1 Solve the inequality $|5x-3| < 2|3x-7|$. [4]

+/- method: Critical values:

+: $5x-3 = 2(3x-7)$

$5x-3 = 6x-14$

$-3 = x-14$

$x = 11$

−: $-(5x-3) = 2(3x-7)$

$-5x+3 = 6x-14$

$3 = 11x-14$

$11x = 17$

$x = \frac{17}{11}$

$x < \frac{17}{11}$	$\frac{17}{11} < x < 11$	$x > 11$
e.g. $x = 0$	eg. $x = 5$	eg. $x = 12$
$\lvert 5(0)-3\rvert < 2\lvert 3(0)-7\rvert$	$\lvert 5(5)-3\rvert < 2\lvert 3(5)-7\rvert$	$\lvert 5(12)-3\rvert < 2\lvert 3(12)-7\rvert$
$\lvert -3\rvert < 2\lvert -7\rvert$	$\lvert 22\rvert < 2\lvert 8\rvert$	$\lvert 57\rvert < 2\lvert 29\rvert$
$3 < 2\times 7$	$\lvert 22\rvert < 2\times 8$	$57 < 2\times 29$
$3 < 14$	$22 < 16$	$57 < 58$
✓	✗	✓

So $x < \frac{17}{11}$ or $x > 11$

2 Solve the equation $\ln(2x^2 - 3) = 2\ln x - \ln 2$, giving your answer in an exact form. [3]

$\ln(2x^2 - 3) = \ln x^2 - \ln 2$

$\ln(2x^2 - 3) = \ln\left(\frac{x^2}{2}\right)$

$2x^2 - 3 = \frac{x^2}{2}$

$\times 2 \quad \times 2 \quad \times 2$

$4x^2 - 6 = x^2$

$3x^2 - 6 = 0$

$x^2 - 2 = 0$

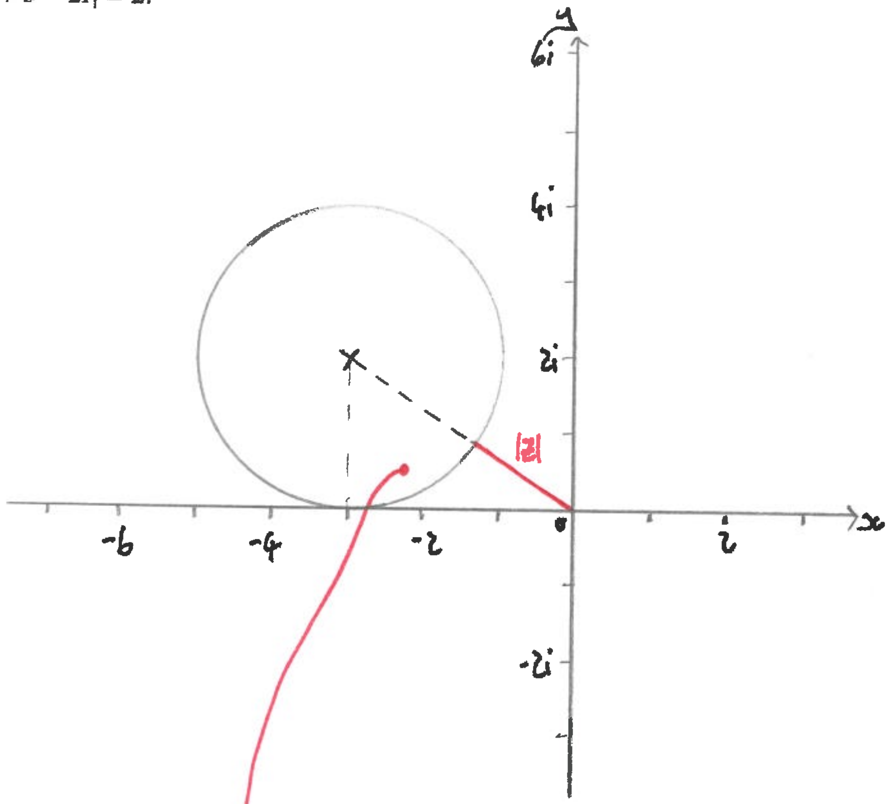
$x^2 = 2$

$x = \pm\sqrt{2}$

$\underline{x = \sqrt{2}}$ $\qquad$ $x = -\sqrt{2}$ x

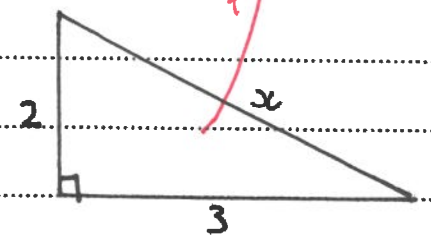
gives $2\ln(-\sqrt{2})$ in $2\ln x$ and logs of negatives don't exist.

