3 **(a)** Given that $\cos(x - 30°) = 2\sin(x + 30°)$, show that $\tan x = \dfrac{2-\sqrt{3}}{1-2\sqrt{3}}$. [4]

$\cos(x - 30) = 2\sin(x + 30)$

$\cos x \cos 30 + \sin x \sin 30 = 2[\sin x \cos 30 + \cos x \sin 30]$

$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right)$

$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \sqrt{3}\sin x + \cos x$

$\sqrt{3}\cos x + \sin x = 2\sqrt{3}\sin x + 2\cos x$

$\sin x - 2\sqrt{3}\sin x = 2\cos x - \sqrt{3}\cos x$

$\sin x(1 - 2\sqrt{3}) = \cos x(2 - \sqrt{3})$

$\frac{\sin x}{\cos x} = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$

$\tan x = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$ QED

(b) Hence solve the equation

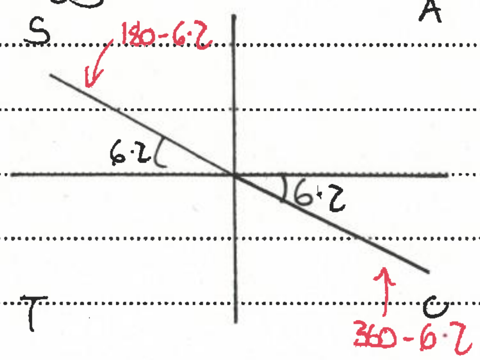
$$\cos(x - 30°) = 2\sin(x + 30°),$$

for $0° < x < 360°$. [2]

from (a): $\tan x = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$

$x = \tan^{-1}\left(\frac{2-\sqrt{3}}{1-2\sqrt{3}}\right)$

$= -6.20\ldots$ $\boxed{\text{STO}}$



$x = \underline{173.8°}, \underline{353.8°}$

2 Solve the equation $3\cos 2\theta = 3\cos\theta + 2$, for $0° \leqslant \theta \leqslant 360°$. [5]

$$3\cos 2\Theta = 3\cos\Theta + 2$$

$$3(2\cos^2\Theta - 1) = 3\cos\Theta + 2$$

$$6\cos^2\Theta - 3 = 3\cos\Theta + 2$$

$$6\cos^2\Theta - 3\cos\Theta - 5 = 0$$

let $y = \cos\Theta$:

$$6y^2 - 3y - 5 = 0$$

$$y = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(-5)}}{2(6)}$$

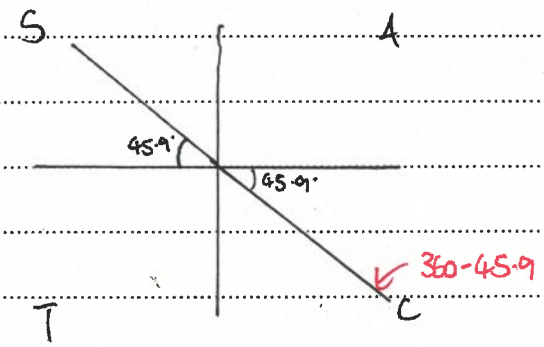
$$y = \frac{3 + \sqrt{129}}{12} \quad \text{or} \quad y = \frac{3 - \sqrt{129}}{12}$$

$$\cos\Theta = \frac{3 + \sqrt{129}}{12} \quad \text{or} \quad \cos\Theta = \frac{3 - \sqrt{129}}{12}$$

