

- 6 (a) Using the expansions of $\sin(3x + 2x)$ and $\sin(3x - 2x)$, show that

$$\frac{1}{2}(\sin 5x + \sin x) \equiv \sin 3x \cos 2x.$$

[3]

$$\sin 5x \rightarrow \sin(3x + 2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$$

$$\sin x \rightarrow \sin(3x - 2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$$

$$\frac{\sin 5x + \sin x}{2} = \frac{2 \sin 3x \cos 2x}{2}$$

$$\frac{1}{2}(\sin 5x + \sin x) = \sin 3x \cos 2x \quad \underline{\text{QED}}$$

- 2 Solve the equation $\cos(\theta - 60^\circ) = 3 \sin \theta$, for $0^\circ \leq \theta \leq 360^\circ$.

[5]

$$\cos(\theta - 60) = 3 \sin \theta$$

$$\cos \theta \cos 60 + \sin \theta \sin 60 = 3 \sin \theta$$

$$\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta = 3 \sin \theta$$

$$\cos \theta + \sqrt{3} \sin \theta = 6 \sin \theta$$

$$\cos \theta = 6 \sin \theta - \sqrt{3} \sin \theta$$

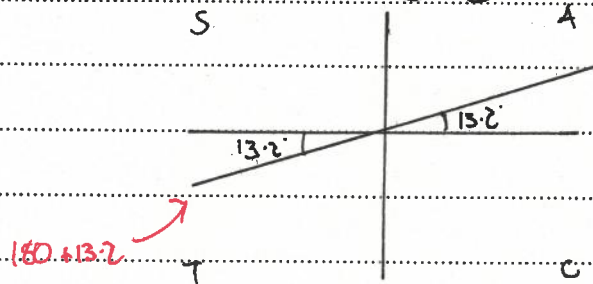
$$\cos \theta = (6 - \sqrt{3}) \sin \theta$$

$$1 = (6 - \sqrt{3}) \frac{\sin \theta}{\cos \theta}$$

$$\frac{\sin \theta}{\cos \theta} = \frac{1}{6 - \sqrt{3}}$$

$$\tan \theta = \frac{1}{6 - \sqrt{3}}$$

$$\theta = 13.2^\circ$$



$$\theta = \underline{13.2^\circ}, \underline{193.2^\circ}$$

- 4 Solve the equation $2\cos x - \cos \frac{1}{2}x = 1$ for $0 \leq x \leq 2\pi$.

[5]

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos x = 2\cos^2\left(\frac{1}{2}x\right) - 1$$

$$2\cos x = 4\cos^2\left(\frac{1}{2}x\right) - 2$$

$$\rightarrow 4\cos^2\left(\frac{1}{2}x\right) - 2 - \cos\left(\frac{1}{2}x\right) = 1$$

$$4\cos^2\left(\frac{1}{2}x\right) - \cos\left(\frac{1}{2}x\right) - 3 = 0$$

$$\text{Let } Y = \cos\left(\frac{1}{2}x\right):$$

$$4Y^2 - Y - 3 = 0$$

$$(4Y + 3)(Y - 1) = 0$$

$$4Y + 3 = 0 \quad \text{or} \quad Y - 1 = 0$$

$$4Y = -3$$

$$Y = 1$$

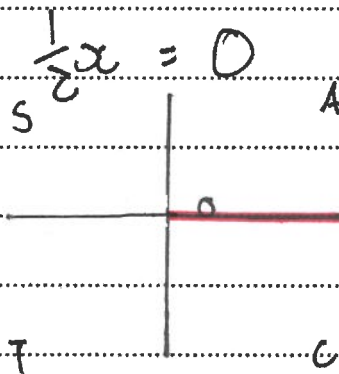
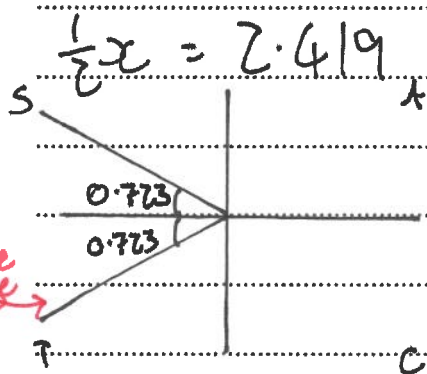
$$Y = -\frac{3}{4}$$