3 **(a)** On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z+3-2\mathrm{i}| = 2$. [2]



(b) Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [2]

Least value of $|z|$ is closest point on circle to origin.



$x^2 = 2^2 + 3^2$

$x^2 = 13$

$x = \sqrt{13}$

$|z| = \sqrt{13} - \text{radius of circle}$

$= \underline{\sqrt{13} - 2}$

4 Solve the equation $2\cos x - \cos\frac{1}{2}x = 1$ for $0 \leqslant x \leqslant 2\pi$. [5]

$\cos 2A = 2\cos^2 A - 1$

$\cos x = 2\cos^2(\frac{1}{2}x) - 1$

$2\cos x = 4\cos^2(\frac{1}{2}x) - 2$

$\rightarrow 4\cos^2(\frac{1}{2}x) - 2 - \cos(\frac{1}{2}x) = 1$

$4\cos^2(\frac{1}{2}x) - \cos(\frac{1}{2}x) - 3 = 0$

Let $Y = \cos(\frac{1}{2}x)$:

$4Y^2 - Y - 3 = 0$

$(4Y + 3)(Y - 1) = 0$

$4Y + 3 = 0$ or $Y - 1 = 0$

$4Y = -3$ $\quad Y = 1$

$Y = \frac{-3}{4}$

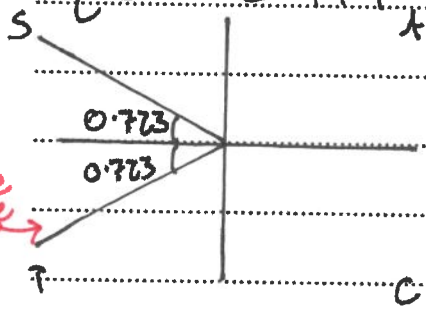
$\cos(\frac{1}{2}x) = \frac{-3}{4}$ or $\cos(\frac{1}{2}x) = 1$

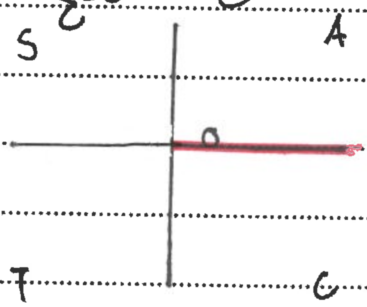
$0 \leqslant x \leqslant 2\pi$

$0 \leqslant \frac{1}{2}x \leqslant \pi$

$\frac{1}{2}x = 2.419$ $\qquad \frac{1}{2}x = 0$

outside range





$\frac{1}{2}x = 0$ only

$\frac{1}{2}x = 0, 2.419$

$x = 0, 4.84$

5 The complex number $2 + y\mathrm{i}$ is denoted by a, where y is a real number and $y < 0$. It is given that $\mathrm{f}(a) = a^3 - a^2 - 2a$.

(a) Find a simplified expression for $\mathrm{f}(a)$ in terms of y. [3]

$(2+iy)^3 - (2+iy)^2 - 2(2+iy)$

$(2+iy)(4+4iy-y^2) - (4+4iy-y^2) - 4 - 2iy$

$8 + 8iy - 2y^2 + 4iy - 4y^2 - iy^3 - 4 - 4iy + y^2 - 4 - 2iy$

$8 + 12iy - 2y^2 - 4y^2 - iy^3 - 8 - 6iy + y^2$

$6iy - y^2 - 4y^2 - iy^3$

$-y^2 - 4y^2 + 6iy - iy^3$

$-5y^2 + (6y - y^3)i$

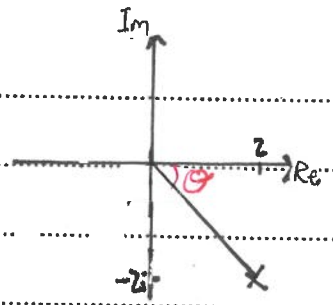
(b) Given that $\mathrm{Re}(\mathrm{f}(a)) = -20$, find $\arg a$. [3]

