1 Find the exact value of $\int_3^\infty \frac{2}{x^2}\,dx$. [3]

$$\int_3^\infty 2x^{-2}\,dx = \left[-2x^{-1}\right]_3^\infty$$

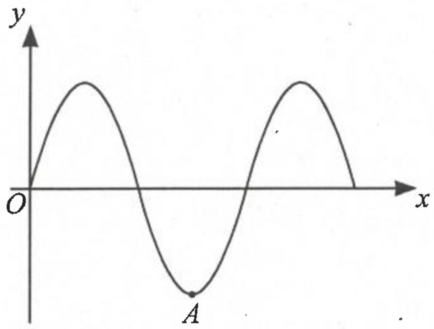
$$= \left[\frac{-2}{x}\right]_3^\infty$$

$$= \left[\frac{-2}{\infty}\right] - \left[\frac{-2}{3}\right]$$

$$= \left[0\right] - \left[\frac{-2}{3}\right]$$

$$= \frac{2}{3}$$

2



The diagram shows part of the curve with equation $y = k\sin\frac{1}{2}x$, where k is a positive constant and x is measured in radians. The curve has a minimum point A.

(a) State the coordinates of A. [1]

On a normal sine graph, the coordinates of A would be $\left(\frac{3\pi}{2}, -1\right)$. Given the transformation $kf(\frac{1}{2}x)$, the y-coords have been multiplied by k and the x-coords have been multiplied by 2, so: $(3\pi, -k)$

(b) A sequence of transformations is applied to the curve in the following order.

Translation of 2 units in the negative y-direction

Reflection in the x-axis

Find the equation of the new curve and determine the coordinates of the point on the new curve corresponding to A. [3]

$f(x) \xrightarrow{\binom{0}{-2}} f(x) - 2 \xrightarrow{\times -1} -f(x) + 2$

Equation: $k\sin(\frac{1}{2}x) \rightarrow k\sin(\frac{1}{2}x) - 2 \rightarrow \underline{-k\sin(\frac{1}{2}x) + 2}$

point A: $\left(\frac{3\pi}{2}, -1\right) \rightarrow \left(\frac{3\pi}{2}, -3\right) \rightarrow \underline{\left(\frac{3\pi}{2}, 3\right)}$

3 A curve is such that $\frac{dy}{dx} = 3(4x+5)^{\frac{1}{2}}$. It is given that the points $(1, 9)$ and $(5, a)$ lie on the curve.

Find the value of a. [5]

Integrate: $y = \frac{2}{3} \times \frac{1}{4} \times 3(4x+5)^{3/2} + C$

$= \frac{1}{2}(4x+5)^{3/2} + C$

Sub. $(1,9)$: $9 = \frac{1}{2}(4+5)^{3/2} + C$

$9 = \frac{1}{2}(9)^{3/2} + C$

$9 = \frac{27}{2} + C$

$C = -\frac{9}{2}$

$\rightarrow \quad y = \frac{1}{2}(4x+5)^{3/2} - \frac{9}{2}$

Sub. $(5,a)$: $a = \frac{1}{2}(20+5)^{3/2} - \frac{9}{2}$

$= \frac{1}{2}(25)^{3/2} - \frac{9}{2}$

$= \frac{125}{2} - \frac{9}{2}$

$= \frac{116}{2}$

$= \underline{58}$

4 (a) Prove that $\dfrac{(\sin\theta+\cos\theta)^2-1}{\cos^2\theta} \equiv 2\tan\theta$. [3]