X no sol^n s

$$\Theta = 134.1°$$

STO



$$\Theta = \underline{134.1°}, \ \underline{314.1°}$$

5 By first expressing the equation

$$\tan\theta\tan(\theta + 45°) = 2\cot 2\theta$$

as a quadratic equation in $\tan\theta$, solve the equation for $0° < \theta < 90°$. [6]

$$\tan\theta \tan(\theta + 45) = 2\cot 2\theta$$

$$\tan\theta \times \frac{\tan\theta + \tan 45}{1 - \tan\theta \tan 45} = 2 \times \frac{1}{\tan 2\theta}$$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$$\tan\theta \times \frac{\tan\theta + 1}{1 - \tan\theta} = 2 \times \frac{1-\tan^2\theta}{2\tan\theta}$$

$$\frac{\tan^2\theta + \tan\theta}{1 - \tan\theta} = \frac{2 - 2\tan^2\theta}{2\tan\theta}$$

$$2\tan\theta(\tan^2\theta + \tan\theta) = (1-\tan\theta)(2 - 2\tan^2\theta)$$

$$\cancel{2\tan^3\theta} + 2\tan^2\theta = 2 - 2\tan^2\theta - 2\tan\theta + \cancel{2\tan^3\theta}$$

$$4\tan^2\theta + 2\tan\theta - 2 = 0$$

$$2\tan^2\theta + \tan\theta - 1 = 0$$

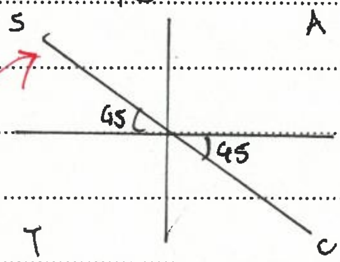
let $y = \tan\theta$: $2y^2 + y - 1 = 0$

$$(2y - 1)(y + 1) = 0$$

$y = \frac{1}{2}$ or $y = -1$

$\tan\theta = \frac{1}{2}$ or $\tan\theta = -1$

$\theta = 26.6°$ $\theta = -45°$



180-45

also outside domain

4 **(a)** Show that the equation $\tan(\theta + 60°) = 2\cot\theta$ can be written in the form

$$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0.$$ [3]

$$\tan(\theta + 60) = 2\cot\theta$$

$$\frac{\tan\theta + \tan 60}{1 - \tan\theta \tan 60} = \frac{2}{\tan\theta}$$

$$\frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} = \frac{2}{\tan\theta}$$

$$\tan\theta(\tan\theta + \sqrt{3}) = 2(1 - \sqrt{3}\tan\theta)$$

$$\tan^2\theta + \sqrt{3}\tan\theta = 2 - 2\sqrt{3}\tan\theta$$

$$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$$ QED

(b) Hence solve the equation $\tan(\theta + 60°) = 2\cot\theta$, for $0° < \theta < 180°$. [3]

from part (a): $\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$

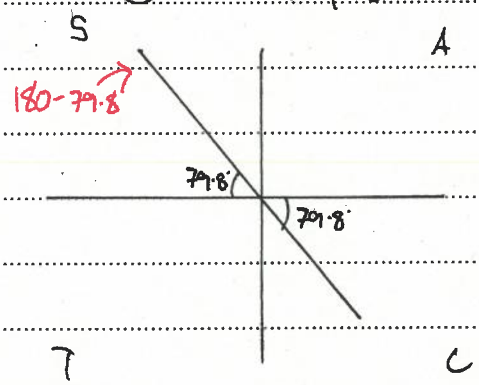
let $y = \tan\theta$: $y^2 + 3\sqrt{3}y - 2 = 0$

$$y = \frac{-(3\sqrt{3}) \pm \sqrt{(3\sqrt{3})^2 - 4(1)(-2)}}{2(1)}$$

$$y = \frac{\sqrt{35} - 3\sqrt{3}}{2} \quad \text{or} \quad y = \frac{-\sqrt{35} - 3\sqrt{3}}{2}$$

$$\tan\theta = \frac{\sqrt{35} - 3\sqrt{3}}{2} \quad \text{or} \quad \tan\theta = \frac{-\sqrt{35} - 3\sqrt{3}}{2}$$