$-5y^2 = -20$

$y^2 = 4$

$y = 2$ or $y = -2$ ✓
×
$y < 0$

$a: 2 - 2i$



$\theta = -\tan^{-1}\left(\frac{2}{2}\right)$

$= -\frac{\pi}{4}$

6 The equation $\cot \frac{1}{2}x = 3x$ has one root in the interval $0 < x < \pi$, denoted by α.

(a) Show by calculation that α lies between 0.5 and 1. [2]

Set = 0:

$\cot\frac{1}{2}x - 3x = 0$

$x = 0.5$:

$\cot\left(\frac{1}{2}\times 0.5\right) - 3(0.5) = 2.416$

$x = 1$:

$\cot\left(\frac{1}{2}\times 1\right) - 3(1) = -1.170$

Sign change (and continuous function) between $x = 0.5$ and $x = 1$ shows there's a root between 0.5 and 1.

(b) Show that, if a sequence of positive values given by the iterative formula

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$$

converges, then it converges to α. [2]

$x = \frac{1}{3}\left(x + 4\tan^{-1}\left(\frac{1}{3x}\right)\right)$

$3x = x + 4\tan^{-1}\left(\frac{1}{3x}\right)$

$2x = 4\tan^{-1}\left(\frac{1}{3x}\right)$

$\frac{x}{2} = \tan^{-1}\left(\frac{1}{3x}\right)$

$\tan\left(\frac{x}{2}\right) = \frac{1}{3x}$

$3x = \frac{1}{\tan\left(\frac{x}{2}\right)}$

$3x = \cot\left(\frac{x}{2}\right)$ QED

(c) Use this iterative formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 0.5000$

$x_2 = 0.9507$

$x_3 = 0.7665$

$x_4 = 0.8024$

$x_5 = 0.7924$

$x_6 = 0.7950$

$x_7 = 0.7944$

$x_8 = 0.7945$

$x_9 = 0.7945$

x_7, x_8 and x_9 all round to 0.79 to 2dp,

so $\alpha = 0.79$

7 The equation of a curve is $3x^2 + 4xy + 3y^2 = 5$.

(a) Show that $\frac{dy}{dx} = -\frac{3x+2y}{2x+3y}$. [4]

differentiate $4\overset{u}{x}\overset{v}{y}$:

$u = 4x \qquad v = y$

$\frac{du}{dx} = 4 \qquad \frac{dv}{dx} = \frac{dy}{dx}$

$\rightarrow 4x\frac{dy}{dx} + 4y$

whole thing: $6x + 4x\frac{dy}{dx} + 4y + 6y\frac{dy}{dx} = 0$

$4x\frac{dy}{dx} + 6y\frac{dy}{dx} = -6x - 4y$

$\frac{dy}{dx}(4x + 6y) = -6x - 4y$

$\frac{dy}{dx} = \frac{-6x - 4y}{4x + 6y}$

$= \frac{-3x - 2y}{2x + 3y}$

$= -\frac{3x + 2y}{2x + 3y}$ QED

(b) Hence find the exact coordinates of the two points on the curve at which the tangent is parallel to $y + 2x = 0$. [5]

$\hookrightarrow y = -2x$

$\rightarrow$ gradient $= -2$:

$$-\frac{3x + 2y}{2x + 3y} = -2$$

$$\frac{3x + 2y}{2x + 3y} = 2$$

$$3x + 2y = 2(2x + 3y)$$

$$3x + 2y = 4x + 6y$$

$$2y = x + 6y$$

$$\underline{x = -4y}$$

Sub. into equation:

$$3(-4y)^2 + 4(-4y)y + 3y^2 = 5$$

$$3 \times 16y^2 - 16y^2 + 3y^2 = 5$$

$$48y^2 - 13y^2 = 5$$

$$35y^2 = 5$$

$$7y^2 = 1$$

$$y^2 = \frac{1}{7}$$

$$\underline{y = \pm\sqrt{\frac{1}{7}}}$$

Sub. into $x = -4y$:

$\underline{y = \sqrt{\frac{1}{7}}}$: $x = -4 \times \sqrt{\frac{1}{7}} = \frac{-4}{\sqrt{7}}$