$$\cos\left(\frac{1}{2}x\right) = -\frac{3}{4}$$

$$\text{or} \quad \cos\left(\frac{1}{2}x\right) = 1$$

$$0 \leq x \leq 2\pi$$

$$0 \leq \frac{1}{2}x \leq \pi$$



$$\frac{1}{2}x = 0 \text{ only}$$

$$\frac{1}{2}x = 0, 2.419$$

$$\underline{x = 0, 4.84}$$

$$\tan(A+B)$$

6

- 4 Solve the equation $\tan(x + 45^\circ) = 2 \cot x$ for $0^\circ < x < 180^\circ$.

[5]

$$\frac{\tan x + \tan 45}{1 - \tan x \tan 45} = \frac{2}{\tan x}$$

$$\frac{\tan x + 1}{1 - \tan x} \neq \frac{2}{\tan x}$$

$$\tan x (\tan x + 1) = 2(1 - \tan x)$$

$$\tan^2 x + \tan x = 2 - 2 \tan x$$

$$\tan^2 x + 3 \tan x - 2 = 0$$

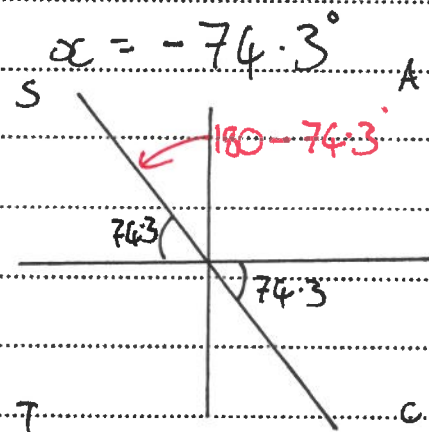
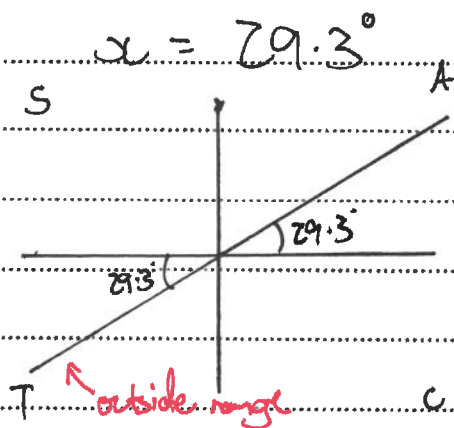
let $Y = \tan x$:

$$Y^2 + 3Y - 2 = 0$$

$$Y = \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)}$$

$$Y = \frac{-3 + \sqrt{17}}{2} \quad \text{or} \quad Y = \frac{-3 - \sqrt{17}}{2}$$

$$\tan x = \frac{-3 + \sqrt{17}}{2} \quad \text{or} \quad \tan x = \frac{-3 - \sqrt{17}}{2}$$



$$x = 105.7^\circ$$

$$x = \underline{29.3^\circ}, \underline{105.7^\circ}$$

- 5 (a) Show that $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x} = 2 \cot 2x$.

[4]

$$\text{LHS: } \frac{\cos 3x \cos x}{\sin x \cos x} + \frac{\sin 3x \sin x}{\sin x \cos x}$$

$$= \frac{\cos 3x \cos x + \sin 3x \sin x}{\sin x \cos x}$$

using $\cos(A+B)$ and $\sin(A+B)$

$$= \frac{(\cos 2x \cos x - \sin 2x \sin x) \cos x + (\sin 2x \cos x + \cos 2x \sin x) \sin x}{\sin x \cos x}$$

$$= \frac{\cos 2x \cos^2 x - \sin 2x \sin x \cos x + \sin 2x \sin x \cos x + \cos 2x \sin^2 x}{\sin x \cos x}$$

$$= \frac{\cos 2x \cos^2 x + \cos 2x \sin^2 x}{\sin x \cos x}$$

$$= \frac{\cos 2x (\cos^2 x + \sin^2 x)}{\sin x \cos x}$$

$$= \frac{\cos 2x}{\sin x \cos x}$$

$\sin 2x = 2 \sin x \cos x$
 $\frac{1}{2} \sin 2x = \sin x \cos x$

$$= \frac{\cos 2x \times 2}{\frac{1}{2} \sin 2x \times 2}$$

$$= \frac{2 \cos 2x}{\sin 2x}$$

$$= \underline{2 \cot 2x} = \text{RHS}$$

- (b) Hence solve the equation $\frac{\cos 3x}{\sin x} + \frac{\sin 3x}{\cos x} = 4$, for $0 < x < \pi$.

