1 Solve the equation

$$3e^{2x} - 4e^{-2x} = 5.$$

Give the answer correct to 3 decimal places. [3]

$$3e^{2x} - \frac{4}{e^{2x}} = 5$$

$\times e^{2x}$ $\times e^{2x}$ $\times e^{2x}$

$$3e^{4x} - 4 = 5e^{2x}$$

$$3e^{4x} - 5e^{2x} - 4 = 0$$

Let $Y = e^{2x}$:

$$3Y^2 - 5Y - 4 = 0$$

$$Y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-4)}}{2(3)}$$

$$Y = \frac{5+\sqrt{73}}{6} \quad \text{or} \quad Y = \frac{5-\sqrt{73}}{6}$$

$$e^{2x} = \frac{5+\sqrt{73}}{6} \qquad e^{2x} = \frac{5-\sqrt{73}}{6}$$

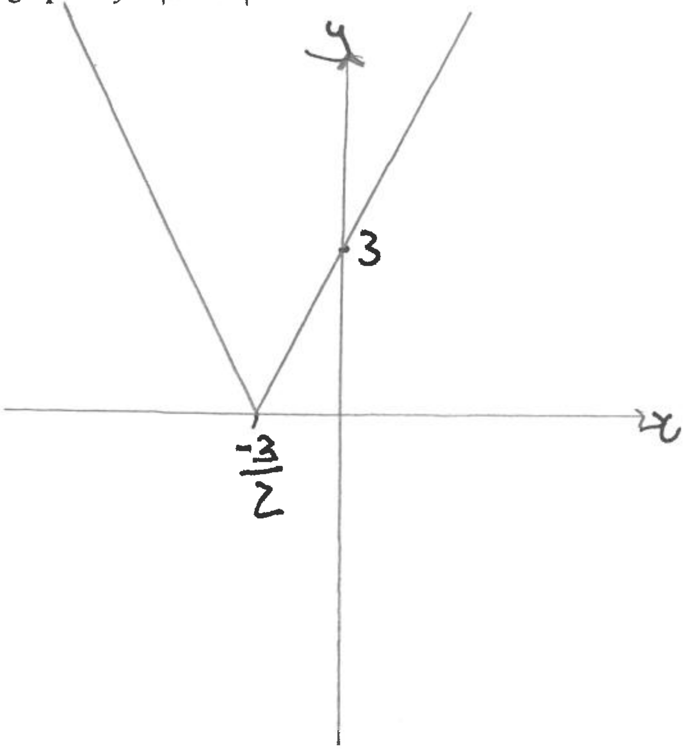
X no solutions

$$2x = \ln\left(\frac{5+\sqrt{73}}{6}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{5+\sqrt{73}}{6}\right)$$

$$= \underline{0.407}$$

2 **(a)** Sketch the graph of $y = |2x + 3|$. [1]



(b) Solve the inequality $3x + 8 > |2x + 3|$. [3]

+/- method: critical values:

+: $3x + 8 = 2x + 3$ —: $3x + 8 = -(2x + 3)$

$x + 8 = 3$ $3x + 8 = -2x - 3$

$x = -5$ $5x + 8 = -3$

$5x = -11$

$x = -11/5$

$x < -5$	$-5 < x < \frac{-11}{5}$	$x > \frac{-11}{5}$
eg. $x = -10$	eg. $x = -3$	eg. $x = 0$
$3(-10) + 8 > \|2(-10) + 3\|$	$3(-3) + 8 > \|2(-3) + 3\|$	$3(0) + 8 > \|2(0) + 3\|$
$-30 + 8 > \|-17\|$	$-9 + 8 > \|-3\|$	$8 > \|3\|$
$-22 > 17$	$-1 > 3$	$8 > 3$
✗	✗	✓

so $x > \frac{-11}{5}$

3 Find the coefficient of x^3 in the binomial expansion of $(3+x)\sqrt{1+4x}$. [4]

$$(1+4x)^{\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right)(4x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(4x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(4x)^3 + \dots$$