$\underline{y = -\sqrt{\frac{1}{7}}}$: $x = -4 \times -\sqrt{\frac{1}{7}} = \frac{4}{\sqrt{7}}$

$\underline{\left(\sqrt{\frac{1}{7}}, \frac{-4}{\sqrt{7}}\right)}$ and $\underline{\left(-\sqrt{\frac{1}{7}}, \frac{4}{\sqrt{7}}\right)}$

8 (a) The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{4+9y^2}{e^{2x+1}}.$$

It is given that $y = 0$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

$$\int \frac{1 \ \div 9}{4 + 9y^2 \ \div 9} \, dy = \int \frac{1}{e^{2x+1}} \, dx$$

$$\int \frac{\frac{1}{9}}{\frac{4}{9} + y^2} \, dy = \int e^{-2x-1} \, dx$$

$$\frac{1}{9} \int \frac{1}{\frac{4}{9} + y^2} \, dy = \int e^{-2x-1} \, dx$$

$\int \frac{1}{y^2 + a^2}$ with $a^2 = \frac{4}{9}$ so $a = \frac{2}{3}$

$$\frac{1}{9} \times \frac{3}{2} \tan^{-1}\left(\frac{3}{2}y\right) = \frac{-1}{2} e^{-2x-1} + C$$

$y=0, x=1:$

$$\frac{1}{6} \tan^{-1}(0) = \frac{-1}{2} e^{-3} + c$$

$$0 = \frac{-1}{2} e^{-3} + c$$

$$c = \frac{1}{2} e^{-3}$$

$$\rightarrow \frac{1}{6} \tan^{-1}\left(\frac{3}{2}y\right) = \frac{-1}{2} e^{-2x-1} + \frac{1}{2} e^{-3}$$

$\times 6$ $\quad$ $\times 6$ $\quad$ $\times 6$

$$\tan^{-1}\left(\frac{3y}{2}\right) = -3e^{-2x-1} + 3e^{-3}$$

$$\frac{3y}{2} = \tan\left(-3e^{-2x-1} + 3e^{-3}\right)$$

cont..

$3y = 2\tan\left(-3e^{-2x-1} + 3e^{-3}\right)$

$y = \frac{2}{3}\tan\left(-3e^{-2x-1} + 3e^{-3}\right)$

(b) State what happens to the value of y as x tends to infinity. Give your answer in an exact form. [1]

as $x \to \infty$, $-3e^{-2x-1} \to 0$

so $y \to \frac{2}{3}\tan\left(3e^{-3}\right)$

9 Let $f(x) = \dfrac{2x^2 + 17x - 17}{(1 + 2x)(2 - x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{2x^2+17x-17}{(1+2x)(2-x)^2} \equiv \frac{A}{1+2x} + \frac{B}{2-x} + \frac{C}{(2-x)^2}$$

$$2x^2 + 17x - 17 \equiv A(2-x)^2 + B(1+2x)(2-x) + C(1+2x)$$

$x=2$:

$$2(2)^2 + 17(2) - 17 = 0 + 0 + C(1+4)$$
$$8 + 34 - 17 = 5C$$
$$25 = 5C$$
$$C = 5$$

$x = \frac{-1}{2}$:

$$2\left(\tfrac{-1}{2}\right)^2 + 17\left(\tfrac{-1}{2}\right) - 17 = A\left(2 - -\tfrac{1}{2}\right)^2 + 0 + 0$$
$$\frac{1}{2} - \frac{17}{2} - 17 = A\left(\frac{5}{2}\right)^2$$
$$-25 = \frac{25}{4}A$$
$$-100 = 25A$$
$$A = -4$$

$x=0$:

$$0 + 0 - 17 = A(2)^2 + B(1)(2) + C(1)$$
$$-17 = 4A + 2B + C$$
$$-17 = -16 + 2B + 5$$
$$-17 = -11 + 2B$$
$$2B = -6$$
$$B = -3$$

$$\rightarrow \frac{-4}{1+2x} - \frac{3}{2-x} + \frac{5}{(2-x)^2}$$

(b) Hence show that $\int_0^1 f(x)\,dx = \frac{5}{2} - \ln 72$. [5]

$$\int_0^1 \left(\frac{-4}{1+2x} - \frac{3}{2-x} + \frac{5}{(2-x)^2} \right) dx$$

$$= -2\int_0^1 \frac{2}{1+2x}\,dx + 3\int_0^1 \frac{-1}{2-x}\,dx + 5\int_0^1 (2-x)^{-2}\,dx$$