[3]

from (a): $2 \cot 2x = 4$

$$\cot 2x = \frac{4}{2}$$

$$\cot 2x = 2$$

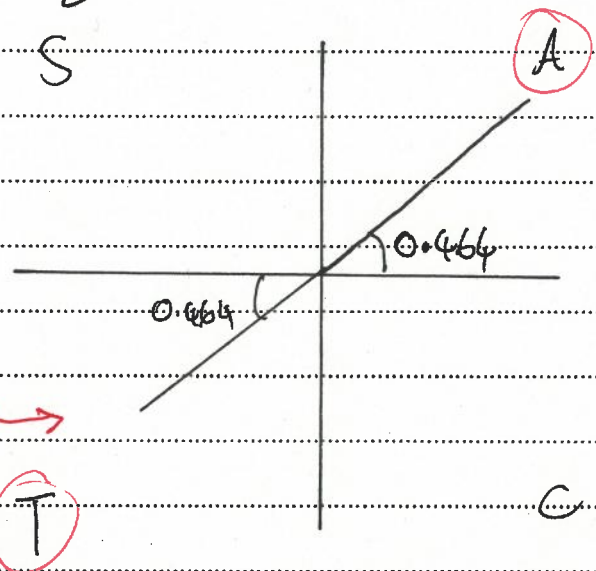
$$\frac{1}{\tan 2x} = 2$$

$$\tan 2x = \frac{1}{2}$$

$$2x = \tan^{-1}\left(\frac{1}{2}\right) \quad S$$

$$2x = 0.464$$

STO



$$\Rightarrow 2x = 0.464, 3.605 \dots$$

STO

$$x = \underline{0.232}, \underline{1.80}$$

5 Solve the equation $\sin \theta = 3 \cos 2\theta + 2$, for $0^\circ \leq \theta \leq 360^\circ$.

[5]

$$\sin \theta = 3 \cos 2\theta + 2$$

$$\sin \theta = 3(1 - 2\sin^2 \theta) + 2$$

$$\sin \theta = 3 - 6\sin^2 \theta + 2$$

$$\sin \theta = 5 - 6\sin^2 \theta$$

$$6\sin^2 \theta + \sin \theta - 5 = 0$$

let $y = \sin \theta$:

$$6y^2 + y - 5 = 0$$

$$(6y - 5)(y + 1) = 0$$

$$6y - 5 = 0$$

$$\text{or } y + 1 = 0$$

$$y = \frac{5}{6}$$

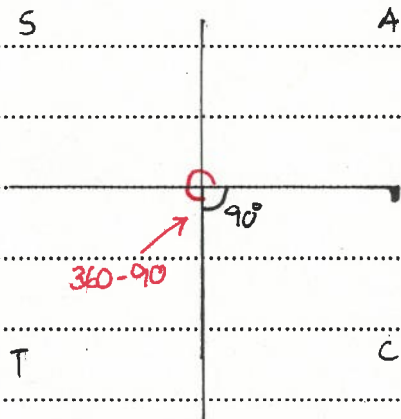
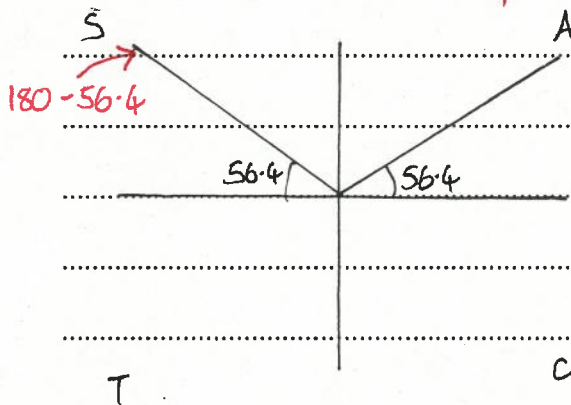
$$\text{or } y = -1$$

