

$$cx^2 - 4x + c - 3 = 0 \quad ①$$

Line is a tangent to the curve, so $b^2 - 4ac = 0$:

$$(-4)^2 - 4(c)(c-3) = 0$$

$$16 - 4c^2 + 12c = 0$$

$$-4c^2 + 12c + 16 = 0 \quad \div -4$$

$$c^2 - 3x - 4 = 0$$

$$(c-4)(c+1) = 0$$

$$\underline{c = 4} \quad \text{or} \quad \underline{c = -1}$$

$\underline{c = 4}$: Sub into ①:

$$4x^2 - 4x + 1 = 0$$

$$(2x-1)(2x-1) = 0$$

$$\underline{x = \tfrac{1}{2}}$$

Sub. into $y = 6x - c$:

$$y = 6x - 4$$

$$= 6(\tfrac{1}{2}) - 4$$

$$= \underline{-1}$$

$c = -1$: Sub into ①:

$$-x^2 - 4x - 4 = 0$$

$$x^2 + 4x + 4 = 0$$

$$(x+2)(x+2) = 0$$

$$\underline{x = -2}$$

Sub into $y = 6x - c$

$$y = 6x + 1$$

$$= 6(-2) + 1$$

$$= \underline{-11}$$

When $c = 4$, $P: (\tfrac{1}{2}, -1)$

When $c = -1$, $P: (-2, -11)$

7 The function f is defined by $f(x) = 1 + \dfrac{3}{x-2}$ for $x > 2$.

(a) State the range of f. [1]

$x = 2.0001$: $f(x) = 1 + \dfrac{3}{0.0001} = 30001$ so $f(x) \to \infty$

$x \to \infty$: $f(x) = 1 + \dfrac{3}{\infty - 2} = 1$ so $f(x) \to 1$

$f(x) > 1$

(b) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

$y = 1 + \dfrac{3}{x-2}$

$y - 1 = \dfrac{3}{x-2}$

$(y-1)(x-2) = 3$

$xy - 2y - x + 2 = 3$

$xy - x = 3 - 2 + 2y$

$xy - x = 1 + 2y$

$x(y-1) = 1 + 2y$

$x = \dfrac{1+2y}{y-1}$

$f^{-1}(x) = \dfrac{1+2x}{x-1}$

domain = range of $f(x)$

so $x > 1$

The function g is defined by $g(x) = 2x - 2$ for $x > 0$.

(c) Obtain a simplified expression for $gf(x)$. [2]

$gf(x) = 2(f(x)) - 2$

$= 2\left(1 + \dfrac{3}{x-2}\right) - 2$

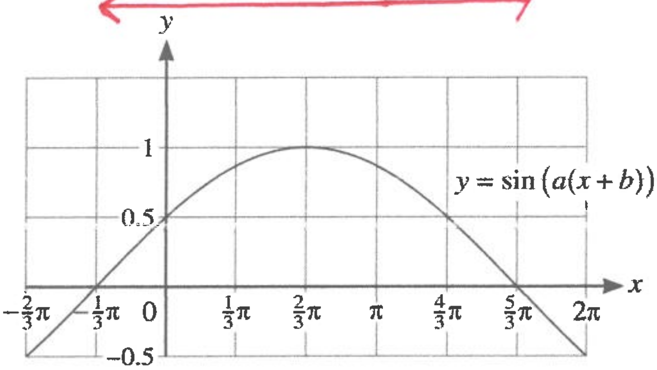
$= 2 + \dfrac{6}{x-2} - 2$

$= \dfrac{6}{x-2}$

8

$\frac{5\pi}{3} - -\frac{\pi}{3}$

2π



The diagram shows part of the graph of $y = \sin(a(x+b))$, where a and b are positive constants.

(a) State the value of a and one possible value of b. [2]

Distance between $-\frac{\pi}{3}$ and $\frac{5\pi}{3}$ is 2π, so period of graph $= 4\pi$

→ Stretch factor of 2, so $a = \frac{1}{2}$.