$$= \left[-2\ln|1+2x| + 3\ln|2-x| + 5(2-x)^{-1} \right]_0^1$$

$$= \left[-2\ln(3) + 3\ln(1) + 5(1)^{-1} \right] - \left[-2\ln(1) + 3\ln(2) + 5(2)^{-1} \right]$$

$$= -2\ln 3 + 0 + 5 + 0 - 3\ln 2 - \frac{5}{2}$$

$$= -2\ln 3 - 3\ln 2 + \frac{5}{2}$$

$$= \ln 3^{-2} - \ln 2^{3} + \frac{5}{2}$$

$$= \ln\left(\frac{1}{9}\right) - \ln 8 + \frac{5}{2}$$

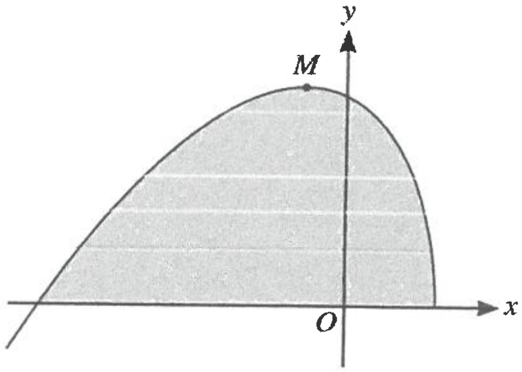
$$= \ln\left(\frac{\frac{1}{9}\times 9}{8 \times 9}\right) + \frac{5}{2}$$

$$= \ln\frac{1}{72} + \frac{5}{2}$$

$$= -\ln 72 + \frac{5}{2}$$

$$= \frac{5}{2} - \ln 72 \quad \text{QED}$$

10



The diagram shows the curve $y = (x+5)\sqrt{3-2x}$ and its maximum point M.

u v

(a) Find the exact coordinates of M. [5]

$u = x + 5$ $\qquad v = (3-2x)^{\frac{1}{2}}$

$\frac{du}{dx} = 1$ $\qquad \frac{dv}{dx} = \frac{1}{2} \times -2(3-2x)^{-\frac{1}{2}}$

$= -(3-2x)^{-\frac{1}{2}}$

$\frac{dy}{dx} = -(x+5)(3-2x)^{-\frac{1}{2}} + (3-2x)^{\frac{1}{2}}$

SP, so $\frac{dy}{dx} = 0$:

$\frac{-(x+5)}{(3-2x)^{\frac{1}{2}}} + (3-2x)^{\frac{1}{2}} = 0$

$\frac{x+5}{(3-2x)^{\frac{1}{2}}} = (3-2x)^{\frac{1}{2}}$

$x + 5 = 3 - 2x$

$3x = -2$

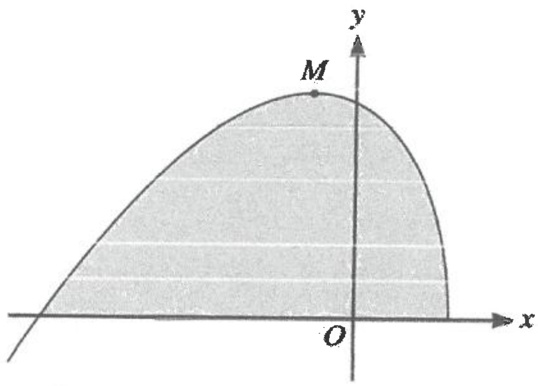
$x = \frac{-2}{3}$

sub. in to find y:

$y = \left(\frac{-2}{3} + 5\right)\sqrt{3 - 2\left(\frac{-2}{3}\right)}$

$= \frac{13}{3}\sqrt{\frac{13}{3}}$

$= \frac{13\sqrt{39}}{9}$ $\rightarrow \left(\frac{-2}{3}, \frac{13\sqrt{39}}{9}\right)$



The diagram shows the curve $y = (x+5)\sqrt{3-2x}$ and its maximum point M.