$$\sin \theta = \frac{5}{6}$$

$$\text{or } \sin \theta = -1$$

$$\theta = 56.4^\circ$$

$$\text{or } \theta = -90^\circ$$



$$\theta = 56.4^\circ, 123.6^\circ$$

$$\theta = 270^\circ$$

$$\theta = 56.4^\circ, 123.6^\circ, 270^\circ$$

- 3 Express the equation $\tan(\theta + 60^\circ) = 2 + \tan(60^\circ - \theta)$ as a quadratic equation in $\tan \theta$, and hence solve the equation for $0^\circ \leq \theta \leq 180^\circ$. [6]

$$\tan(\theta + 60^\circ) = 2 + \tan(60^\circ - \theta)$$

$$\tan(\theta + 60^\circ) - \tan(60^\circ - \theta) = 2$$

$$\frac{\tan \theta + \tan 60^\circ}{1 - \tan \theta \tan 60^\circ} - \frac{\tan 60^\circ - \tan \theta}{1 + \tan 60^\circ \tan \theta} = 2$$

$$\frac{\tan \theta + \sqrt{3}}{1 - \sqrt{3} \tan \theta} - \frac{\sqrt{3} - \tan \theta}{1 + \sqrt{3} \tan \theta} = 2$$

$$\frac{(\tan \theta + \sqrt{3})(1 + \sqrt{3} \tan \theta) - (\sqrt{3} - \tan \theta)(1 - \sqrt{3} \tan \theta)}{(1 - \sqrt{3} \tan \theta)(1 + \sqrt{3} \tan \theta)} = 2$$

$$\frac{\tan \theta + \sqrt{3} + \sqrt{3} \tan \theta + 3 + 3 \tan \theta - \sqrt{3} + 3 \tan \theta + \tan \theta - \sqrt{3} \tan^2 \theta}{1 - 3 \tan^2 \theta} = 2$$

$$\frac{8 \tan \theta}{1 - 3 \tan^2 \theta} = 2$$

$$8 \tan \theta = 2(1 - 3 \tan^2 \theta)$$

$$8 \tan \theta = 2 - 6 \tan^2 \theta$$

$$6 \tan^2 \theta + 8 \tan \theta - 2 = 0$$

$$3 \tan^2 \theta + 4 \tan \theta - 1 = 0$$

$$\text{Let } y = \tan \theta: 3y^2 + 4y - 1 = 0$$

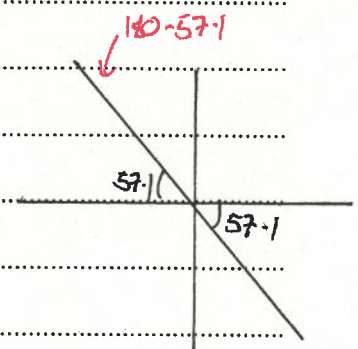
$$y = \frac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(3)}$$

$$y = \frac{-2 + \sqrt{7}}{3} \quad \text{or} \quad y = \frac{-2 - \sqrt{7}}{3}$$

$$\tan \theta = \frac{-2 + \sqrt{7}}{3} \quad \text{or} \quad \tan \theta = \frac{-2 - \sqrt{7}}{3}$$

$$\theta = 12.1^\circ \quad \text{or} \quad \theta = -57.1^\circ$$

$$\theta = 122.9^\circ$$



- 3 By first expressing the equation $\tan(x + 45^\circ) = 2 \cot x + 1$ as a quadratic equation in $\tan x$, solve the equation for $0^\circ < x < 180^\circ$. [6]

$$\begin{aligned} \tan(x + 45) &= 2 \cot x + 1 \\ \frac{\tan x + \tan 45}{1 - \tan x \tan 45} &= 2 \cot x + 1 \\ \frac{\tan x + 1}{1 - \tan x} &= 2 \cot x + 1 \\ \tan x + 1 &= (2 \cot x + 1)(1 - \tan x) \\ \tan x + 1 &= 2 \cot x - 2 \cot x \tan x + 1 - \tan x \\ \tan x + 1 &= 2 \frac{1}{\tan x} - 2 \frac{1}{\tan x} \tan x + 1 - \tan x \\ \tan x + 1 &= 2 \frac{1}{\tan x} - 2 + 1 - \tan x \\ \tan x + 1 &= \frac{2}{\tan x} - 1 - \tan x \\ 2 \tan x + 2 &= \frac{2}{\tan x} \\ 2 \tan^2 x + 2 \tan x &= 2 \\ 2 \tan^2 x + 2 \tan x - 2 &= 0 \\ \tan^2 x + \tan x - 1 &= 0 \end{aligned}$$