LHS: $\sin^2\theta+\cos^2\theta=1$

$$\frac{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 1}{\cos^2\theta}$$

$$\frac{2\sin\theta\cos\theta + 1 - 1}{\cos^2\theta}$$

$$\frac{2\sin\theta\cos\theta}{\cos^2\theta} \quad \begin{matrix}\div\cos\theta\\ \div\cos\theta\end{matrix}$$

$$= \frac{2\sin\theta}{\cos\theta}$$

$$= 2\tan\theta$$

(b) Hence solve the equation $\dfrac{(\sin\theta+\cos\theta)^2-1}{\cos^2\theta} = 5\tan^3\theta$ for $-90° < \theta < 90°$. [3]

from (a):

$$2\tan\theta = 5\tan^3\theta$$

$$5\tan^3\theta - 2\tan\theta = 0$$

$$\tan\theta(5\tan^2\theta - 2) = 0$$

$\tan\theta = 0$ or $5\tan^2\theta - 2 = 0$

$\theta = 0$ $\qquad \tan^2\theta = \frac{2}{5}$

$\tan\theta = \sqrt{\frac{2}{5}}$ or $\tan\theta = -\sqrt{\frac{2}{5}}$

$\theta = 32.3°$ $\qquad \theta = -32.3°$

No other solutions in given range, so no need for CAST diagram.

$$\theta = -32.3°, 0°, 32.3°$$

5 A curve has the equation $y = \dfrac{3}{2x^2 - 5}$.

Find the equation of the normal to the curve at the point $(2, 1)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. [6]

$y = 3(2x^2 - 5)^{-1}$

Chain rule:

$y = 3t^{-1}$ $\quad t = 2x^2 - 5$

$\dfrac{dy}{dt} = -3t^{-2}$ $\quad \dfrac{dt}{dx} = 4x$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$= -3t^{-2} \times 4x$

$= -3(2x^2 - 5)^{-2} \times 4x$

gradient at $(2,1)$:

$m = -3(2(2)^2 - 5)^{-2} \times 4(2)$

$= -3(3)^{-2} \times 8$

$= \dfrac{-3}{9} \times 8$

$= -8/3$

gradient of normal:

$m_{norm} = 3/8$

$y - y_1 = m(x - x_1)$

$y - 1 = 3/8(x - 2)$ $\quad \times 8$

$8y - 8 = 3(x - 2)$

$8y - 8 = 3x - 6$

$3x - 8y + 2 = 0$

6 It is given that the coefficient of x^3 in the expansion of

$$(2+ax)^4(5-ax)$$

is 432.

Find the value of the constant a. [5]

$(2+ax)^4 = 2^4 + 4\times2^3\times(ax) + 6\times2^2(ax)^2 + 4\times2(ax)^3 + \dots$

$= 16 + 32ax + 24a^2x^2 + 8a^3x^3 + \dots$

$(2+ax)^4(5-ax)$:

$(16 + 32ax + 24a^2x^2 + 8a^3x^3)(5-ax)$

x^3 term:

$-24a^3x^3 + 40a^3x^3$

$= 16a^3x^3$

coeff of x^3: $\underline{16a^3}$

$16a^3 = 432$

$a^3 = 27$

$\underline{a = 3}$

7 The straight line $y = x + 5$ meets the curve $2x^2 + 3y^2 = k$ at a single point P.

(a) Find the value of the constant k. [4]

Solve simultaneously:

$$2x^2 + 3(x+5)^2 = k$$

$$2x^2 + 3(x^2 + 10x + 25) = k$$

$$2x^2 + 3x^2 + 30x + 75 = k$$

$$5x^2 + 30x + 75 - k = 0$$

They meet at a single point, so $b^2 - 4ac = 0$:

$$30^2 - 4(5)(75 - k) = 0$$

$$900 - 1500 + 20k = 0$$

$$-600 + 20k = 0$$

$$20k = 600$$

$$k = 30$$

(b) Find the coordinates of P. [2]

Sub. $k = 30$: $2x^2 + 3y^2 = 30$

Solve simultaneously:

$$2x^2 + 3(x+5)^2 = 30$$

$$2x^2 + 3(x^2 + 10x + 25) = 30$$

$$2x^2 + 3x^2 + 30x + 75 = 30$$

$$5x^2 + 30x + 45 = 0 \quad \div 5$$

$$x^2 + 6x + 9 = 0$$

$$(x + 3)^2 = 0$$

$$x = -3$$

Sub. into $y = x + 5$: $y = -3 + 5$

$= 2$

$(-3, 2)$

8 (a) An arithmetic progression is such that its first term is 6 and its tenth term is 19.5 .