$y = \sin\left(\frac{1}{2}(x+b)\right)$ → $0 = \frac{1}{2}\left(-\frac{\pi}{3}+b\right)$ → $\frac{b}{2} = \frac{\pi}{6}$

Sub. $\left(-\frac{\pi}{3}, 0\right)$: $0 = \sin\left(\frac{1}{2}\left(-\frac{\pi}{3}+b\right)\right)$ $0 = -\frac{\pi}{6} + \frac{b}{2}$ $b = \frac{\pi}{3}$

Another curve, with equation $y = f(x)$, has a single stationary point at the point (p, q), where p and q are constants. This curve is transformed to a curve with equation

$$y = -3f\left(\tfrac{1}{4}(x+8)\right). \rightarrow y = -3f\left(\tfrac{1}{4}x+2\right)$$

(b) For the transformed curve, find the coordinates of the stationary point, giving your answer in terms of p and q. [3]

Careful: two transformations to x-coords, so the order isn't what you expect: Translate first.

x-coord:

$p \xrightarrow{\binom{-2}{0}} p-2 \xrightarrow{\times 4} 4p-8$

y-coord:

$q \xrightarrow{\times -3} -3q$

$(4p-8, -3q)$

9 A curve has equation $y = 2x^{\frac{1}{2}} - 1$.

(a) Find the equation of the normal to the curve at the point A (4, 3), giving your answer in the form $y = mx + c$. [3]

$\frac{dy}{dx} = x^{-\frac{1}{2}}$

Gradient at (4,3):

$m = 4^{-\frac{1}{2}}$

$= \frac{1}{2}$

Gradient of Normal $= -2$

$y - y_1 = m(x - x_1)$

$y - 3 = -2(x - 4)$

$y - 3 = -2x + 8$

$y = -2x + 11$

A point is moving along the curve $y = 2x^{\frac{1}{2}} - 1$ in such a way that at A the rate of increase of the x-coordinate is $3\,\text{cm}\,\text{s}^{-1}$. ①

(b) Find the rate of increase of the y-coordinate at A. [2]

①: $\frac{dx}{dt} = 3$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{1}{2}$ from part (a)

$\frac{1}{2} = \frac{dy}{dt} \times \frac{1}{3}$

$\frac{dy}{dt} = \frac{3}{2}$

At A the moving point suddenly changes direction and speed, and moves down the normal in such a way that the rate of decrease of the y-coordinate is constant at $5\,\text{cm}\,\text{s}^{-1}$. ②

(c) As the point moves down the normal, find the rate of change of its x-coordinate. [3]

②: $\frac{dy}{dt} = -5$

If the point is now on the normal, then: $y = -2x + 11$

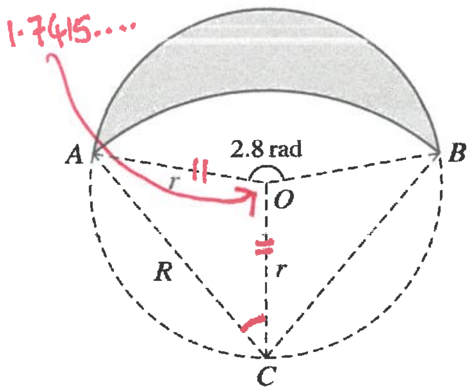
and: $\frac{dy}{dx} = -2$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$-2 = -5 \times \frac{dt}{dx}$

$\frac{dt}{dx} = \frac{2}{5} \rightarrow \frac{dx}{dt} = \frac{5}{2}$

10



The diagram shows points A, B and C lying on a circle with centre O and radius r. Angle AOB is 2.8 radians. The shaded region is bounded by two arcs. The upper arc is part of the circle with centre O and radius r. The lower arc is part of a circle with centre C and radius R.