let $y = \tan x$: $y^2 + y - 1 = 0$

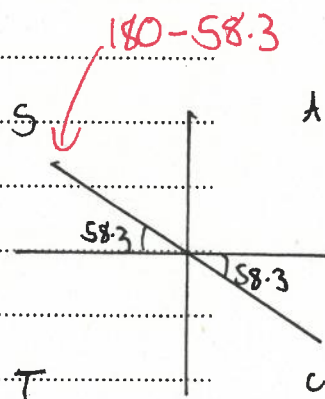
$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$

$$y = \frac{-1 + \sqrt{5}}{2} \quad \text{or} \quad y = \frac{-1 - \sqrt{5}}{2}$$

$$\tan x = \frac{-1 + \sqrt{5}}{2} \quad \text{or} \quad \tan x = \frac{-1 - \sqrt{5}}{2}$$

$$x = 31.7^\circ \quad \text{or} \quad x = -58.3^\circ = 121.7^\circ$$

$$x = 31.7^\circ, 121.7^\circ$$



- 5 The angles α and β lie between 0° and 180° and are such that

$$\tan(\alpha + \beta) = 2 \quad \text{and} \quad \tan \alpha = 3 \tan \beta.$$

Find the possible values of α and β .

[6]

$$\textcircled{1} \tan(\alpha + \beta) = 2$$

$$\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = 2$$

$$\textcircled{2} \frac{3 \tan \beta + \tan \beta}{1 - 3 \tan^2 \beta} = 2$$

$$4 \tan \beta = 2(1 - 3 \tan^2 \beta)$$

$$4 \tan \beta = 2 - 6 \tan^2 \beta$$

$$6 \tan^2 \beta + 4 \tan \beta - 2 = 0$$

$$3 \tan^2 \beta + 2 \tan \beta - 1 = 0$$

$$\text{let } y = \tan \beta:$$

$$3y^2 + 2y - 1 = 0$$

$$(3y - 1)(y + 1) = 0$$

$$3y - 1 = 0 \quad \text{or} \quad y + 1 = 0$$

$$y = \frac{1}{3} \quad \text{or} \quad y = -1$$

$$\tan \beta = \frac{1}{3} \quad \text{or} \quad \tan \beta = -1$$

$$\beta = 18.4^\circ$$

no others in domain

$$\beta = -45^\circ$$

$$= 135^\circ$$

$$\textcircled{2} \text{ if } \tan \beta = \frac{1}{3}, \tan \alpha = 1:$$

$$\alpha = 45^\circ$$

$$\textcircled{2} \text{ if } \tan \beta = -1, \tan \alpha = -3$$

$$\alpha = -71.6^\circ$$

$$\alpha = 108.4^\circ$$

so:

$$\alpha = 45^\circ \text{ and } \beta = 18.4^\circ$$

or:

$$\alpha = 108.4^\circ \text{ and } \beta = 135^\circ$$

5 (a) Given that

$$\sin\left(x + \frac{1}{6}\pi\right) - \sin\left(x - \frac{1}{6}\pi\right) = \cos\left(x + \frac{1}{3}\pi\right) - \cos\left(x - \frac{1}{3}\pi\right),$$

find the exact value of $\tan x$.

[4]

$$\begin{aligned}\sin\left(x + \frac{\pi}{6}\right) &= \sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x\end{aligned}$$

$$\begin{aligned}\sin\left(x - \frac{\pi}{6}\right) &= \sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x\end{aligned}$$

$$\begin{aligned}\rightarrow \sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x - \left(\frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x\right) \\ &= \cos x\end{aligned}$$

$$\begin{aligned}\cos\left(x + \frac{\pi}{3}\right) &= \cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3} \\ &= \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x\end{aligned}$$

$$\begin{aligned}\cos\left(x - \frac{\pi}{3}\right) &= \cos x \cos \frac{\pi}{3} + \sin x \sin \frac{\pi}{3} \\ &= \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x\end{aligned}$$

$$\begin{aligned}\rightarrow \cos\left(x + \frac{\pi}{3}\right) - \cos\left(x - \frac{\pi}{3}\right) &= \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x - \left(\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x\right) \\ &= -\sqrt{3} \sin x\end{aligned}$$

