1 **(a)** Expand $(1 + 3x)^6$ in ascending powers of x up to, and including, the term in x^2. [2]

$(1+3x)^6 = 1^6 + 6 \times 1^5 \times 3x + 15 \times 1^4 \times (3x)^2 + \ldots$

$= 1 + 18x + 15 \times 9x^2 + \ldots$

$= \underline{1 + 18x + 135x^2 + \ldots}$

(b) Hence find the coefficient of x^2 in the expansion of $(1 - 7x + x^2)(1 + 3x)^6$. [2]

$(1 - 7x + x^2)(1 + 18x + 135x^2)$

x^2 term: $135x^2 - 126x^2 + x^2$

$= 10x^2$

Coeff $= \underline{10}$

2 A line has equation $y = 2cx + 3$ and a curve has equation $y = cx^2 + 3x - c$, where c is a constant.

Showing all necessary working, determine which of the following statements is correct.

A The line and curve intersect only for a particular set of values of c.

B The line and curve intersect for all values of c.

C The line and curve do not intersect for any values of c. [4]

Solve simultaneously:

$$cx^2 + 3x - c = 2cx + 3$$

$$cx^2 + 3x - 2cx - c - 3 = 0$$

$$cx^2 + (3 - 2c)x + (-c - 3) = 0$$

$b^2 - 4ac$:

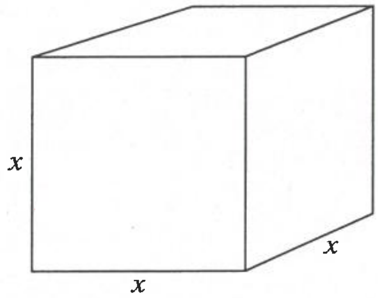
$$(3 - 2c)^2 - 4 \times c \times (-c - 3)$$

$$= 9 - 12c + 4c^2 + 4c^2 + 12c$$

$$= 8c^2 + 9$$

$8c^2 + 9$ is positive for all values of c, so the discriminant is > 0. This means the line and curve intersect for all values of c.

3



The diagram shows a cubical closed container made of a thin elastic material which is filled with water and frozen. During the freezing process the length, x cm, of each edge of the container increases at the constant rate of 0.01 cm per minute. The volume of the container at time t minutes is $V\text{cm}^3$.

1

Find the rate of increase of V when $x = 20$. [3]

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$\frac{dV}{dx}$: $V = x^3$

$$\frac{dV}{dx} = 3x^2$$

$\frac{dx}{dt}$: 1:

$$\frac{dx}{dt} = 0.01$$

→ $\frac{dV}{dt} = 3x^2 \times 0.01$

$= 0.03x^2$

sub. $x = 20$:

$= 0.03 \times 20^2$

$= \underline{12}$

4 The transformation R denotes a reflection in the x-axis and the transformation T denotes a translation of $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(a) Find the equation, $y = \mathrm{g}(x)$, of the curve with equation $y = x^2$ after it has been transformed by the sequence of transformations R followed by T. [2]

$f(x) \xrightarrow[R]{} -f(x) \xrightarrow[T]{} -f(x-3) - 1$

→ $g(x) = -(x-3)^2 - 1$

(b) Find the equation, $y = \mathrm{h}(x)$, of the curve with equation $y = x^2$ after it has been transformed by the sequence of transformations T followed by R. [2]

$f(x) \xrightarrow[T]{} f(x-3) - 1 \xrightarrow[R]{} -f(x-3) + 1$

→ $h(x) = -(x-3)^2 + 1$

(c) State fully the transformation that maps the curve $y = \mathrm{g}(x)$ onto the curve $y = \mathrm{h}(x)$. [2]