$$= 1 + 2x - \frac{1}{8}\times 16x^2 + \frac{1}{16}\times 64x^3 + \dots$$

$$= 1 + 2x - 2x^2 + 4x^3 + \dots$$

$$(3+x)(1+4x)^{\frac{1}{2}} = (3+x)(1 + 2x - 2x^2 + 4x^3 + \dots)$$

x^3 term: $12x^3 - 2x^3$

$= 10x^3$

Coefficient $= \underline{10}$

4 **(a)** Show that the equation $\sin 2\theta + \cos 2\theta = 2\sin^2\theta$ can be expressed in the form

$$\cos^2\theta + 2\sin\theta\cos\theta - 3\sin^2\theta = 0.$$ **[2]**

$\sin 2\theta + \cos 2\theta = 2\sin^2\theta$

$2\sin\theta\cos\theta$ → ↖ $\cos^2\theta - \sin^2\theta$

$2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta = 2\sin^2\theta$

$2\sin\theta\cos\theta + \cos^2\theta - 3\sin^2\theta = 0$

$\cos^2\theta + 2\sin\theta\cos\theta - 3\sin^2\theta = 0$ QED

(b) Hence solve the equation $\sin 2\theta + \cos 2\theta = 2\sin^2\theta$ for $0° < \theta < 180°$. **[4]**

Factorise by splitting the middle: two numbers that multiply to -3 and sum to $+2$: -1 and $+3$

$\cos^2\theta + 3\sin\theta\cos\theta - 1\sin\theta\cos\theta - 3\sin^2\theta = 0$

$\cos\theta(\cos\theta + 3\sin\theta) - \sin\theta(\cos\theta + 3\sin\theta) = 0$

$(\cos\theta - \sin\theta)(\cos\theta + 3\sin\theta) = 0$

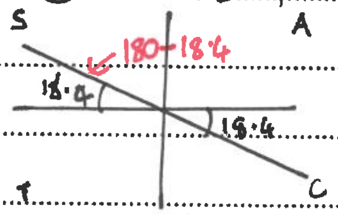
$\cos\theta - \sin\theta = 0$ or $\cos\theta + 3\sin\theta = 0$

$\sin\theta = \cos\theta$ $\qquad$ $3\sin\theta = -\cos\theta$

$\frac{\sin\theta}{\cos\theta} = 1$ $\qquad$ $\frac{\sin\theta}{\cos\theta} = \frac{-1}{3}$

$\tan\theta = 1$ $\qquad$ $\tan\theta = -\frac{1}{3}$

$\theta = 45°$ $\qquad$ $\theta = -18.4$



$\theta = 161.6°$

$\theta = 45°, 161.6°$

5 The equation of a curve is $x^2y - ay^2 = 4a^3$, where a is a non-zero constant.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2xy}{2ay - x^2}$. **[4]**

differentiate: $\overset{u}{x^2}\overset{v}{y}$

$u = x^2 \qquad v = y$

$\dfrac{du}{dx} = 2x \qquad \dfrac{dv}{dx} = \dfrac{dy}{dx}$

$\rightarrow x^2\dfrac{dy}{dx} + 2xy$

Whole thing:

$x^2\dfrac{dy}{dx} + 2xy - 2ay\dfrac{dy}{dx} = 0$

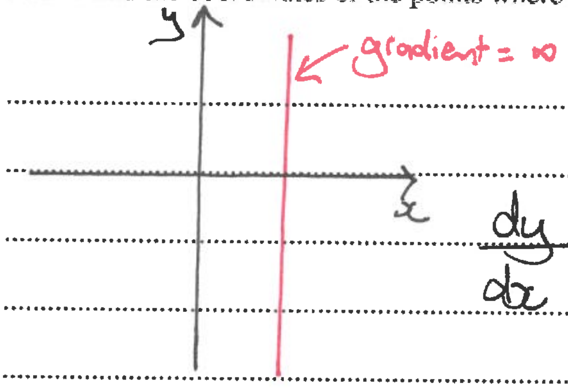
$x^2\dfrac{dy}{dx} - 2ay\dfrac{dy}{dx} = -2xy$