Original equation becomes:

$$\cos x = -\sqrt{3} \sin x$$

$$1 = \frac{-\sqrt{3} \sin x}{\cos x}$$

$$\tan x = \frac{-1}{\sqrt{3}}$$

(b) Hence find the exact roots of the equation

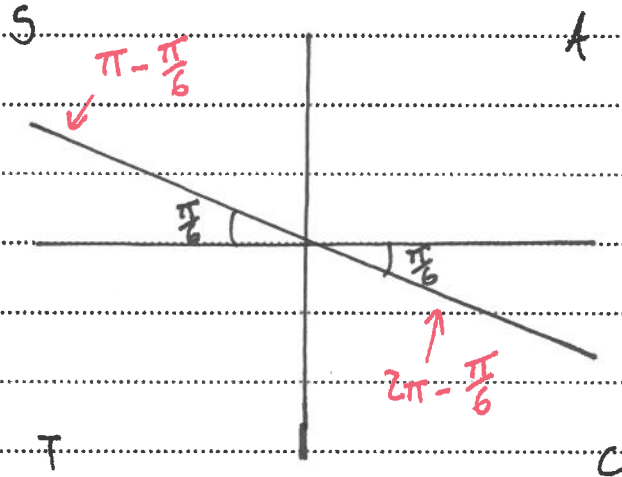
$$\sin\left(x + \frac{1}{6}\pi\right) - \sin\left(x - \frac{1}{6}\pi\right) = \cos\left(x + \frac{1}{3}\pi\right) - \cos\left(x - \frac{1}{3}\pi\right)$$

for $0 \leq x \leq 2\pi$.

[2]

from (a): $\tan x = \frac{-1}{\sqrt{3}}$

$$x = -\frac{\pi}{6}$$



$$\underline{x = \frac{5\pi}{6}, \frac{11\pi}{6}}$$

- 4 (a) Show that the equation $\sin 2\theta + \cos 2\theta = 2 \sin^2 \theta$ can be expressed in the form

$$\cos^2 \theta + 2 \sin \theta \cos \theta - 3 \sin^2 \theta = 0.$$

[2]

$$\begin{array}{l} \sin 2\theta + \cos 2\theta = 2 \sin^2 \theta \\ \text{2sin}\theta\text{cos}\theta \rightarrow \quad \quad \quad \leftarrow \cos^2\theta - \sin^2\theta \end{array}$$

$$2 \sin \theta \cos \theta + \cos^2 \theta - \sin^2 \theta = 2 \sin^2 \theta$$

$$2 \sin \theta \cos \theta + \cos^2 \theta - 3 \sin^2 \theta = 0$$

$$\cos^2 \theta + 2 \sin \theta \cos \theta - 3 \sin^2 \theta = 0 \quad \text{QED}$$

- (b) Hence solve the equation $\sin 2\theta + \cos 2\theta = 2 \sin^2 \theta$ for $0^\circ < \theta < 180^\circ$.

[4]

Factorise by splitting the middle: two numbers that multiply to -3 and sum to $+2$: -1 and $+3$

$$\cos^2 \theta + 3 \sin \theta \cos \theta - 1 \sin \theta \cos \theta - 3 \sin^2 \theta = 0$$

$$\cos \theta (\cos \theta + 3 \sin \theta) - \sin \theta (\cos \theta + 3 \sin \theta) = 0$$

$$(\cos \theta - \sin \theta)(\cos \theta + 3 \sin \theta) = 0$$

$$\cos \theta - \sin \theta = 0 \quad \text{or} \quad \cos \theta + 3 \sin \theta = 0$$

$$\sin \theta = \cos \theta$$

$$3 \sin \theta = -\cos \theta$$

$$\frac{\sin \theta}{\cos \theta} = 1$$

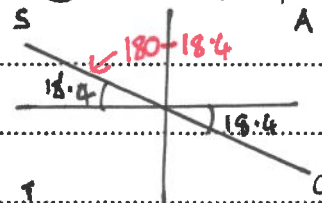
$$\frac{\sin \theta}{\cos \theta} = -\frac{1}{3}$$