$g(x) = -(x-3)^2 - 1$
$h(x) = -(x-3)^2 + 1$ $+2$

→ Translation by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

5 **(a)** Show that the equation

$$4\sin x + \frac{5}{\tan x} + \frac{2}{\sin x} = 0$$

may be expressed in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found. [3]

$$4\sin x + \frac{5 \times\cos x}{\frac{\sin x}{\cos x}\times\cos x} + \frac{2}{\sin x} = 0$$

$$4\sin x + \frac{5\cos x}{\sin x} + \frac{2}{\sin x} = 0$$

$\times\sin x$ $\times\sin x$ $\times\sin x$ $\times\sin x$

$$4\sin^2 x + 5\cos x + 2 = 0$$

$\sin^2 x \to 1-\cos^2 x$

$$4(1-\cos^2 x) + 5\cos x + 2 = 0$$

$$4 - 4\cos^2 x + 5\cos x + 2 = 0$$

$$-4\cos^2 x + 5\cos x + 6 = 0$$

$\times -1$ $\times -1$ $\times -1$ $\times -1$

$$\underline{4\cos^2 x - 5\cos x - 6 = 0}$$

(b) Hence solve the equation $4\sin x + \dfrac{5}{\tan x} + \dfrac{2}{\sin x} = 0$ for $0° \leqslant x \leqslant 360°$. [3]

from (a): $4\cos^2 x - 5\cos x - 6 = 0$

Let $Y = \cos x$: $4Y^2 - 5Y - 6 = 0$

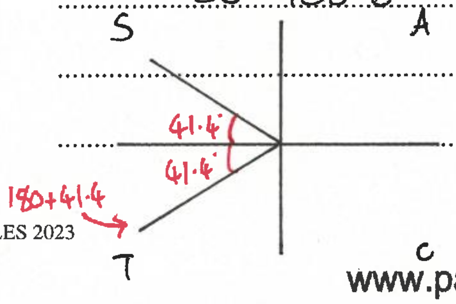
$$(4Y+3)(Y-2) = 0$$

$4Y+3=0$ or $Y-2=0$

$Y = \frac{-3}{4}$ $\qquad$ $Y = 2$

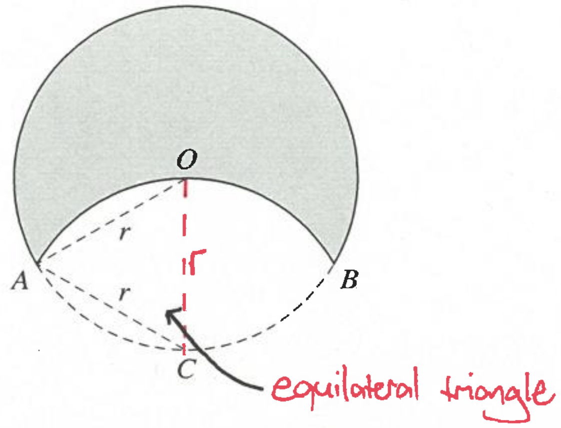
$\cos x = \frac{-3}{4}$ $\qquad$ $\cos x = 2$

$x = 138.6°$ $\qquad$ × no sol^{n}s



$\rightarrow x = \underline{138.6°, 221.4°}$

6



The diagram shows a motif formed by the major arc AB of a circle with radius r and centre O, and the minor arc AOB of a circle, also with radius r but with centre C. The point C lies on the circle with centre O.

(a) Given that angle $ACB = k\pi$ radians, state the value of the fraction k. [1]

ACO is an equilateral triangle, so $ACO = 60^\circ = \frac{\pi}{3}$ radians

$ACB = 2 \times ACO$

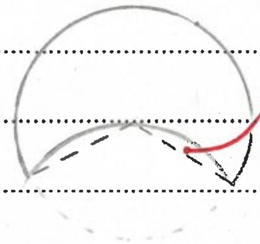
$= \frac{2\pi}{3}$ radians

$\rightarrow k = \frac{2}{3}$

(b) State the perimeter of the shaded motif in terms of π and r. [1]