Find the sum of the first 100 terms of this arithmetic progression. [4]

u_1: $a = 6$

u_{10}: $a + 9d = 19.5$

$6 + 9d = 19.5$

$9d = 13.5$

$d = 1.5$

S_{100}:

$S_{100} = \frac{100}{2}[2(6) + (100-1)(1.5)]$

$= 50[12 + 99 \times 1.5]$

$= 8025$

(b) A geometric progression $a_1, a_2, a_3, \ldots$ is such that $a_1 = 24$ and the common ratio is $\frac{1}{2}$.

The sum to infinity of this geometric progression is denoted by S. The sum to infinity of the even-numbered terms (i.e. $a_2, a_4, a_6, \ldots$) is denoted by S_E.

Find the values of S and S_E. [4]

All terms: $S = \frac{a}{1-r}$

(progression: $24, 12, 6, 3, 1.5, 0.75 \ldots$)

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

$= \frac{24}{1-\frac{1}{2}}$

$= 48$

Even numbered terms: $12 \xrightarrow{\times\frac{1}{4}} 3 \xrightarrow{\times\frac{1}{4}} 0.75, \ldots$

so $a = 12$ and $r = \frac{1}{4}$

$S_E = \frac{12}{1-\frac{1}{4}}$

$= \frac{12}{3/4}$

$= 16$

9 The functions f and g are defined for all real values of x by

$$f(x) = (3x-2)^2 + k \quad \text{and} \quad g(x) = 5x - 1,$$

where k is a constant.

(a) Given that the range of the function gf is $gf(x) \geqslant 39$, find the value of k. [4]

If the range is $gf(x) \geqslant 39$, then the minimum point must be at $y = 39$. We need to find $gf(x)$ and the minimum point:

$$gf(x) = 5[f(x)] - 1$$
$$= 5[(3x-2)^2 + k] - 1$$
$$= 5(3x-2)^2 + 5k - 1$$

Minimum point is when $3x - 2 = 0$:

min. pt: $gf(x) = 5(0)^2 + 5k - 1$
$= 5k - 1$

Min-pt = 39: $5k - 1 = 39$
$5k = 40$
$k = 8$

(b) For this value of k, determine the range of the function fg. [2]

$$fg(x) = [3(5x-1) - 2]^2 + 8$$
$$= (15x - 3 - 5)^2 + 8$$
$$= (15x - 5)^2 + 8$$

Minimum point is when this = 0, at which point $fg(x) = 8$

so $fg(x) \geqslant 8$

(c) The function h is defined for all real values of x and is such that $gh(x) = 35x + 19$.

Find an expression for $g^{-1}(x)$ and hence, or otherwise, find an expression for $h(x)$. [3]

$g^{-1}(x)$: $y = 5x - 1$

$y + 1 = 5x$

$x = \frac{y+1}{5}$

$g^{-1}(x) = \frac{x+1}{5}$

$gh(x) = 35x + 19$

To get rid of g, we can't divide by g, we have to do g^{-1} on both sides:

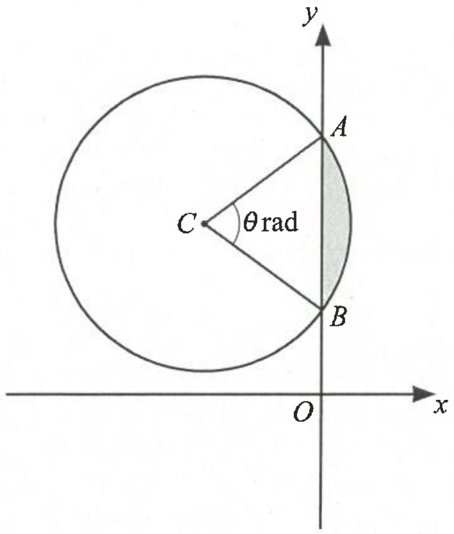
$g^{-1}(gh(x)) = g^{-1}(35x + 19)$

cancel out $\cancel{g^{-1}g}\,h(x) = \frac{(35x+19)+1}{5}$

$h(x) = \frac{35x + 20}{5}$

$h(x) = 7x + 4$

10



The diagram shows the circle with centre $C(-4, 5)$ and radius $\sqrt{20}$ units. The circle intersects the y-axis at the points A and B. The size of angle ACB is θ radians.

(a) Find the equation of the tangent to the circle at the point $(-6, 9)$. [3]

Gradient of radius $= \dfrac{9-5}{-6--4} = \dfrac{4}{-2} = \underline{-2}$

Gradient of tangent $= \underline{\frac{1}{2}}$

$y - y_1 = m(x - x_1)$

$y - 9 = \frac{1}{2}(x - -6)$

$y - 9 = \frac{1}{2}x + 3$

$\underline{y = \frac{1}{2}x + 12}$

(b) Find the equation of the circle in the form $x^2 + y^2 + ax + by + c = 0$. [2]

$C: (-4, 5)$, $r = \sqrt{20}$:

$(x+4)^2 + (y-5)^2 = (\sqrt{20})^2$

$x^2 + 8x + 16 + y^2 - 10y + 25 = 20$

$x^2 + y^2 + 8x - 10y + 41 = 20$

$\underline{x^2 + y^2 + 8x - 10y + 21 = 0}$

(c) Find the value of θ correct to 4 significant figures. [3]

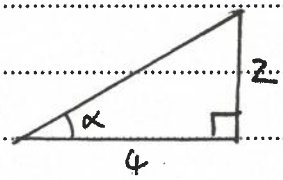
Find A and B by substituting $x = 0$:

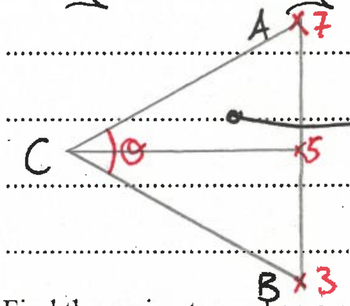
$0^2 + y^2 + 0 - 10y + 21 = 0$

$y^2 - 10y + 21 = 0$

$(y-3)(y-7) = 0$

$y = 3$ or $y = 7$





$\tan\alpha = \frac{2}{4}$

$\alpha = 0.4636...$

$\theta = 2\alpha = \underline{0.9273}$ STO

(d) Find the perimeter and area of the segment shaded in the diagram. [4]

Perimeter:

Arc length $= r\theta$

$= \sqrt{20} \times 0.9273$

Perimeter $= 4 + \sqrt{20} \times 0.9273$

$= \underline{8.15}$

Area:

Area of sector ACB $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}(\sqrt{20})^2 \times 0.9273$

$= \underline{9.27295....}$

Area of triangle ACB $= \frac{1}{2} \times r \times r \times \sin\theta$

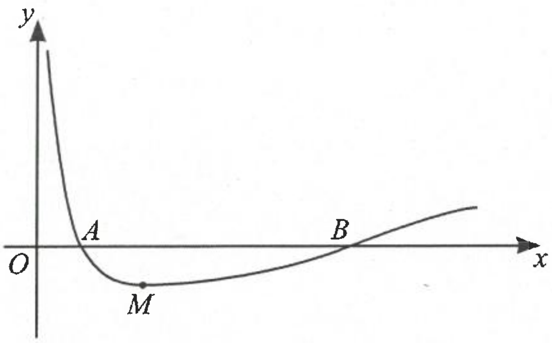
$= \frac{1}{2}(\sqrt{20})^2 \times \sin(0.9273)$

$= \underline{8}$

Shaded Area $= 9.27295 - 8$

$= \underline{1.27}$

11



The diagram shows the curve with equation $y = 2x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} + 1$ for $x > 0$. The curve crosses the x-axis at points A and B and has a minimum point M.