$\theta = 19.8°$ or $\theta = -79.8°$



$\theta = 19.8°, 100.2°$

5 **(a)** Show that the equation

$$\cot 2\theta + \cot \theta = 2$$

can be expressed as a quadratic equation in $\tan \theta$. [3]

$$\frac{1}{\tan 2\theta} + \frac{1}{\tan\theta} = 2$$

$$\frac{1-\tan^2\theta}{2\tan\theta} + \frac{1}{\tan\theta} = 2$$

$\times 2\tan\theta$ $\quad\times 2\tan\theta$ $\quad\times 2\tan\theta$

$$1 - \tan^2\theta + 2 = 4\tan\theta$$

$$3 - \tan^2\theta = 4\tan\theta$$

$$-\tan^2\theta - 4\tan\theta + 3 = 0$$

$$\tan^2\theta + 4\tan\theta - 3 = 0$$

QED

(b) Hence solve the equation $\cot 2\theta + \cot\theta = 2$, for $0 < \theta < \pi$, giving your answers correct to 3 decimal places. [3]

$$\tan^2\theta + 4\tan\theta - 3 = 0$$

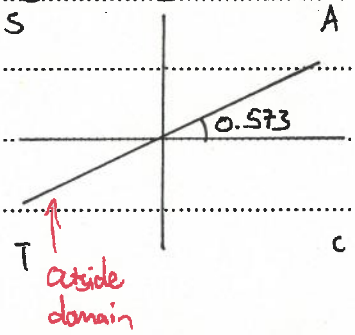
let $y = \tan\theta$: $\quad y^2 + 4y - 3 = 0$

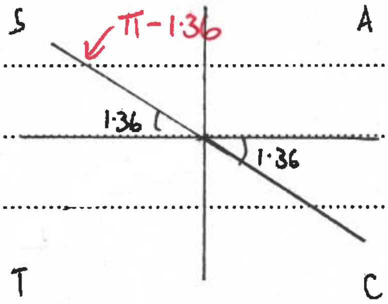
$$y = \frac{-4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$$

$y = -2 + \sqrt{7}$ or $y = -2 - \sqrt{7}$

$\tan\theta = -2 + \sqrt{7}$ $\qquad \tan\theta = -2 - \sqrt{7}$

$\theta = 0.573$ $\qquad \theta = -1.36$





$\theta = 0.573$ $\qquad \theta = 1.783$

6 (a) Show that the equation $\cot^2\theta + 2\cos 2\theta = 4$ can be written in the form

$$4\sin^4\theta + 3\sin^2\theta - 1 = 0.$$ [3]

$$\frac{\cos^2\theta}{\sin^2\theta} + 2\cos 2\theta = 4$$

$$\cos^2\theta + 2\cos 2\theta \times \sin^2\theta = 4\sin^2\theta$$

$\uparrow\ 1-2\sin^2\theta$

$$\cos^2\theta + 2(1-2\sin^2\theta)\sin^2\theta = 4\sin^2\theta$$

$\uparrow\ 1-\sin^2\theta$

$$1-\sin^2\theta + 2\sin^2\theta - 4\sin^4\theta = 4\sin^2\theta$$

$$1 + \sin^2\theta - 4\sin^4\theta = 4\sin^2\theta$$

$$1 - 3\sin^2\theta - 4\sin^4\theta = 0$$

$\times -1$ $\quad \times -1$ $\quad \times -1$ $\quad \times -1$

$$-1 + 3\sin^2\theta + 4\sin^4\theta = 0$$

$$4\sin^4\theta + 3\sin^2\theta - 1 = 0$$ QED

(b) Hence solve the equation $\cot^2\theta + 2\cos 2\theta = 4$, for $0° < \theta < 360°$. [3]

from (a): $4\sin^4\theta + 3\sin^2\theta - 1 = 0$

Let $Y = \sin^2\theta$:

$4Y^2 + 3Y - 1 = 0$

$(4Y - 1)(Y + 1) = 0$

$4Y - 1 = 0$ or $Y + 1 = 0$

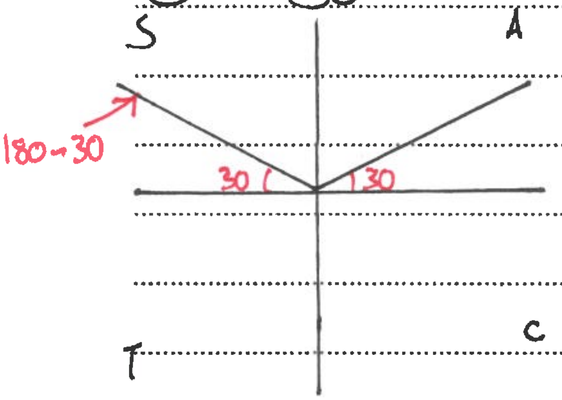
$Y = \frac{1}{4}$ $\qquad$ $Y = -1$

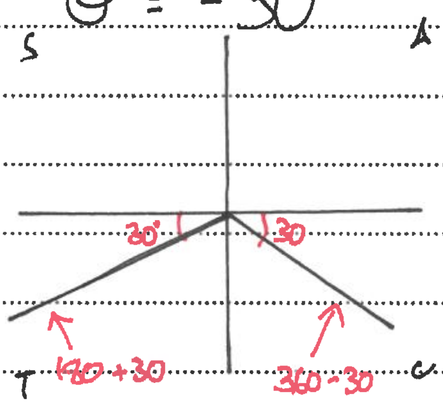
$\sin^2\theta = \frac{1}{4}$ $\qquad$ $\sin^2\theta = -1$

X no sol^{n}s

$\sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{-1}{2}$

$\theta = 30°$ $\qquad$ $\theta = -30°$





$\theta = 30°, 150°$ $\qquad$ $\theta = 210°, 330°$

→ $\underline{\theta = 30°, 150°, 210°, 330°}$

6 (a) Prove the identity $\cos 4\theta + 4\cos 2\theta + 3 \equiv 8\cos^4\theta$. [4]

LHS: $\cos 4\theta + 4\cos 2\theta + 3$

$\cos 2A = 2\cos^2 A - 1$ $\quad$ $\cos 2\theta = 2\cos^2\theta - 1$

$\cos 4\theta = 2\cos^2 2\theta - 1$

$2\cos^2 2\theta - 1 + 4(2\cos^2\theta - 1) + 3$

$\cos 2\theta = 2\cos^2\theta - 1$

$2(2\cos^2\theta - 1)^2 - 1 + 4(2\cos^2\theta - 1) + 3$

$2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 + 8\cos^2\theta - 4 + 3$

$8\cos^4\theta - 8\cos^2\theta + 2 + 8\cos^2\theta - 2$

$= \underline{8\cos^4\theta}$ $= \text{RHS}$

QED

(b) Hence solve the equation $\cos 4\theta + 4\cos 2\theta = 4$ for $0° \leqslant \theta \leqslant 180°$. [3]

from part (a): $\cos 4\theta + 4\cos 2\theta = 8\cos^4\theta - 3$

sub into new eq^n:

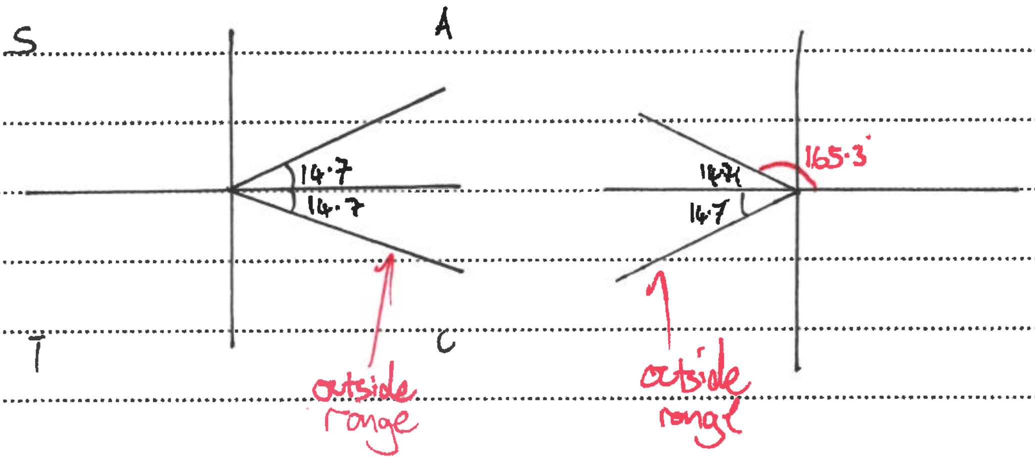
$$8\cos^4\theta - 3 = 4$$

$$8\cos^4\theta = 7$$

$$\cos^4\theta = \frac{7}{8}$$

$\cos\theta = \sqrt[4]{\frac{7}{8}}$ or $\cos\theta = -\sqrt[4]{\frac{7}{8}}$

$\theta = 14.7°$ $\qquad$ $\theta = 165.3°$



$$\theta = \underline{14.7°}, \ \underline{165.3°}$$

3 Solve the equation $2\cot 2x + 3\cot x = 5$, for $0° < x < 180°$. **[6]**

$$\frac{2}{\tan 2x} + \frac{3}{\tan x} = 5$$

$\tan 2x \leftarrow \frac{2\tan x}{1-\tan^2 x}$

$$\frac{2}{\frac{2\tan x}{1-\tan^2 x}} \times \frac{1-\tan^2 x}{1-\tan^2 x} + \frac{3}{\tan x} = 5$$

$$\frac{2(1-\tan^2 x)}{2\tan x} + \frac{3}{\tan x} = 5$$

$$\frac{1-\tan^2 x}{\tan x} + \frac{3}{\tan x} = 5$$

$$\frac{4 - \tan^2 x}{\tan x} = 5$$

$$4 - \tan^2 x = 5\tan x$$

$$\tan^2 x + 5\tan x - 4 = 0$$

let $Y = \tan x$:

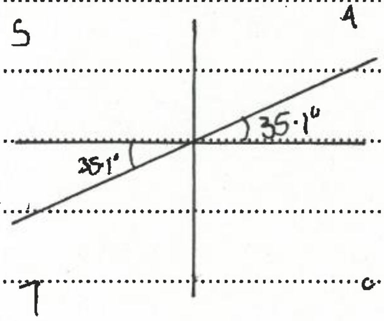
$$Y^2 + 5Y - 4 = 0$$

$$Y = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(-4)}}{2(1)}$$

$$Y = \frac{-5+\sqrt{41}}{2} \quad \text{or} \quad Y = \frac{-5-\sqrt{41}}{2}$$

$$\tan x = \frac{-5+\sqrt{41}}{2} \quad \text{or} \quad \tan x = \frac{-5-\sqrt{41}}{2}$$

$$x = 35.1° \qquad x = -80.1°,\ 99.9°$$



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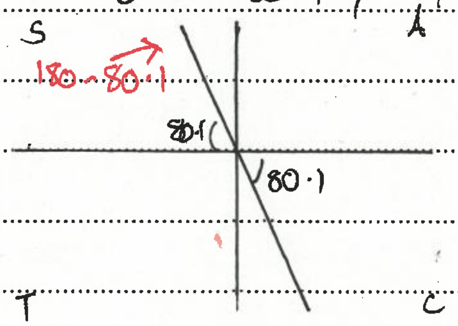


$$x = \underline{35.1°},\ \underline{99.9°}$$

7 (a) By expressing 3θ as $2\theta + \theta$, prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [3]

$\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$2\cos^2\theta - 1$ ↗ ↖ $2\sin\theta\cos\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta \times \sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

↑ $1 - \cos^2\theta$

$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= \underline{4\cos^3\theta - 3\cos\theta}$ QED

(b) Hence solve the equation

$$\cos 3\theta + \cos\theta \cos 2\theta = \cos^2\theta$$

for $0° \leqslant \theta \leqslant 180°$. [5]

from (a): $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\rightarrow 4\cos^3\theta - 3\cos\theta + \cos\theta\cos 2\theta = \cos^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$4\cos^3\theta - 3\cos\theta + \cos\theta(2\cos^2\theta - 1) = \cos^2\theta$