Arc AOB is the same as arc ACB, so the perimeter is the same as the circumference of a circle: $2\pi r$

(c) Find the area of the shaded motif, giving your answer in terms of π, r and $\sqrt{3}$. **[5]**



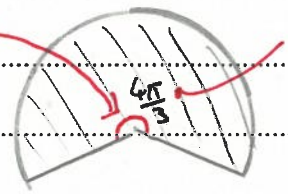
Area of tiny segment:

$= $ Area of sector $-$ Area of triangle

$= \frac{1}{2} r^2 \left(\frac{\pi}{3}\right) - \frac{1}{2} \times r \times r \times \sin\left(\frac{\pi}{3}\right)$

$= \frac{\pi}{6} r^2 - \frac{\sqrt{3}}{4} r^2$

$2\pi - \frac{2\pi}{3}$



Area of major sector:

$= \frac{1}{2} r^2 \times \frac{4\pi}{3}$

$= \frac{2\pi}{3} r^2$

Shaded motif $=$ Major sector $-$ $2 \times$ segments

$= \frac{2\pi}{3} r^2 - 2\left(\frac{\pi}{6} r^2 - \frac{\sqrt{3}}{4} r^2\right)$

$= \frac{2\pi}{3} r^2 - \frac{\pi}{3} r^2 + \frac{\sqrt{3}}{2} r^2$

$= \frac{\pi}{3} r^2 + \frac{\sqrt{3}}{2} r^2$

$= \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) r^2$

7 The sum of the first two terms of a geometric progression is 15 and the sum to infinity is $\frac{125}{7}$. The common ratio of the progression is negative.

Find the third term of the progression. [7]

Sum of first two terms: $a + ar = 15$

$a(1 + r) = 15$ ①

sum to infinity: $\frac{a}{1 - r} = \frac{125}{7}$

$a = \frac{125}{7}(1 - r)$ ②

Sub. ② into ①: $\frac{125}{7}(1-r)(1+r) = 15$

$125(1-r)(1+r) = 105$

$125[1 + r - r - r^2] = 105$

$125(1 - r^2) = 105$

$125 - 125r^2 = 105$

$125r^2 = 20$

$r^2 = \frac{4}{25}$

$r = \pm\frac{2}{5}$

Common ratio is negative: $r = -\frac{2}{5}$

Sub. into ①: $a\left(1 + -\frac{2}{5}\right) = 15$

$a \times \frac{3}{5} = 15$

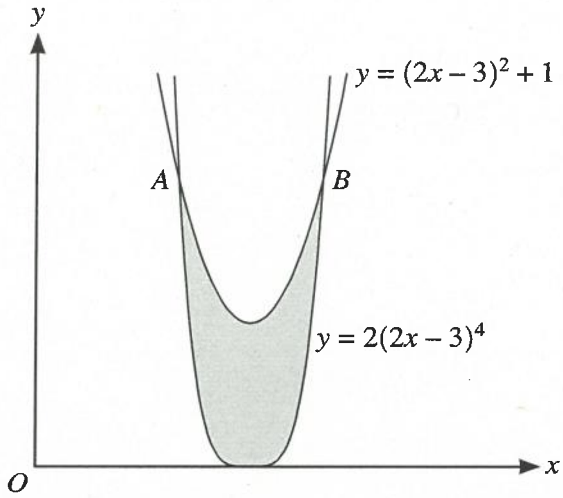
$a = 25$

Third term: $u_3 = ar^2$

$= 25 \times \left(\frac{-2}{5}\right)^2$

$= 4$

8



The diagram shows the curves with equations $y = 2(2x-3)^4$ and $y = (2x-3)^2 + 1$ meeting at points A and B.