$$\tan \theta = 1$$

$$\tan \theta = -\frac{1}{3}$$

$$\theta = 45^\circ$$

$$\theta = -18.4^\circ$$



$$\theta = 161.6^\circ$$

$$\theta = 45^\circ, 161.6^\circ$$

- 5 (a) By first expanding $\tan(2\theta + 2\theta)$, show that the equation $\tan 4\theta = \frac{1}{2} \tan \theta$ may be expressed as $\tan^4 \theta + 2 \tan^2 \theta - 7 = 0$. [4]

$$\begin{aligned} \tan(2\theta + 2\theta) &= \frac{\tan 2\theta + \tan 2\theta}{1 - \tan 2\theta \tan 2\theta} \\ &= \frac{2 \tan 2\theta}{1 - \tan^2 2\theta} \quad \leftarrow \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ &= \frac{2 \left(\frac{2 \tan \theta}{1 - \tan^2 \theta} \right)}{1 - \left(\frac{2 \tan \theta}{1 - \tan^2 \theta} \right)^2} \\ &= \frac{4 \tan \theta}{1 - \tan^2 \theta} \quad \times (1 - \tan^2 \theta)^2 \\ &= \frac{4 \tan \theta}{(1 - \tan^2 \theta)^2} \quad \times (1 - \tan^2 \theta)^2 \end{aligned}$$

$$\tan 4\theta = \frac{1}{2} \tan \theta \quad \rightarrow \quad \frac{1}{2} \tan \theta = \frac{4 \tan \theta (1 - \tan^2 \theta)}{(1 - \tan^2 \theta)^2 - 4 \tan^2 \theta}$$

$$\frac{1}{2} \tan \theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 2 \tan^2 \theta + \tan^4 \theta - 4 \tan^2 \theta}$$

$$\frac{1}{2} \tan \theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

$$\frac{1}{2} \tan \theta (1 - 6 \tan^2 \theta + \tan^4 \theta) = 4 \tan \theta - 4 \tan^3 \theta$$

$$\frac{1}{2} \tan \theta - 3 \tan^3 \theta + \frac{1}{2} \tan^5 \theta = 4 \tan \theta - 4 \tan^3 \theta$$

$$\tan \theta - 6 \tan^3 \theta + \tan^5 \theta = 8 \tan \theta - 8 \tan^3 \theta$$

$$\tan^5 \theta + 2 \tan^3 \theta - 7 \tan \theta = 0$$

$\div \tan \theta$ $\div \tan \theta$ $\div \tan \theta$ $\div \tan \theta$

$$\tan^4 \theta + 2 \tan^2 \theta - 7 = 0$$

(b) Hence solve the equation $\tan 4\theta = \frac{1}{2} \tan \theta$, for $0^\circ < \theta < 180^\circ$.

[3]

From part (a): $\tan^4 \theta + 2 \tan^2 \theta - 7 = 0$

let $y = \tan^2 \theta$: $y^2 + 2y - 7 = 0$

$$y = \frac{-2 \pm \sqrt{2^2 - 4(1)(-7)}}{2(1)}$$

$$y = -1 + 2\sqrt{2}$$

or

$$y = -1 - 2\sqrt{2}$$

$$\tan^2 \theta = -1 + 2\sqrt{2}$$

or

$$\tan^2 \theta = -1 - 2\sqrt{2}$$

X no sol's

$$\tan \theta = 1.35 \dots$$

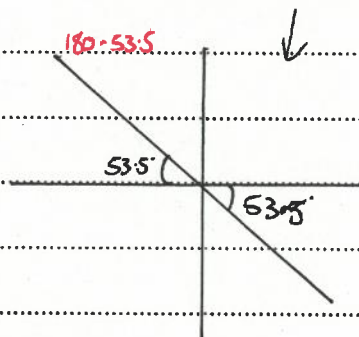
or

$$\tan \theta = -1.35 \dots$$

$$\theta = 53.5^\circ$$

or

$$\theta = -53.5^\circ, 126.5^\circ$$



$$\theta = 53.5^\circ, 126.5^\circ$$