$\dfrac{dy}{dx}(x^2 - 2ay) = -2xy$

$\dfrac{dy}{dx} = \dfrac{-2xy}{x^2 - 2ay}$

$\dfrac{dy}{dx} = \dfrac{2xy}{2ay - x^2}$ QED

(b) Hence find the coordinates of the points where the tangent to the curve is parallel to the y-axis. [4]



$$\frac{dy}{dx} = \frac{2xy}{2ay - x^2} \leftarrow \text{denominator} = 0$$

$$2ay - x^2 = 0$$
$$x^2 = 2ay$$

Sub. into equation:

$$x^2y - ay^2 = 4a^3$$
$$(2ay)y - ay^2 = 4a^3$$
$$2ay^2 - ay^2 = 4a^3$$
$$ay^2 = 4a^3$$
$$y^2 = 4a^2$$
$$\underline{y = \pm 2a}$$

Sub into $x^2 = 2ay$:

$y = 2a$: $x^2 = 2a(+2a)$ $\qquad$ $y = -2a$: $x^2 = 2a(-2a)$

$x^2 = 4a$ $\qquad$ $x^2 = -4a$

$\underline{x = \pm 2a}$ $\qquad$ no solutions

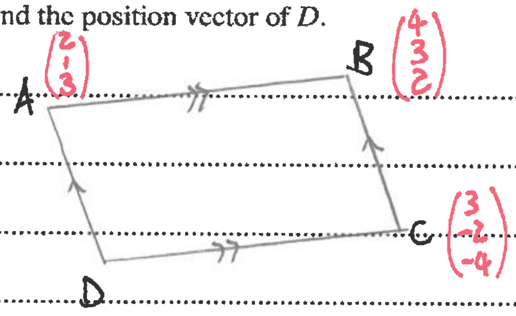
$\rightarrow (2a, 2a)$ and $(-2a, 2a)$

6 Relative to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}.$$

The quadrilateral $ABCD$ is a parallelogram.

(a) Find the position vector of D. [3]



$O^{\times}$

$$\overrightarrow{CD} = \overrightarrow{BA}$$
$$= \underline{a} - \underline{b}$$
$$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$$
$$= \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$
$$= \underline{\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}}$$

(b) The angle between BA and BC is θ.

Find the exact value of $\cos\theta$. [3]

$\vec{BA} = \begin{pmatrix}-2\\-2\\1\end{pmatrix}$ (from part (a))

$\vec{BC} = \underline{c} - \underline{b}$

$= \begin{pmatrix}3\\-2\\-4\end{pmatrix} - \begin{pmatrix}4\\3\\2\end{pmatrix}$

$= \begin{pmatrix}-1\\-5\\-6\end{pmatrix}$

$\vec{BA}\cdot\vec{BC} = |\vec{BA}||\vec{BC}|\cos\theta$

$\begin{pmatrix}-2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\-5\\-6\end{pmatrix} = \sqrt{(-2)^2+(-2)^2+1^2}\sqrt{(-1)^2+(-5)^2+(-6)^2}\cos\theta$

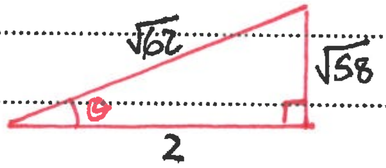
$2+10-6 = \sqrt{9}\sqrt{62}\cos\theta$

$6 = 3\sqrt{62}\cos\theta$

$\cos\theta = \dfrac{2}{\sqrt{62}}$

(c) Hence find the area of $ABCD$, giving your answer in the form $p\sqrt{q}$, where p and q are integers. [4]

Need to use $\frac{1}{2}ab\sin C$ formula, so need exact value of $\sin\theta$:



$= \sqrt{62-2^2}$ (pythag)

$\sin\theta = \dfrac{\sqrt{58}}{\sqrt{62}}$

Area of triangle ABC:

$\text{Area} = \frac{1}{2}|\vec{BA}||\vec{BC}|\sin\theta$

$= \frac{1}{2}\sqrt{9}\sqrt{62}\times\dfrac{\sqrt{58}}{\sqrt{62}}$

$= \frac{1}{2}\times 3\times\sqrt{58}$

Area of parallelogram = 2x area of triangle:

$\text{Area} = 2\times\frac{1}{2}\times 3\times\sqrt{58}$

$= \underline{3\sqrt{58}}$

7 The variables x and y satisfy the differential equation

$$\cos 2x \frac{dy}{dx} = \frac{4\tan 2x}{\sin^2 3y},$$

where $0 \leqslant x < \frac{1}{4}\pi$. It is given that $y = 0$ when $x = \frac{1}{6}\pi$.

