1 It is given that $x = \ln(2y - 3) - \ln(y + 4)$.

Express y in terms of x. [3]

$$x = \ln\left(\frac{2y-3}{y+4}\right)$$

$$e^x = \frac{2y-3}{y+4}$$

$$(y+4)e^x = 2y - 3$$

$$ye^x + 4e^x = 2y - 3$$

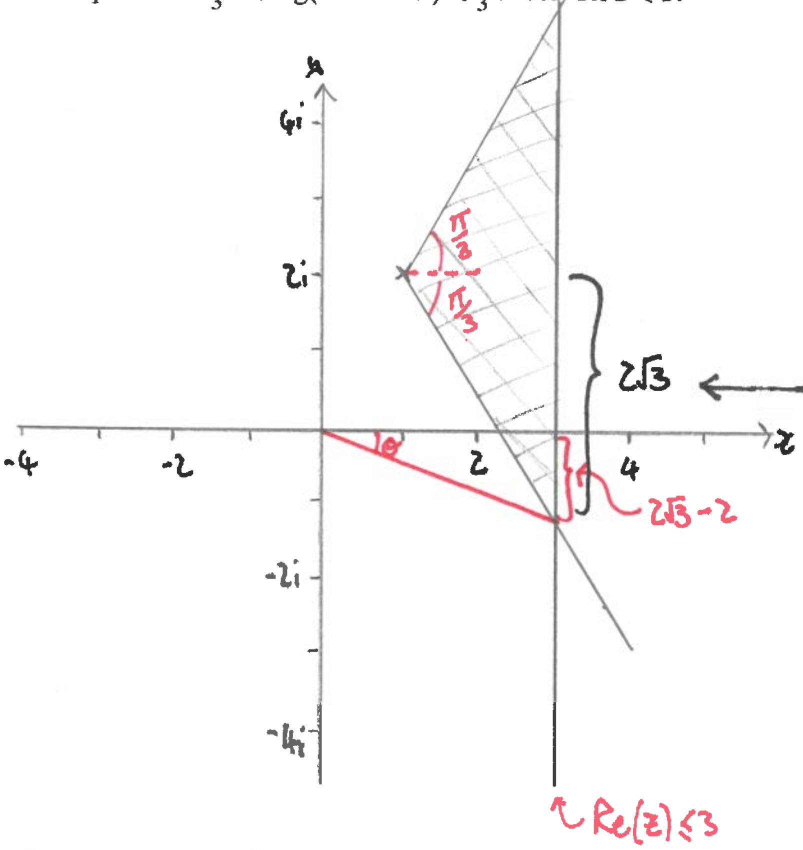
$$ye^x - 2y = -4e^x - 3$$

$$y(e^x - 2) = -4e^x - 3$$

$$y = \frac{-4e^x - 3}{e^x - 2}$$

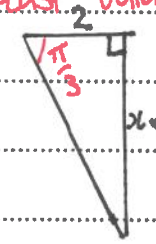
or $$y = \frac{4e^x + 3}{2 - e^x}$$

2 (a) On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leqslant \arg(z - 1 - 2i) \leqslant \frac{1}{3}\pi$ and $\operatorname{Re} z \leqslant 3$. [3]



(b) Calculate the least value of $\arg z$ for points in the region from (a). Give your answer in radians correct to 3 decimal places. [2]

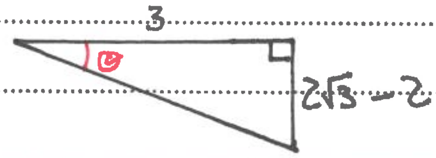
Least value of arg(z) is lowest point in region (θ).



$\tan\left(\frac{\pi}{3}\right) = \frac{x}{2}$

$x = 2\tan\left(\frac{\pi}{3}\right)$

$= 2\sqrt{3}$



$\tan\theta = \frac{2\sqrt{3} - 2}{3}$

$\theta = 0.454$

but below x-axis, so:

$\theta = -0.454$

3 The polynomial $2x^4 + ax^3 + bx - 1$, where a and b are constants, is denoted by $\mathrm{p}(x)$. When $\mathrm{p}(x)$ is divided by $x^2 - x + 1$ the remainder is $3x + 2$.