$4\cos^3\theta - 3\cos\theta + 2\cos^3\theta - \cos\theta = \cos^2\theta$

$6\cos^3\theta - 4\cos\theta = \cos^2\theta$

$6\cos^3\theta - \cos^2\theta - 4\cos\theta = 0$

$\cos\theta(6\cos^2\theta - \cos\theta - 4) = 0$

$\cos\theta = 0$

$\theta = 90°$

let $Y = \cos\theta$:

$6Y^2 - Y - 4 = 0$

$Y = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-4)}}{2(6)}$

$Y = \dfrac{1 + \sqrt{97}}{12}$ or $Y = \dfrac{1 - \sqrt{97}}{12}$

$\cos\theta = \dfrac{1 + \sqrt{97}}{12}$ $\quad$ $\cos\theta = \dfrac{1 - \sqrt{97}}{12}$

$\theta = 25.3°$ $\quad$ $\theta = 137.5°$

$\theta = 25.3°, 90°, 137.5°$

8 **(a)** By first expanding $(\cos^2\theta + \sin^2\theta)^2$, show that

$$\cos^4\theta + \sin^4\theta \equiv 1 - \tfrac{1}{2}\sin^2 2\theta. \qquad [3]$$

$(\cos^2\theta + \sin^2\theta)^2 = 1$

$(\cos^2\theta + \sin^2\theta)(\cos^2\theta + \sin^2\theta) = 1$

$\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta = 1$

$\cos^4\theta + \sin^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

$\cos^4\theta + \sin^4\theta = 1 - 2(\sin\theta\cos\theta)^2$

$= 1 - 2\left(\frac{1}{2}\sin 2\theta\right)^2$

$= 1 - 2\times\frac{1}{4}\sin^2 2\theta$

$= 1 - \frac{1}{2}\sin^2 2\theta$

$\cos^4\theta + \sin^4\theta = 1 - \frac{1}{2}\sin^2 2\theta$ QED

(b) Hence solve the equation

$$\cos^4\theta + \sin^4\theta = \tfrac{5}{9},$$

for $0° < \theta < 180°$. [4]

from part (a): $1 - \frac{1}{2}\sin^2 2\theta = \frac{5}{9}$

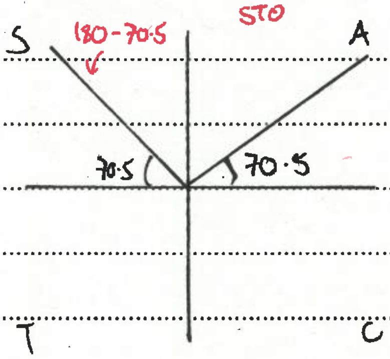
$-\frac{1}{2}\sin^2 2\theta = \frac{-4}{9}$

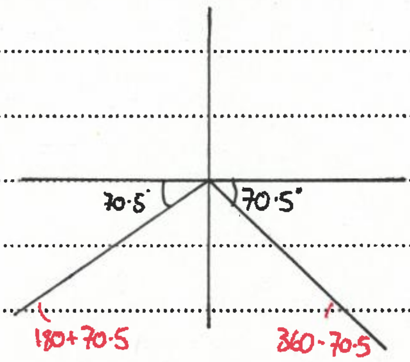
$\sin^2 2\theta = \frac{8}{9}$

$\boxed{\begin{array}{l} 0 < \theta < 180 \\ 0 < 2\theta < 360 \end{array}}$

$\sin 2\theta = \sqrt{\frac{8}{9}}$ or $\sin 2\theta = -\sqrt{\frac{8}{9}}$

$2\theta = 70.5°$ $\qquad$ $2\theta = -70.5°$





$2\theta = 70.5°, 109.5°$ $\qquad$ $2\theta = -70.5, 250.5, 289.5°$

$\theta = 35.3°, 54.7°$ $\qquad$ $\theta = \cancel{-35.3°}, 125.3°, 144.7°$

$\theta = 35.3°, 54.7°, 125.3°, 144.7°$