Solve the differential equation to obtain the value of x when $y = \frac{1}{6}\pi$. Give your answer correct to 3 decimal places. [8]

$$\int \sin^2 3y \, dy = \int \frac{4\tan 2x}{\cos 2x} \, dx$$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 6y = 1 - 2\sin^2 3y$

$2\sin^2 3y = 1 - \cos 6y$

$\sin^2 3y = \frac{1}{2} - \frac{1}{2}\cos 6y$

$$\int \left(\frac{1}{2} - \frac{1}{2}\cos 6y\right) dy = \int 4\tan 2x \sec 2x \, dx$$

integral of $\sec x \tan x$ is $\sec x$ (formula book)

$$\int \left(\frac{1}{2} - \frac{1}{2}\cos 6y\right) dy = 2\sec 2x + C$$

$$\frac{1}{2}y - \frac{1}{12}\sin 6y = 2\sec 2x + C$$

$y = 0, x = \frac{\pi}{6}$:

$$0 - 0 = 2\sec\left(\frac{\pi}{3}\right) + C$$

$$0 = \frac{2}{\cos\left(\frac{\pi}{3}\right)} + C$$

$$0 = 4 + C$$

$$C = -4$$

$$\frac{1}{2}y - \frac{1}{12}\sin 6y = 2\sec 2x - 4$$

cont.

$y = \frac{\pi}{6}$:

$$\frac{1}{2}\left(\frac{\pi}{6}\right) - \frac{1}{12}\sin(\pi) = 2\sec 2x - 4$$

$$\frac{\pi}{12} - 0 = 2\sec 2x - 4$$

$$2\sec 2x = \frac{\pi}{12} + 4$$

$$\sec 2x = \frac{\pi}{24} + 2$$

$$\frac{1}{\cos 2x} = \frac{\pi}{24} + \frac{48}{24}$$

$$\frac{1}{\cos 2x} = \frac{\pi + 48}{24}$$

$$\cos 2x = \frac{24}{\pi + 48}$$

$$2x = 1.082$$

$$x = 0.541$$

8 Let $f(x) = \dfrac{3 - 3x^2}{(2x+1)(x+2)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{3-3x^2}{(2x+1)(x+2)^2} \equiv \frac{A}{2x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$3 - 3x^2 \equiv A(x+2)^2 + B(2x+1)(x+2) + C(2x+1)$$

$x=-2$:

$$3 - 3(-2)^2 = 0 + 0 + C(2(-2)+1)$$

$$3 - 12 = -3C$$

$$-9 = -3C$$

$$C = 3$$

$x = \frac{-1}{2}$:

$$3 - 3\left(\tfrac{-1}{2}\right)^2 = A\left(\tfrac{-1}{2}+2\right)^2 + 0 + 0$$

$$3 - \tfrac{3}{4} = A\left(\tfrac{3}{2}\right)^2$$

$$\tfrac{9}{4} = \tfrac{9}{4}A$$

$$A = 1$$

$x=0$:

$$3 - 0 = A(2)^2 + B(1)(2) + C(1)$$

$$3 = 4A + 2B + C$$

$$3 = 4 + 2B + 3$$

$$3 = 7 + 2B$$

$$2B = -4$$

$$B = -2$$

$$\rightarrow \frac{1}{2x+1} - \frac{2}{x+2} + \frac{3}{(x+2)^2}$$

8 Let $f(x) = \dfrac{3 - 3x^2}{(2x+1)(x+2)^2}$.

(b) Hence find the exact value of $\int_0^4 f(x)\,dx$, giving your answer in the form $a + b\ln c$, where a, b and c are integers. [5]

$$\int_0^4 \left(\frac{1}{2x+1} - \frac{2}{x+2} + \frac{3}{(x+2)^2} \right) dx$$

$$\frac{1}{2}\int_0^4 \frac{2}{2x+1}\,dx - 2\int_0^4 \frac{1}{x+2}\,dx + 3\int_0^4 (x+2)^{-2}\,dx$$