Find the values of a and b. [5]

$$
\begin{array}{r|llllll}
 & 2x^2 & + (a+2)x & + a & & & \\
\hline
x^2 - x + 1 & 2x^4 & + ax^3 & + 0x^2 & + bx & - 1 \\
 & 2x^4 & - 2x^3 & + 2x^2 & \downarrow & \\
\hline
 & & (a+2)x^3 & - 2x^2 & + bx & \\
 & & (a+2)x^3 & - (a+2)x^2 & + (a+2)x & \\
\hline
 & & & ax^2 & + (b-a-2)x & - 1 \\
 & & & ax^2 & - ax & + a \\
\hline
 & & & & (b-2)x & -1-a
\end{array}
$$

$\rightarrow (b-2)x - 1 - a = 3x + 2$

Compare coeffs: $b - 2 = 3$ $\quad -1 - a = 2$

$\underline{b = 5}$ $\quad -1 = 2 + a$

$\underline{a = -3}$

Alternative Method:

$$\frac{2x^4 + ax^3 + bx - 1}{x^2 - x + 1} \equiv \alpha x^2 + \beta x + \gamma + \frac{3x+2}{x^2 - x + 1}$$

$2x^4 + ax^3 + bx - 1 \equiv (\alpha x^2 + \beta x + \gamma)(x^2 - x + 1) + 3x + 2$

$2x^4 + ax^3 + bx - 1 \equiv \alpha x^4 - \alpha x^3 + \alpha x^2 + \beta x^3 - \beta x^2 + \beta x$
$\quad + \gamma x^2 - \gamma x + \gamma + 3x + 2$

$\equiv \alpha x^4 + (\beta - \alpha)x^3 + (\alpha - \beta + \gamma)x^2 + (3 + \beta - \gamma)x + \gamma + 2$

Compare coeffs:

x^4: $\underline{2 = \alpha}$ $\quad$ const: $-1 = \gamma + 2$ $\quad$ x^2: $0 = \alpha - \beta + \gamma$

$\underline{\gamma = -3}$ $\quad 0 = 2 - \beta - 3$

$\underline{\beta = -1}$

x^3: $a = \beta - \alpha$ $\quad$ x: $b = 3 + \beta - \gamma$

$= -1 - 2$ $\quad = 3 + -1 - -3$

$\underline{a = -3}$ $\quad = 3 - 1 + 3$

$\rightarrow \underline{b = 5}$

4 Solve the equation

$$\frac{5z}{1+2i} - zz^* + 30 + 10i = 0,$$

giving your answers in the form $x + iy$, where x and y are real. [5]

Substitute $z = x + iy$ and $z^* = x - iy$:

$$\frac{5(x+iy)}{1+2i} - (x+iy)(x-iy) + 30 + 10i = 0$$

$$\frac{5x + 5iy}{1+2i} \times \frac{(1-2i)}{(1-2i)} - (x^2 + y^2) + 30 + 10i = 0$$

$$\frac{(5x+5iy)(1-2i)}{(1+2i)(1-2i)} - x^2 - y^2 + 30 + 10i = 0$$

$$\frac{5x - 10ix + 5iy + 10y}{1 - \cancel{2i} + \cancel{2i} + 4} - x^2 - y^2 + 30 + 10i = 0$$

$$\frac{5x + 10y - 10ix + 5iy}{5} - x^2 - y^2 + 30 + 10i = 0$$

$$x + 2y - 2ix + iy - x^2 - y^2 + 30 + 10i = 0$$

$$x + 2y - x^2 - y^2 + 30 + (-2x + y + 10)i = 0$$

Re: $x + 2y - x^2 - y^2 + 30 = 0$ (1) Im: $-2x + y + 10 = 0$

$y = 2x - 10$ (2)

Sub. (2) into (1):

$$x + 2(2x-10) - x^2 - (2x-10)^2 + 30 = 0$$

$$x + 4x - 20 - x^2 - (4x^2 - 40x + 100) + 30 = 0$$

$$5x + 10 - x^2 - 4x^2 + 40x - 100 = 0$$

$$-5x^2 + 45x - 90 = 0$$

$$x^2 - 9x + 18 = 0$$

$$(x-6)(x-3) = 0$$

$x = 6$ or $x = 3$

(2): $x = 6$: $y = 12 - 10 = 2$

$\rightarrow$ $z = 6 + 2i$

$x = 3$: $y = 6 - 10 = -4$

$\rightarrow$ $z = 3 - 4i$

5 The parametric equations of a curve are

$$x = te^{2t}, \qquad y = t^2 + t + 3.$$

(a) Show that $\frac{dy}{dx} = e^{-2t}$. [3]