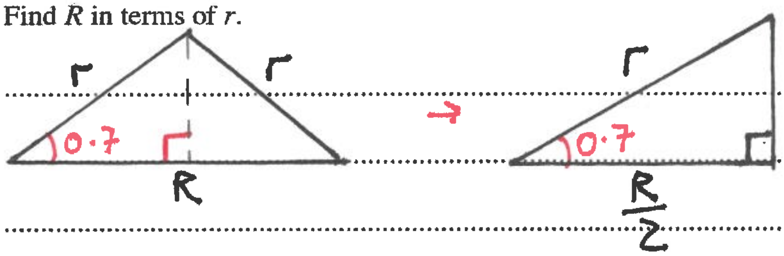
(a) State the size of angle ACO in radians. [1]

$A\hat{O}C$: $A\hat{O}C = \frac{2\pi - 2.8}{2} = \pi - 1.4$ (or 1.7415...)

$A\hat{C}O$: $A\hat{C}O = \frac{\pi - (\pi - 1.4)}{2}$ (isosceles triangle)

$= \frac{1.4}{2} = 0.7$

(b) Find R in terms of r. [1]



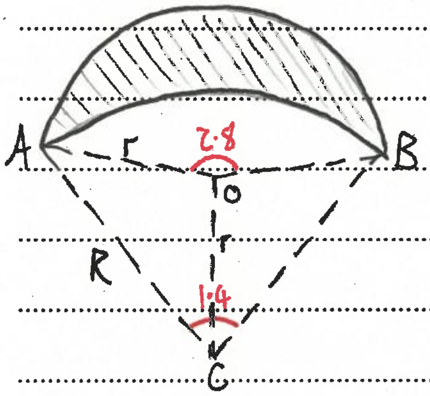
$\cos(0.7) = \frac{\frac{R}{2}}{r}$ ×2

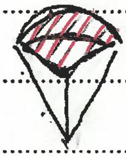
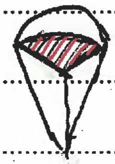
$\cos(0.7) = \frac{R}{2r}$

$R = 2r\cos(0.7)$

$= 1.53r$ STO

(c) Find the area of the shaded region in terms of r. [7]



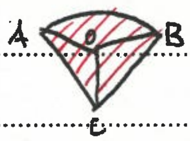
Shaded Area =  − 

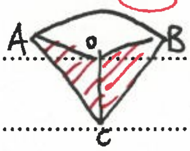
= Sector AOB − Segment AOB

Sector AOB: Area $= \frac{1}{2} r^2 \times 2{\cdot}8$

$= 1{\cdot}4r^2$

Segment AOB = Sector ACB − 2×triangles





Sector ACB: Area $= \frac{1}{2} R^2 \times 1{\cdot}4$

$= 0{\cdot}7R^2$

$= 0{\cdot}7(1{\cdot}53r)^2$

$= \underline{1{\cdot}638r^2}$ STO

1 triangle (AOC) $= \frac{1}{2} \times r \times r \times \sin(1{\cdot}7415\ldots)$

$= \underline{0{\cdot}4927r^2}$ STO

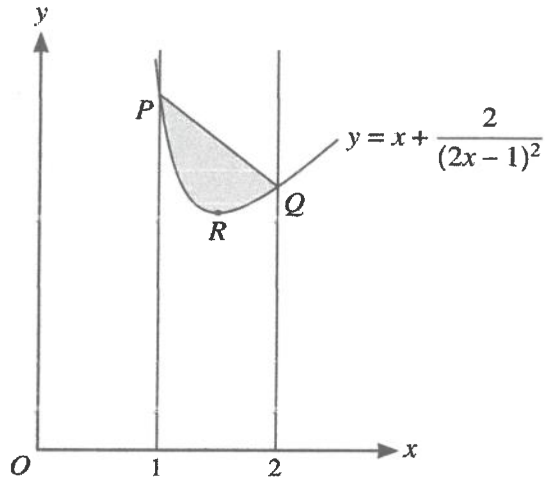
Segment AOB: $= 1{\cdot}638r^2 - 2 \times 0{\cdot}4927r^2$

$= \underline{0{\cdot}6525r^2}$ STO

Shaded Area $= 1{\cdot}4r^2 - 0{\cdot}6525r^2$

$= \underline{\underline{0{\cdot}747r^2}}$

11



The diagram shows part of the curve with equation $y = x + \frac{2}{(2x-1)^2}$. The lines $x = 1$ and $x = 2$ intersect the curve at P and Q respectively and R is the stationary point on the curve.