(a) By using the substitution $u = 2x - 3$ find, by calculation, the coordinates of A and B. [4]

Solve simultaneously: $2(2x-3)^4 = (2x-3)^2 + 1$

use substitution: $2u^4 = u^2 + 1$

$2u^4 - u^2 - 1 = 0$

$(2u^2 + 1)(u^2 - 1) = 0$

$2u^2 + 1 = 0$ or $u^2 - 1 = 0$

$u^2 = \frac{-1}{2}$ $\quad u^2 = 1$

× no sol^ns $\quad u = 1$ or $u = -1$

$2x - 3 = 1$ or $2x - 3 = -1$

$2x = 4$ $\quad 2x = 2$

$x = 2$ $\quad x = 1$

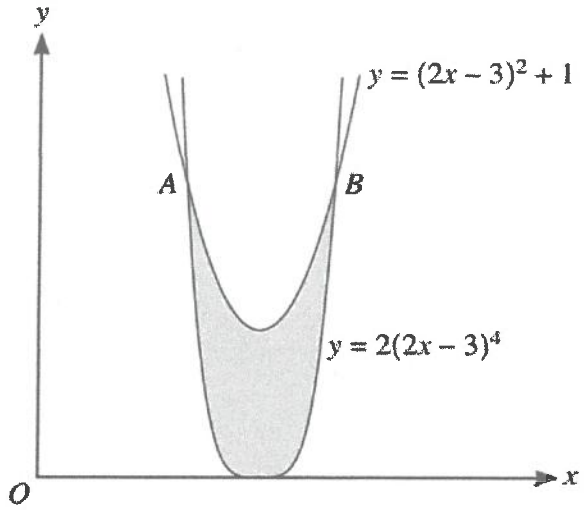
Sub into $y = (2x-3)^2 + 1$:

$x = 1$: $y = (2-3)^2 + 1$

$y = 2$ $\rightarrow$ $(1, 2)$

$x = 2$: $y = (4-3)^2 + 1$

$y = 2$ $\rightarrow$ $(2, 2)$



from (a) A:(1,2) B:(2,2)

The diagram shows the curves with equations $y = 2(2x-3)^4$ and $y = (2x-3)^2 + 1$ meeting at points A and B.

(b) Find the exact area of the shaded region. [5]

$$\text{Area} = \int_1^2 \left((2x-3)^2 + 1 - 2(2x-3)^4\right) dx$$

$$= \left[\frac{1}{3}\times\frac{1}{2}(2x-3)^3 + x - 2\times\frac{1}{5}\times\frac{1}{2}(2x-3)^5\right]_1^2$$

$$= \left[\frac{1}{6}(2x-3)^3 + x - \frac{1}{5}(2x-3)^5\right]_1^2$$

$$= \left[\frac{1}{6}(2(2)-3)^3 + 2 - \frac{1}{5}(2(2)-3)^5\right] - \left[\frac{1}{6}(2(1)-3)^3 + 1 - \frac{1}{5}(2(1)-3)^5\right]$$

$$= \left[\frac{1}{6}\times 1^3 + 2 - \frac{1}{5}\times 1^5\right] - \left[\frac{1}{6}\times(-1)^3 + 1 - \frac{1}{5}(-1)^5\right]$$

$$= \left[\frac{1}{6} + 2 - \frac{1}{5}\right] - \left[\frac{-1}{6} + 1 + \frac{1}{5}\right]$$

$$= \frac{59}{30} - \frac{31}{30}$$

$$= \frac{28}{30}$$

$$= \frac{14}{15}$$

9 (a) Express $4x^2 - 12x + 13$ in the form $(2x + a)^2 + b$, where a and b are constants. [2]

$(2x + a)^2 + b = 4x^2 + 4ax + a^2 + b$

$\rightarrow 4x^2 + 4ax + a^2 + b = 4x^2 - 12x + 13$

Compare coefficients:

x^2: $4 = 4$

x: $4a = -12$, $a = -3$

constant: $a^2 + b = 13$, $(-3)^2 + b = 13$, $9 + b = 13$, $b = 4$

$\rightarrow (2x - 3)^2 + 4$

The function f is defined by $f(x) = 4x^2 - 12x + 13$ for $p < x < q$, where p and q are constants. The function g is defined by $g(x) = 3x + 1$ for $x < 8$.