(a) Find the exact coordinates of M. [4]

$$\frac{dy}{dx} = -\frac{2}{3} \times 2x^{-\frac{5}{3}} - \frac{-1}{3} \times 3x^{-\frac{4}{3}}$$

$$= \frac{-4}{3}x^{-\frac{5}{3}} + x^{-\frac{4}{3}}$$

SP so $\frac{dy}{dx} = 0$:

$$\frac{-4}{3}x^{-5/3} + x^{-4/3} = 0$$

factorise: $x^{-\frac{4}{3}}\left(\frac{-4}{3}x^{-\frac{1}{3}} + 1\right) = 0$

$x^{-\frac{4}{3}} = 0$ or $\frac{-4}{3}x^{-\frac{1}{3}} + 1 = 0$

No sol^s

$$\frac{4}{3\sqrt[3]{x}} = 1$$

$$\frac{4}{3} = \sqrt[3]{x}$$

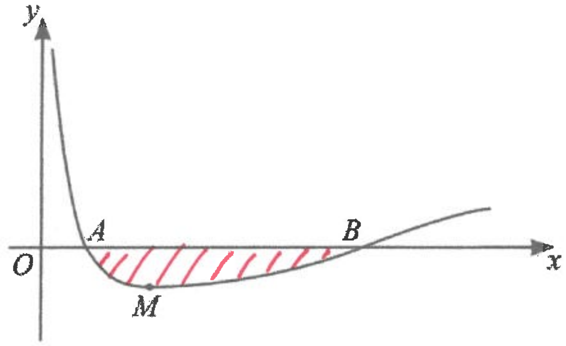
$$x = \left(\frac{4}{3}\right)^3$$

$$= \frac{64}{27}$$

Sub. $x = \frac{64}{27}$: $y = 2\left(\frac{64}{27}\right)^{-\frac{2}{3}} - 3\left(\frac{64}{27}\right)^{-\frac{1}{3}} + 1$

$$y = 2\left(\frac{9}{16}\right) - 3\left(\frac{3}{4}\right) + 1$$

$= \frac{-1}{8}$ $\rightarrow$ $\left(\frac{64}{27}, \frac{-1}{8}\right)$



The diagram shows the curve with equation $y = 2x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} + 1$ for $x > 0$. The curve crosses the x-axis at points A and B and has a minimum point M.

(b) Find the area of the region bounded by the curve and the line segment AB. [7]

Find A and B: $0 = 2x^{-2/3} - 3x^{-1/3} + 1$

Let $Y = x^{-\frac{1}{3}}$: $2Y^2 - 3Y + 1 = 0$

$(2Y - 1)(Y - 1) = 0$

$Y = \frac{1}{2}$ or $Y = 1$

$x^{-\frac{1}{3}} = \frac{1}{2}$ $\quad$ $x^{-\frac{1}{3}} = 1$

$x = 8$ $\quad$ $x = 1$

$\int_1^8 (2x^{-2/3} - 3x^{-1/3} + 1)\,dx$

$= \left[3 \times 2x^{\frac{1}{3}} - \frac{3}{2} \times 3x^{\frac{2}{3}} + x\right]_1^8$

$= \left[6x^{\frac{1}{3}} - \frac{9}{2}x^{\frac{2}{3}} + x\right]_1^8$

$= \left[6(8)^{\frac{1}{3}} - \frac{9}{2}(8)^{\frac{2}{3}} + 8\right] - \left[6(1)^{\frac{1}{3}} - \frac{9}{2}(1)^{\frac{2}{3}} + 1\right]$

$= \left[6 \times 2 - \frac{9}{2} \times 4 + 8\right] - \left[6 - \frac{9}{2} + 1\right]$

$= [2] - \left[\frac{5}{2}\right] = -0.5$

Area can't be negative so $= 0.5$