(a) Verify that the x-coordinate of R is $\frac{3}{2}$ and find the y-coordinate of R. [4]

$y = x + 2(2x-1)^{-2}$

$\frac{dy}{dx} = 1 - 2 \times 2 \times 2(2x-1)^{-3}$

$= 1 - 8(2x-1)^{-3}$

SP so $\frac{dy}{dx} = 0$: $1 - 8(2x-1)^{-3} = 0$

$8(2x-1)^{-3} = 1$

$(2x-1)^{-3} = \frac{1}{8}$

$(2x-1)^{3} = 8$

$2x - 1 = \sqrt[3]{8}$

$2x - 1 = 2$

$2x = 3$

$x = \frac{3}{2}$ QED

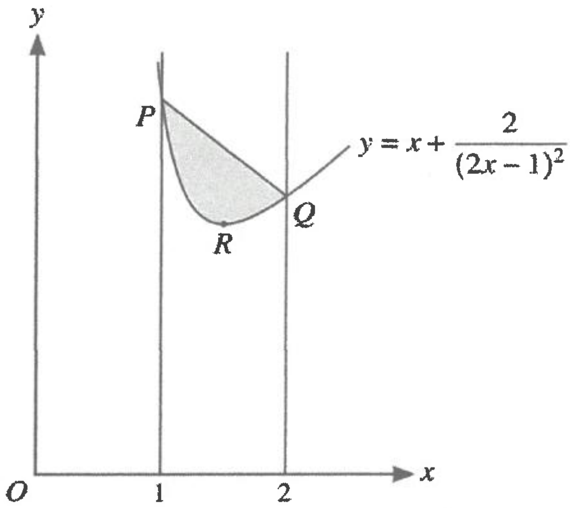
Sub. $x = \frac{3}{2}$:

$y = \frac{3}{2} + \frac{2}{(2(\frac{3}{2}) - 1)^2}$

$= \frac{3}{2} + \frac{2}{2^2}$

$= \frac{3}{2} + \frac{2}{4}$

$y = 2$



The diagram shows part of the curve with equation $y = x + \frac{2}{(2x-1)^2}$. The lines $x = 1$ and $x = 2$ intersect the curve at P and Q respectively and R is the stationary point on the curve.

(b) Find the exact value of the area of the shaded region. [6]

Find coords of P and Q:

$x=1$: $y = 1 + \frac{2}{(2-1)^2}$ $\qquad$ $x=2$: $y = 2 + \frac{2}{(4-1)^2}$

$= 1 + \frac{2}{1}$ $\qquad$ $= 2 + \frac{2}{9}$

$= 3$ $\qquad$ $= \frac{20}{9}$

Shaded Area = Area of Trapezium - Area under Curve

Area of Trapezium $= \frac{1}{2}\left(3 + \frac{20}{9}\right) \times 1$

$= \frac{47}{18}$

Area under Curve:

$\int_1^2 \left(x + 2(2x-1)^{-2}\right) dx$

$= \left[\frac{1}{2}x^2 - 1 \times \frac{1}{2} \times 2(2x-1)^{-1}\right]_1^2$

$$= \left[\frac{1}{2}x^2 - (2x-1)^{-1}\right]_1^2$$

$$= \left[\frac{1}{2}(2)^2 - (4-1)^{-1}\right] - \left[\frac{1}{2}(1)^2 - (2-1)^{-1}\right]$$

$$= \left[\frac{1}{2}\times 4 - 3^{-1}\right] - \left[\frac{1}{2}\times 1 - 1^{-1}\right]$$

$$= \left[2 - \frac{1}{3}\right] - \left[\frac{1}{2} - 1\right]$$

$$= \frac{5}{3} - -\frac{1}{2}$$

$$= \frac{13}{6}$$

Shaded Area $= \frac{47}{18} - \frac{13}{6}$

$$= \frac{4}{9}$$