(b) Given that it is possible to form the composite function gf, find the least possible value of p and the greatest possible value of q. [3]

Range of f becomes the domain of g:

$x \xrightarrow{f} f(x) \xrightarrow{g} gf(x)$

Domain of g is $x < 8$, so range of f must be $f(x) < 8$.

$f(x) < 8$

$4x^2 - 12x + 13 < 8$

$4x^2 - 12x + 5 < 0$

critical values:

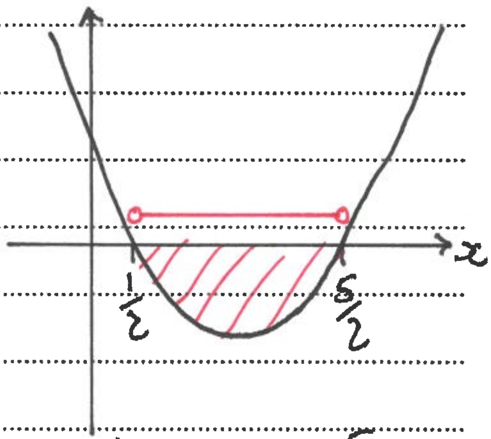
$4x^2 - 12x + 5 = 0$

$(2x - 5)(2x - 1) = 0$

$2x - 5 = 0$ or $2x - 1 = 0$

$2x = 5$ $\quad$ $2x = 1$

$x = \frac{5}{2}$ $\quad$ $x = \frac{1}{2}$



$\frac{1}{2} < x < \frac{5}{2}$

(c) Find an expression for $\mathrm{gf}(x)$. [1]

$\mathrm{gf}(x) = 3(4x^2 - 12x + 13) + 1$

$= 12x^2 - 36x + 39 + 1$

$= 12x^2 - 36x + 40$

The function h is defined by $\mathrm{h}(x) = 4x^2 - 12x + 13$ for $x < 0$.

(d) Find an expression for $\mathrm{h}^{-1}(x)$. [3]

from (a): $h(x) = (2x - 3)^2 + 4$

$y = (2x - 3)^2 + 4$

$y - 4 = (2x - 3)^2$

$\pm\sqrt{y - 4} = 2x - 3$

Domain of $h(x)$ is $x < 0$. $2x - 3$ is negative for all $x < 0$, so we want the negative square root.

$-\sqrt{y-4} = 2x - 3$

$3 - \sqrt{y-4} = 2x$

$\frac{3 - \sqrt{y-4}}{2} = x$

$h^{-1}(x) = \frac{3 - \sqrt{x-4}}{2}$

10 A curve has a stationary point at $(2, -10)$ and is such that $\frac{d^2y}{dx^2} = 6x$.

(a) Find $\frac{dy}{dx}$. [3]

Integrate $\frac{d^2y}{dx^2}$: $\frac{dy}{dx} = 3x^2 + c$

$\frac{dy}{dx} = 0$ at $(2, -10)$:

$0 = 3(2)^2 + c$

$0 = 3 \times 4 + c$

$c = -12$

$\rightarrow \frac{dy}{dx} = 3x^2 - 12$

(b) Find the equation of the curve. [3]

Integrate $\frac{dy}{dx}$:

$y = x^3 - 12x + c$

Sub. $(2, -10)$:

$-10 = 2^3 - 12(2) + c$

$-10 = 8 - 24 + c$

$-10 = -16 + c$

$c = 6$

$\rightarrow y = x^3 - 12x + 6$

(c) Find the coordinates of the other stationary point and determine its nature. [3]