$x = \overset{u}{t}\,\overset{v}{e^{2t}}$ $\qquad y = t^2 + t + 3$

$u = t \qquad v = e^{2t}$

$\frac{du}{dt} = 1 \qquad \frac{dv}{dt} = 2e^{2t} \qquad \frac{dy}{dt} = 2t + 1$

$\frac{dx}{dt} = 2te^{2t} + e^{2t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= (2t+1) \times \frac{1}{2te^{2t} + e^{2t}}$

$= \frac{2t+1}{2te^{2t} + e^{2t}}$

$= \frac{\cancel{2t+1}}{e^{2t}(\cancel{2t+1})}$

$= \frac{1}{e^{2t}}$

$= \underline{e^{-2t}}$ QED

(b) Hence show that the normal to the curve, where $t = -1$, passes through the point $\left(0,\ 3 - \frac{1}{e^4}\right)$. [3]

gradient at $t=-1$:

$m = e^{-2(-1)}$

$= e^2$

gradient of normal:

$m_{norm} = \frac{-1}{e^2}$

Coordinates when $t=-1$:

$x = -1 \times e^{-2}$ $\quad$ $y = (-1)^2 + -1 + 3$

$= -e^{-2}$ $\quad$ $= 1 - 1 + 3$

$= \frac{-1}{e^2}$ $\quad$ $= 3$

eqn of normal:

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{-1}{e^2}\left(x - -\frac{1}{e^2}\right)$

$y - 3 = \frac{-x}{e^2} - \frac{1}{e^4}$

Sub. $x=0$:

$y - 3 = 0 - \frac{1}{e^4}$

$y = 3 - \frac{1}{e^4}$

So it goes through $\left(0, 3 - \frac{1}{e^4}\right)$ QED

6 **(a)** Express $5 \sin \theta + 12 \cos \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. [3]

$5\sin\theta + 12\cos\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$12\cos\theta + 5\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\cos\alpha = 12$ ① $\quad R\sin\alpha = 5$ ②

② ÷ ①: $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{5}{12}$

$\tan\alpha = \dfrac{5}{12}$

$\alpha = 0.395$ STO

①^2 + ②^2: $R^2 = 12^2 + 5^2$

$R^2 = 144 + 25$

$R^2 = 169$

$R = 13$

→ $13\cos(\theta - 0.395)$

(b) Hence solve the equation $5\sin 2x + 12\cos 2x = 6$ for $0 \leqslant x \leqslant \pi$. **[4]**

from (a): $5\sin\theta + 12\cos\theta = 13\cos(\theta - 0.395)$

replace θ with $2x$:

$$5\sin 2x + 12\cos 2x = 13\cos(2x - 0.395)$$

$\rightarrow \quad 13\cos(2x - 0.395) = 6$

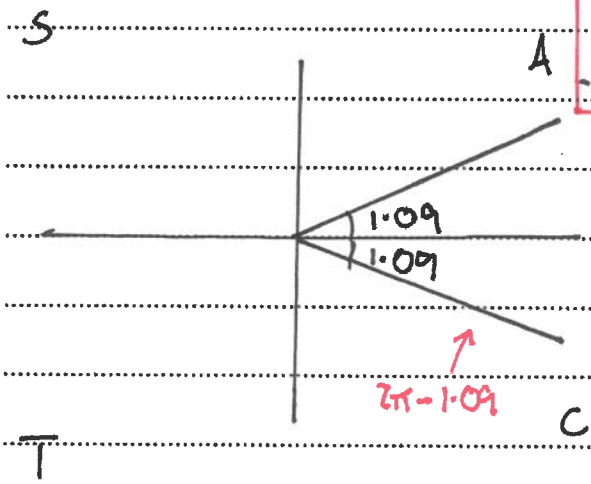
$$\cos(2x - 0.395) = \frac{6}{13}$$

$$2x - 0.395 = 1.09 \text{ STO}$$

$$0 \leqslant x \leqslant \pi$$

$$0 \leqslant 2x \leqslant 2\pi$$

$$-0.395 \leqslant 2x - 0.395 \leqslant 2\pi - 0.395$$

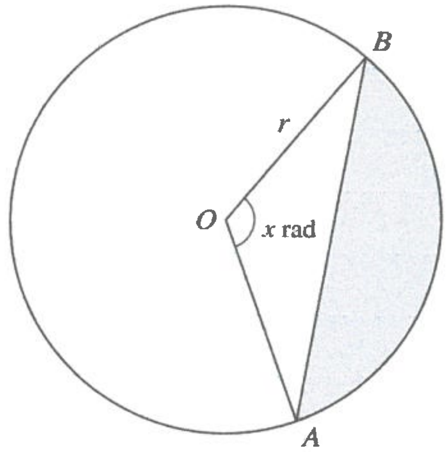


$$2x - 0.395 = 1.09,\ 5.19$$

$$2x = 1.486,\ 5.587$$

$$\underline{x = 0.743,\ 2.79}$$

7



The diagram shows a circle with centre O and radius r. The angle of the **minor** sector AOB of the circle is x radians. The area of the **major** sector of the circle is 3 times the area of the shaded region.

(a) Show that $x = \frac{3}{4}\sin x + \frac{1}{2}\pi$. **[4]**

Area of shaded region $= \lhd - \lhd$

$= \frac{1}{2}r^2\theta - \frac{1}{2}ab\sin C$

$= \frac{1}{2}r^2 x - \frac{1}{2}r^2\sin x$

Area of major sector:

Angle of major sector $= 2\pi - x$

Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}r^2(2\pi - x)$

Area of major sector $= 3 \times$ Shaded region

$\frac{1}{2}\cancel{r^2}(2\pi - x) = 3 \times \left(\frac{1}{2}\cancel{r^2}x - \frac{1}{2}\cancel{r^2}\sin x\right)$

$\frac{1}{2}(2\pi - x) = \frac{3}{2}x - \frac{3}{2}\sin x$

$2\pi - x = 3x - 3\sin x$

$2\pi + 3\sin x = 4x$

$4x = 2\pi + 3\sin x$

$x = \frac{\pi}{2} + \frac{3}{4}\sin x$

QED

(b) Show by calculation that the root of the equation in **(a)** lies between 2 and 2.5. [2]

set $=0$: $\frac{3}{4}\sin x + \frac{\pi}{2} - x = 0$

$x=2$: $\frac{3}{4}\sin 2 + \frac{\pi}{2} - 2 = 0.253$

$x=2.5$: $\frac{3}{4}\sin 2.5 + \frac{\pi}{2} - 2.5 = -0.480$

Sign change (and continuous function) between $x=2$ and $x=2.5$ shows there's a root between 2 and 2.5

(c) Use an iterative formula based on the equation in **(a)** to calculate this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_{n+1} = \frac{3}{4}\sin x_n + \frac{\pi}{2}$

$x_1 = 2.0000$

$x_2 = 2.2528$ x_7, x_8 and x_9 all

$x_3 = 2.1530$ round to 2.18 to

$x_4 = 2.1972$ 2dp, so $\underline{x = 2.18}$

$x_5 = 2.1784$

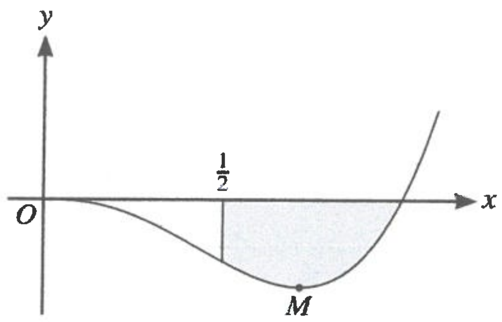
$x_6 = 2.1866$

$x_7 = 2.1830$

$x_8 = 2.1846$

$x_9 = 2.1839$

8



u v

The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M.