$$= \left[\frac{1}{2}\ln|2x+1| - 2\ln|x+2| - 3(x+2)^{-1} \right]_0^4$$

$$= \left[\frac{1}{2}\ln(9) - 2\ln(6) - 3(6)^{-1} \right] - \left[\frac{1}{2}\ln(1) - 2\ln(2) - 3(2)^{-1} \right]$$

$$= \frac{1}{2}\ln 9 - 2\ln 6 - \frac{3}{6} - 0 + 2\ln 2 + \frac{3}{2}$$

$$= \frac{1}{2}\ln 9 - 2\ln 6 + 2\ln 2 - \frac{1}{2} + \frac{3}{2}$$

$$= \ln 9^{\frac{1}{2}} - \ln 6^2 + \ln 2^2 + 1$$

$$= \ln 3 - \ln 36 + \ln 4 + 1$$

$$= \ln\left(\frac{3 \times 4}{36}\right) + 1$$

$$= \ln\left(\frac{1}{3}\right) + 1$$

$$= \ln 3^{-1} + 1$$

$$= -\ln 3 + 1$$

$$= 1 - \ln 3$$

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9 The constant a is such that $\int_0^a \overset{u}{x}\overset{dv}{e^{-2x}}\,dx = \frac{1}{8}$.

(a) Show that $a = \frac{1}{2}\ln(4a+2)$. [5]

$u = x \qquad v = -\frac{1}{2}e^{-2x}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = e^{-2x}$

$\rightarrow -\frac{1}{2}xe^{-2x} + \int_0^a \frac{1}{2}e^{-2x}\,dx$

$= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^a$

$= \left[\frac{-1}{2}(a)e^{-2a} - \frac{1}{4}e^{-2a}\right] - \left[0 - \frac{1}{4}e^{0}\right]$

$= \frac{-a}{2}e^{-2a} - \frac{1}{4}e^{-2a} + \frac{1}{4}$

integral $= \frac{1}{8}$:

$\frac{-a}{2}e^{-2a}{}_{\times 8} - \frac{1}{4}e^{-2a}{}_{\times 8} + \frac{1}{4}{}_{\times 8} = \frac{1}{8}{}_{\times 8}$

$-4ae^{-2a} - 2e^{-2a} + 2 = 1$

$-4ae^{-2a} - 2e^{-2a} + 1 = 0$

$4ae^{-2a} + 2e^{-2a} = 1$

$e^{-2a}(4a+2) = 1$

$e^{-2a} = \frac{1}{4a+2}$

$e^{2a} = 4a+2$

$2a = \ln(4a+2)$

$\underline{a = \frac{1}{2}\ln(4a+2)}$ QED

(b) Verify by calculation that a lies between 0.5 and 1. [2]

Set $= 0$:

$$a - \tfrac{1}{2}\ln(4a + 2) = 0$$

$a = 0.5$:

$$0.5 - \tfrac{1}{2}\ln(2 + 2) = -0.193$$

$a = 1$:

$$1 - \tfrac{1}{2}\ln(4 + 2) = 0.104$$

Sign change (and continuous function) between $a = 0.5$ and $a = 1$ shows there's a root between 0.5 and 1.

(c) Use an iterative formula based on the equation in **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \tfrac{1}{2}\ln(4a_n + 2)$$

$a_1 = 0.5000$
$a_2 = 0.6931$
$a_3 = 0.7814$
$a_4 = 0.8171$
$a_5 = 0.8309$
$a_6 = 0.8361$
$a_7 = 0.8380$
$a_8 = 0.8388$

a_6, a_7 and a_8 all round to 0.84 to 2dp, so $a = 0.84$

10 The polynomial $x^3 + 5x^2 + 31x + 75$ is denoted by $p(x)$.

(a) Show that $(x + 3)$ is a factor of $p(x)$. [2]

Sub. $x = -3$:

$(-3)^3 + 5(-3)^2 + 31(-3) + 75$

$-27 + 5 \times 9 - 93 + 75$

$-27 + 45 - 93 + 75$

$18 - 18 = 0$

So $(x+3)$ is a factor.