(b) Using the substitution $u = 3 - 2x$, find by integration the area of the shaded region bounded by the curve and the x-axis. Give your answer in the form $a\sqrt{13}$, where a is a rational number. [5]

roots: $(x+5)\sqrt{3-2x} = 0$

$x+5=0$ or $\sqrt{3-2x}=0$

$\underline{x=-5}$ $\quad 3-2x=0$

$\underline{x=\frac{3}{2}}$

$u = 3-2x$

$\frac{du}{dx} = -2$

$du = -2\,dx$

$\boxed{dx = \frac{du}{-2}}$

$u=3-2(\frac{3}{2}) = 0$

$u = 3-2(-5)$, $u=13$

$$\int_{x=-5}^{x=\frac{3}{2}} (x+5)\sqrt{3-2x}\,dx$$

$$\int_{13}^{0} (x+5)u^{\frac{1}{2}} \times \frac{du}{-2}$$

$u = 3-2x$

$2x = 3-u$

$x = \frac{3-u}{2}$

$x+5 = \frac{3-u}{2}+5 = \frac{3-u}{2}+\frac{10}{2} = \frac{13-u}{2}$

$$\int_{13}^{0} \left(\frac{13-u}{2}\right)u^{\frac{1}{2}} \times \frac{du}{-2}$$

$$= \frac{-1}{4}\int_{13}^{0}(13-u)u^{\frac{1}{2}}\,du = \frac{-1}{4}\int_{13}^{0}\left(13u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$$= \frac{-1}{4}\left[\frac{26}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_{13}^{0} = \frac{-1}{4}\left[0-0\right] - \frac{-1}{4}\left[\frac{26}{3}(13)^{\frac{3}{2}} - \frac{2}{5}(13)^{\frac{5}{2}}\right]$$

$$= \frac{1}{4}\left[\frac{26}{3}\times 13\sqrt{13} - \frac{2}{5}\times 13^2\sqrt{13}\right] = \frac{1}{4}\left[\frac{338}{3}\sqrt{13} - \frac{338}{5}\sqrt{13}\right]$$

$$= \frac{1}{4}\times\frac{676}{15}\sqrt{13}$$

$$= \underline{\frac{169}{15}\sqrt{13}}$$

[Turn over

11 The points A and B have position vectors $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$.

(a) Show that l does not intersect the line passing through A and B. [5]

line through A and B:

$$\vec{AB} = \underline{b} - \underline{a}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$$

point on line (A) direction vector $\vec{AB}$

Equate the two lines:

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$$

$\underline{\underline{i}}$: $1 + 2\mu = 1 + \lambda$

$\lambda = 2\mu$ ①

$\underline{\underline{j}}$: $-1 - 3\mu = 2 - 3\lambda$

$-3\mu = 3 - 3\lambda$

$3\lambda - 3\mu = 3$

$\lambda - \mu = 1$ ②

Sub. ① into ②: $2\mu - \mu = 1$

$\underline{\mu = 1}$

$\to$ ①: $\underline{\lambda = 2}$

Now see if these values work in k:

$\underline{\underline{k}}$: $3 + 4\mu = -2 + 3\lambda$

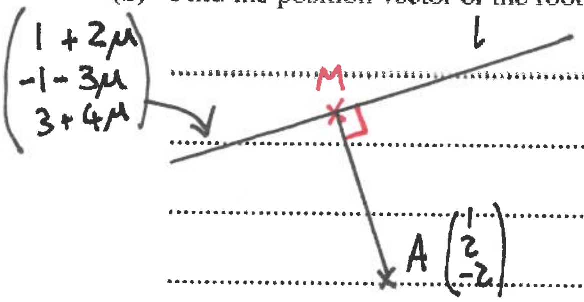
$3 + 4(1) = -2 + 3(2)$

$3 + 4 = -2 + 6$

$7 \neq 4$

Not equal, therefore lines don't intersect.

(b) Find the position vector of the foot of the perpendicular from A to l. **[4]**



$$\vec{AM} = \underline{m} - \underline{a}$$

$$= \begin{pmatrix} 1+2\mu \\ -1-3\mu \\ 3+4\mu \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\mu \\ -3-3\mu \\ 5+4\mu \end{pmatrix}$$

$$\begin{pmatrix} 2\mu \\ -3-3\mu \\ 5+4\mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 0$$

$\vec{AM}$ ↗ ↖ direction vector of l

$$4\mu + 9 + 9\mu + 20 + 16\mu = 0$$

$$29\mu + 29 = 0$$

$$29\mu = -29$$

$$\underline{\mu = -1}$$

For position vector, sub. μ into eqⁿ of line:

$$\underline{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}}$$