(a) Find the exact coordinates of M. [4]

$u = x^3$ $\quad$ $v = \ln x$

$\frac{du}{dx} = 3x^2$ $\quad$ $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = x^3 \times \frac{1}{x} + 3x^2 \ln x$

$= x^2 + 3x^2 \ln x$

SP, so $\frac{dy}{dx} = 0$:

$x^2 + 3x^2 \ln x = 0$

$x^2(1 + 3\ln x) = 0$

$x^2 = 0$ or $1 + 3\ln x = 0$

$x = 0$ $\quad$ $\ln x = -\frac{1}{3}$

not the point we want $\quad$ $\underline{x = e^{-\frac{1}{3}}}$

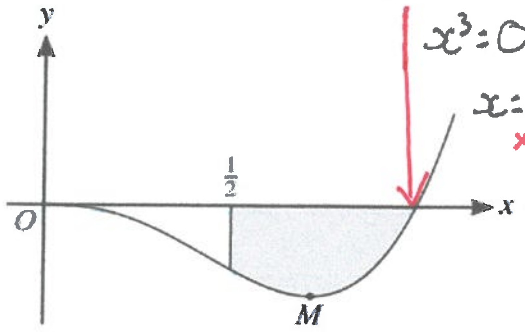
Sub. into eqn to find y:

$y = (e^{-\frac{1}{3}})^3 \times \ln(e^{-\frac{1}{3}})$

$= e^{-1} \times -\frac{1}{3}$

$= \underline{-\frac{1}{3}e^{-1}}$

$\rightarrow \underline{\left(e^{-\frac{1}{3}}, -\frac{1}{3}e^{-1}\right)}$



$0 = x^3 \ln x$

$x^3 = 0$ or $\ln x = 0$

$x = 0$ or $x = e^0$

×

$x = 1$ ✓

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E

The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M.

(b) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = \frac{1}{2}$. [5]

$u = \ln x \qquad v = \frac{1}{4}x^4$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^3$

$= \frac{1}{4}x^4 \ln x - \int_{\frac{1}{2}}^{1} \frac{1}{4}x^4 \times \frac{1}{x}\, dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int_{\frac{1}{2}}^{1} x^3\, dx$

$= \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_{\frac{1}{2}}^{1}$

$= \left[\frac{1}{4}(1)^4 \ln(1) - \frac{1}{16}(1)^4\right] - \left[\frac{1}{4}\left(\frac{1}{2}\right)^4 \ln\left(\frac{1}{2}\right) - \frac{1}{16}\left(\frac{1}{2}\right)^4\right]$

$= \left[0 - \frac{1}{16}\right] - \left[\frac{1}{64}\ln\left(\frac{1}{2}\right) - \frac{1}{256}\right]$

$= -\frac{1}{16} - \frac{1}{64}\ln\left(\frac{1}{2}\right) + \frac{1}{256}$

$= \frac{-15}{256} - \frac{1}{64}\ln 2^{-1}$

$= \underbrace{\frac{-15}{256} + \frac{1}{64}\ln 2}$ this is negative because it is below the x-axis. We want the area, which is positive, so multiply everything by -1:

$\underline{\frac{15}{256} - \frac{1}{64}\ln 2}$

9 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{3y} \sin^2 2x.$$

It is given that $y = 0$ when $x = 0$.

Solve the differential equation and find the value of y when $x = \frac{1}{2}$. [7]

$$\int \frac{1}{e^{3y}}\, dy = \int \sin^2 2x \, dx$$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 4x = 1 - 2\sin^2 2x$

$2\sin^2 2x = 1 - \cos 4x$

$\sin^2 2x = \frac{1}{2} - \frac{1}{2}\cos 4x$

$$\int e^{-3y}\, dy = \int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$-\frac{1}{3}e^{-3y} = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$$

$x=0,\ y=0$:

$$-\frac{1}{3}e^{0} = 0 - 0 + C$$

$$-\frac{1}{3} = C$$

$$\rightarrow \quad -\frac{1}{3}e^{-3y} = \frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{3}$$

$x = \frac{1}{2}$:

$$-\frac{1}{3}e^{-3y} = \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{8}\sin(2) - \frac{1}{3}$$

$$-\frac{1}{3}e^{-3y} = \frac{1}{4} - \frac{1}{3} - \frac{1}{8}\sin(2)$$

$$-\frac{1}{3}e^{-3y} = -\frac{1}{12} - \frac{1}{8}\sin(2)$$

$$e^{-3y} = \frac{1}{4} + \frac{3}{8}\sin(2)$$

$$-3y = \ln\left(\frac{1}{4} + \frac{3}{8}\sin(2)\right)$$

$$y = -\frac{1}{3}\ln\left(\frac{1}{4} + \frac{3}{8}\sin(2)\right)$$

$$y = 0.175$$

10 With respect to the origin O, the points A, B, C and D have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix}.$$