(b) Show that $z = -1 + 2\sqrt{6}\mathrm{i}$ is a root of $p(z) = 0$. [3]

Sub. $z = -1 + 2\sqrt{6}i$:

$(-1 + 2\sqrt{6}i)^3 + 5(-1 + 2\sqrt{6}i)^2 + 31(-1 + 2\sqrt{6}i) + 75$

$(-1 + 2\sqrt{6}i)(1 - 4\sqrt{6}i - 24) + 5(1 - 4\sqrt{6}i - 24) - 31 + 62\sqrt{6}i + 75$

$(-1 + 2\sqrt{6}i)(-23 - 4\sqrt{6}i) + 5(-23 - 4\sqrt{6}i) + 44 + 62\sqrt{6}i$

$23 + 4\sqrt{6}i - 46\sqrt{6}i + 48 - 115 - 20\sqrt{6}i + 44 + 62\sqrt{6}i$

$71 - 42\sqrt{6}i - 115 - 20\sqrt{6}i + 44 + 62\sqrt{6}i$

$-44 - 62\sqrt{6}i + 44 + 62\sqrt{6}i = 0$

So $z = -1 + 2\sqrt{6}i$ is a root. QED

(c) Hence find the complex numbers z which are roots of $p(z^2) = 0$. [7]

From parts (a) and (b), roots of $p(z)$ are ① $z = -3$, ② $z = -1 + 2\sqrt{6}i$ and ③ $z = -1 - 2\sqrt{6}i$ (because coefficients of p are real, so z^* is also a root).

Replace z with z^2 for each root:

① $z^2 = -3$

$z = \pm\sqrt{-3}$

$z = \sqrt{3}i$ or $z = -\sqrt{3}i$

② $z^2 = -1 + 2\sqrt{6}i$

$(x + iy)^2 = -1 + 2\sqrt{6}i$

$x^2 + 2ixy - y^2 = -1 + 2\sqrt{6}i$

Re: $x^2 - y^2 = -1$ ① Im: $2xy = 2\sqrt{6}$

$y = \frac{\sqrt{6}}{x}$ ②

sub. ② into ①: $x^2 - \left(\frac{\sqrt{6}}{x}\right)^2 = -1$

$x^2 - \frac{6}{x^2} = -1$

$x^4 - 6 = -x^2$

$x^4 + x^2 - 6 = 0$

$(x^2 + 3)(x^2 - 2) = 0$

$x^2 = -3$ $x^2 = 2$

no real sol'ns $x = \pm\sqrt{2}$

→②: $x = \sqrt{2}$: $y = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$

$x = -\sqrt{2}$: $y = \frac{\sqrt{6}}{-\sqrt{2}} = -\sqrt{3}$

$z = \sqrt{2} + \sqrt{3}i$ or $z = -\sqrt{2} - \sqrt{3}i$

continued

③ $z^2 = -1 - 2\sqrt{6}i$

$(x+iy)^2 = -1 - 2\sqrt{6}i$

$x^2 + 2ixy - y^2 = -1 - 2\sqrt{6}i$

Re: $x^2 - y^2 = -1$ ① Im: $2xy = -2\sqrt{6}$

$y = \frac{-\sqrt{6}}{x}$ ②

sub. ② into ①: $x^2 - \left(\frac{-\sqrt{6}}{x}\right)^2 = -1$

$x^2 - \frac{6}{x^2} = -1$

same quadratic as before

$\rightarrow x^2 = -3$ $\quad x^2 = 2$

no real sol^ns $\quad x = \pm\sqrt{2}$

→②: $x = \sqrt{2}$: $y = \frac{-\sqrt{6}}{\sqrt{2}} = -\sqrt{3}$

$x = -\sqrt{2}$: $y = \frac{-\sqrt{6}}{-\sqrt{2}} = \sqrt{3}$

$\underline{z = \sqrt{2} - \sqrt{3}i}$ or $\underline{z = -\sqrt{2} + \sqrt{3}i}$

All six answers:

$z_1 = \sqrt{3}i$, $z_2 = -\sqrt{3}i$, $z_3 = \sqrt{2} + \sqrt{3}i$, $z_4 = -\sqrt{2} - \sqrt{3}i$

$z_5 = \sqrt{2} - \sqrt{3}i$, $z_6 = -\sqrt{2} + \sqrt{3}i$