At SP, $\frac{dy}{dx} = 0$:

$$3x^2 - 12 = 0$$
$$3x^2 = 12$$
$$x^2 = 4$$
$$x = \pm 2$$

$x = -2$:

$$y = (-2)^3 - 12(-2) + 6$$
$$= -8 + 24 + 6$$
$$= 22$$

$\rightarrow (-2, 22)$

Sub. into $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = 6(-2)$$
$$= -12$$

$\frac{d^2y}{dx^2} < 0$ so $(-2, 22)$ is a maximum

(d) Find the equation of the tangent to the curve at the point where the curve crosses the y-axis. [2]

At y-axis, $x = 0$:

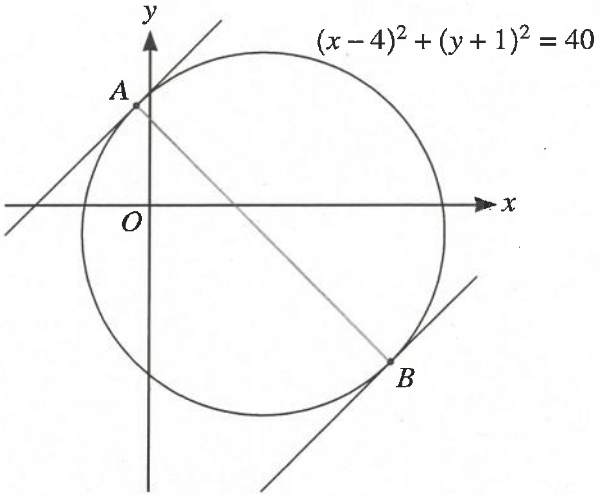
$$y = 0^3 - 12(0) + 6$$
$$y = 6 \rightarrow (0, 6)$$

Gradient at $(0, 6)$:

$$\frac{dy}{dx} = 3(0)^2 - 12$$
$$= -12$$

$$y - y_1 = m(x - x_1)$$
$$y - 6 = -12(x - 0)$$
$$y - 6 = -12x$$
$$y = -12x + 6$$

11



The diagram shows the circle with equation $(x-4)^2+(y+1)^2=40$. Parallel tangents, each with gradient 1, touch the circle at points A and B.

(a) Find the equation of the line AB, giving the answer in the form $y = mx + c$. [3]

• AB must be a diameter, so it passes through the centre $(4,-1)$.

• Gradient of AB $= \frac{-1}{1} = -1$

$\rightarrow \quad y - y_1 = m(x - x_1)$

$y - {}^-1 = -1(x-4)$

$y + 1 = -x + 4$

$\underline{y = -x + 3}$

(b) Find the coordinates of A, giving each coordinate in surd form. [4]

Solve $y=-x+3$ and equation of circle simultaneously:

$$(x-4)^2 + (-x+3+1)^2 = 40$$
$$(x-4)^2 + (-x+4)^2 = 40$$
$$x^2-8x+16+x^2-8x+16 = 40$$
$$2x^2-16x+32 = 40$$
$$2x^2-16x-8 = 0$$
$$x^2-8x-4 = 0$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2-4(1)(-4)}}{2(1)}$$

$x = 4+2\sqrt{5}$ or $x = 4-2\sqrt{5}$ ✓ (point A)

point B (on the right)

Sub. into $y=-x+3$:

$$y = -(4-2\sqrt{5})+3$$
$$= -4+2\sqrt{5}+3$$
$$= -1+2\sqrt{5}$$

$\rightarrow (4-2\sqrt{5}, -1+2\sqrt{5})$

(c) Find the equation of the tangent at A, giving the answer in the form $y = mx + c$, where c is in surd form. [2]

$$y - y_1 = m(x-x_1)$$
$$y-(-1+2\sqrt{5}) = 1(x-(4-2\sqrt{5}))$$
$$y+1-2\sqrt{5} = x-4+2\sqrt{5}$$
$$y = x-5+4\sqrt{5}$$