(a) Find the obtuse angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [3]

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = |\overrightarrow{OA}||\overrightarrow{OB}| \cos\theta$$

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \sqrt{3^2 + (-1)^2 + 2^2}\,\sqrt{1^2 + 2^2 + (-3)^2}\,\cos\theta$$

$$3 - 2 - 6 = \sqrt{14}\,\sqrt{14}\,\cos\theta$$

$$-5 = 14\cos\theta$$

$$\cos\theta = \frac{-5}{14}$$

$$\underline{\theta = 110.9^\circ}$$

The line l passes through the points A and B.

(b) Find a vector equation for the line l. [2]

$$\overrightarrow{AB} = \underline{b} - \underline{a}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$$

point on line (A) direction vector $\overrightarrow{AB}$

other solutions are possible

(c) Find the position vector of the point of intersection of the line l and the line passing through C and D. [4]

line through C and D:

$$\vec{CD} = \underline{d} - \underline{c}$$

$$= \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$$

Find where lines intersect:

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$$

$\underline{i}$: $3 - 2\lambda = 5 + 4\mu$

$-2 - 2\lambda = 4\mu$

$4\mu + 2\lambda = -2$ ①

$\underline{j}$: $-1 + 3\lambda = -6 - 4\mu$

$3\lambda = -5 - 4\mu$

$4\mu + 3\lambda = -5$ ②

② − ①: $\lambda = -3$

Sub. into eqn of line:

$$\underline{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + -3 \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ -9 \\ 15 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ -10 \\ 17 \end{pmatrix}$$

11 Let $f(x) = \dfrac{5x^2 + x + 11}{(4 + x^2)(1 + x)}$.

(a) Express $f(x)$ in partial fractions. **[5]**

$$\frac{5x^2 + x + 11}{(4+x^2)(1+x)} \equiv \frac{Ax + B}{4 + x^2} + \frac{C}{1 + x}$$

$$5x^2 + x + 11 \equiv (Ax + B)(1 + x) + C(4 + x^2)$$

$x = -1$:

$$5(-1)^2 + -1 + 11 = 0 + C(4 + (-1)^2)$$

$$5 - 1 + 11 = 5C$$

$$15 = 5C$$

$$C = 3$$

$x = 0$:

$$0 + 0 + 11 = (0 + B)(1 + 0) + C(4 + 0)$$

$$11 = B + 4C$$

$$11 = B + 12$$

$$B = -1$$

$x = 1$:

$$5(1)^2 + 1 + 11 = (A + B)(2) + C(4 + 1)$$

$$17 = 2A + 2B + 5C$$

$$17 = 2A - 2 + 15$$

$$17 = 2A + 13$$

$$4 = 2A$$

$$A = 2$$

$$\rightarrow \frac{2x - 1}{4 + x^2} + \frac{3}{1 + x}$$

(b) Hence show that $\int_0^2 f(x)\,dx = \ln 54 - \frac{1}{8}\pi$. [5]

$$\int_0^2 \left(\frac{2x-1}{4+x^2} + \frac{3}{1+x}\right) dx$$

$\int \frac{1}{x^2+a^2}$ with $a^2 = 4$

$$= \int_0^2 \left(\frac{2x}{4+x^2} - \frac{1}{4+x^2} + \frac{3}{1+x}\right) dx$$

$$= \left[\ln|4+x^2| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + 3\ln|1+x|\right]_0^2$$

$$= \left[\ln(8) - \frac{1}{2}\tan^{-1}(1) + 3\ln(3)\right] - \left[\ln(4) - \frac{1}{2}\tan^{-1}(0) + 3\ln(1)\right]$$

$$= \ln 8 - \frac{1}{2}\times\frac{\pi}{4} + 3\ln 3 - \ln 4 + 0 - 0$$

$$= \ln 8 - \frac{\pi}{8} + \ln 3^3 - \ln 4$$

$$= \ln 8 + \ln 27 - \ln 4 - \frac{\pi}{8}$$

$$= \ln\left(\frac{8\times 27}{4}\right) - \frac{\pi}{8}$$

$$= \ln 54 - \frac{\pi}{8}$$ QED