4 Express $\dfrac{4x^2 - 13x + 13}{(2x-1)(x-3)}$ in partial fractions. [5]

improper fraction → long division first

$(2x-1)(x-3) = 2x^2 - 7x + 3$

$$\begin{array}{r|l} & 2 \\ 2x^2 - 7x + 3 & 4x^2 - 13x + 13 \\ & \underline{4x^2 - 14x + 6} \\ & x + 7 \end{array}$$

$f(x) = 2 + \dfrac{x+7}{(2x-1)(x-3)}$

$\dfrac{x+7}{(2x-1)(x-3)} \equiv \dfrac{A}{2x-1} + \dfrac{B}{x-3}$

$x + 7 \equiv A(x-3) + B(2x-1)$

$\underline{x=3:}$

$3 + 7 = 0 + 5B$

$10 = 5B$

$\underline{B = 2}$

$\underline{x = \tfrac{1}{2}:}$

$\tfrac{1}{2} + 7 = A\left(\tfrac{1}{2} - 3\right)$

$\tfrac{15}{2} = -\tfrac{5}{2}A$

$15 = -5A$

$\underline{A = -3}$

$f(x) = \underline{2 + \dfrac{-3}{2x-1} + \dfrac{2}{x-3}}$

4 Let $f(x) = \dfrac{15 - 6x}{(1 + 2x)(4 - x)}$.

(a) Express $f(x)$ in partial fractions. [3]

$$\frac{15-6x}{(1+2x)(4-x)} \equiv \frac{A}{1+2x} + \frac{B}{4-x}$$

$$15 - 6x \equiv A(4-x) + B(1+2x)$$

$\underline{x=4:}$ $15 - 6(4) = A(0) + B(1+2(4))$

$15 - 24 = 9B$

$-9 = 9B$

$\underline{B = -1}$

$\underline{x = \frac{-1}{2}:}$ $15 - 6(\frac{-1}{2}) = A(4 - -\frac{1}{2}) + B(0)$

$15 + 3 = A \times \frac{9}{2}$

$18 = \frac{9A}{2}$

$\underline{A = 4}$ → $\underline{f(x) = \frac{4}{1+2x} - \frac{1}{4-x}}$

(b) Hence find $\int_1^2 f(x)\,dx$, giving your answer in the form $\ln\left(\frac{a}{b}\right)$, where a and b are integers. [4]

$$\int_1^2 \left(\frac{4}{1+2x} - \frac{1}{4-x}\right) dx$$

$$= \left[2\ln|1+2x| + \ln|4-x|\right]_1^2$$

$$= [2\ln 5 + \ln 2] - [2\ln 3 + \ln 3]$$

$$= \ln 5^2 + \ln 2 - \ln 3^2 - \ln 3$$

$$= \ln 25 + \ln 2 - \ln 9 - \ln 3$$

$$= \ln 50 - \ln 9 - \ln 3$$

$$= \ln\left(\frac{50}{9}\right) - \ln 3$$

$$= \underline{\ln\left(\frac{50}{27}\right)}$$

6 Let $f(x) = \dfrac{5a}{(2x-a)(3a-x)}$, where a is a positive constant.

(a) Express $f(x)$ in partial fractions. [3]

$$\frac{5a}{(2x-a)(3a-x)} \equiv \frac{A}{2x-a} + \frac{B}{3a-x}$$

$$5a \equiv A(3a-x) + B(2x-a)$$

$x = \frac{a}{2}$: $5a = A\left(3a - \frac{a}{2}\right) + B(0)$

$$5a = A \times \frac{5a}{2}$$

$$10\cancel{a} = A \times 5\cancel{a}$$

$$10 = 5A$$

$$A = 2$$

$x = 3a$: $5a = A(0) + B(2 \times 3a - a)$

$$5a = 5aB$$

$$5 = 5B$$

$$B = 1$$

$$f(x) = \frac{2}{2x-a} + \frac{1}{3a-x}$$

(b) Hence show that $\int_a^{2a} f(x)\,dx = \ln 6$. [4]

$$\int_a^{2a}\left(\frac{2}{2x-a}+\frac{1}{3a-x}\right)dx$$

$$=\Big[\ln|2x-a| - \ln|3a-x|\Big]_a^{2a}$$

$$=\Big[\ln|2(2a)-a| - \ln|3a-2a|\Big]-\Big[\ln|2(a)-a|-\ln|3a-a|\Big]$$

$$=\Big[\ln(3a)-\ln(a)\Big]-\Big[\ln(a)-\ln(2a)\Big]$$

$$=\ln 3a - \ln a - \ln a + \ln 2a$$

$$=\ln 3a + \ln 2a - 2\ln a$$

$$=\ln 3a + \ln 2a - \ln a^2$$

$$=\ln(6a^2)-\ln a^2$$

$$=\ln\left(\frac{6a^2}{a^2}\right)$$

$$=\underline{\ln 6}\quad \text{QED}$$

9 Let $f(x) = \dfrac{2x^2 + 17x - 17}{(1 + 2x)(2 - x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{2x^2 + 17x - 17}{(1+2x)(2-x)^2} \equiv \frac{A}{1+2x} + \frac{B}{2-x} + \frac{C}{(2-x)^2}$$

$$2x^2 + 17x - 17 \equiv A(2-x)^2 + B(1+2x)(2-x) + C(1+2x)$$

$x=2$:

$$2(2)^2 + 17(2) - 17 = 0 + 0 + C(1+4)$$

$$8 + 34 - 17 = 5C$$

$$25 = 5C$$

$$C = 5$$

$x = \frac{-1}{2}$:

$$2\left(\tfrac{-1}{2}\right)^2 + 17\left(\tfrac{-1}{2}\right) - 17 = A\left(2 - -\tfrac{1}{2}\right)^2 + 0 + 0$$

$$\frac{1}{2} - \frac{17}{2} - 17 = A\left(\frac{5}{2}\right)^2$$

$$-25 = \frac{25}{4}A$$

$$-100 = 25A$$

$$A = -4$$

$x=0$:

$$0 + 0 - 17 = A(2)^2 + B(1)(2) + C(1)$$

$$-17 = 4A + 2B + C$$

$$-17 = -16 + 2B + 5$$

$$-17 = -11 + 2B$$

$$2B = -6$$

$$B = -3$$

$$\rightarrow \frac{-4}{1+2x} - \frac{3}{2-x} + \frac{5}{(2-x)^2}$$

(b) Hence show that $\int_0^1 f(x)\,dx = \frac{5}{2} - \ln 72$. [5]

$$\int_0^1 \left(\frac{-4}{1+2x} - \frac{3}{2-x} + \frac{5}{(2-x)^2}\right) dx$$

$$= -2\int_0^1 \frac{2}{1+2x}\,dx + 3\int_0^1 \frac{-1}{2-x}\,dx + 5\int_0^1 (2-x)^{-2}\,dx$$

$$= \Big[-2\ln|1+2x| + 3\ln|2-x| + 5(2-x)^{-1}\Big]_0^1$$

$$= \left[-2\ln(3) + 3\ln(1) + 5(1)^{-1}\right] - \left[-2\ln(1) + 3\ln(2) + 5(2)^{-1}\right]$$

$$= -2\ln 3 + 0 + 5 + 0 - 3\ln 2 - \frac{5}{2}$$

$$= -2\ln 3 - 3\ln 2 + \frac{5}{2}$$

$$= \ln 3^{-2} - \ln 2^3 + \frac{5}{2}$$

$$= \ln\left(\frac{1}{9}\right) - \ln 8 + \frac{5}{2}$$

$$= \ln\left(\frac{\frac{1}{9}\times 9}{8\times 9}\right) + \frac{5}{2}$$

$$= \ln\frac{1}{72} + \frac{5}{2}$$

$$= -\ln 72 + \frac{5}{2}$$

$$= \frac{5}{2} - \ln 72 \quad \text{QED}$$

8 Let $f(x) = \dfrac{3 - 3x^2}{(2x+1)(x+2)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{3-3x^2}{(2x+1)(x+2)^2} \equiv \frac{A}{2x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$3 - 3x^2 \equiv A(x+2)^2 + B(2x+1)(x+2) + C(2x+1)$$

$x = -2$:

$3 - 3(-2)^2 = 0 + 0 + C(2(-2)+1)$

$3 - 12 = -3C$

$-9 = -3C$

$C = 3$

$x = \frac{-1}{2}$:

$3 - 3\left(\frac{-1}{2}\right)^2 = A\left(\frac{-1}{2}+2\right)^2 + 0 + 0$

$3 - \frac{3}{4} = A\left(\frac{3}{2}\right)^2$

$\frac{9}{4} = \frac{9}{4}A$

$A = 1$

$x = 0$:

$3 - 0 = A(2)^2 + B(1)(2) + C(1)$

$3 = 4A + 2B + C$

$3 = 4 + 2B + 3$

$3 = 7 + 2B$

$2B = -4$

$B = -2$

$$\rightarrow \frac{1}{2x+1} - \frac{2}{x+2} + \frac{3}{(x+2)^2}$$

8 Let $f(x) = \dfrac{3-3x^2}{(2x+1)(x+2)^2}$.

(b) Hence find the exact value of $\int_0^4 f(x)\,dx$, giving your answer in the form $a + b\ln c$, where a, b and c are integers. [5]

$$\int_0^4 \left(\frac{1}{2x+1} - \frac{2}{x+2} + \frac{3}{(x+2)^2}\right) dx$$

$$\frac{1}{2}\int_0^4 \frac{2}{2x+1}\,dx - 2\int_0^4 \frac{1}{x+2}\,dx + 3\int_0^4 (x+2)^{-2}\,dx$$

$$= \left[\frac{1}{2}\ln|2x+1| - 2\ln|x+2| - 3(x+2)^{-1}\right]_0^4$$

$$= \left[\frac{1}{2}\ln(9) - 2\ln(6) - 3(6)^{-1}\right] - \left[\frac{1}{2}\ln(1) - 2\ln(2) - 3(2)^{-1}\right]$$

$$= \frac{1}{2}\ln 9 - 2\ln 6 - \frac{3}{6} - 0 + 2\ln 2 + \frac{3}{2}$$

$$= \frac{1}{2}\ln 9 - 2\ln 6 + 2\ln 2 - \frac{1}{2} + \frac{3}{2}$$

$$= \ln 9^{\frac{1}{2}} - \ln 6^2 + \ln 2^2 + 1$$

$$= \ln 3 - \ln 36 + \ln 4 + 1$$

$$= \ln\left(\frac{3 \times 4}{36}\right) + 1$$

$$= \ln\left(\frac{1}{3}\right) + 1$$

$$= \ln 3^{-1} + 1$$

$$= -\ln 3 + 1$$

$$= 1 - \ln 3$$

7 Let $f(x) = \dfrac{2}{(2x-1)(2x+1)}$.

(a) Express $f(x)$ in partial fractions. [2]

$$\frac{2}{(2x-1)(2x+1)} \equiv \frac{A}{2x-1} + \frac{B}{2x+1}$$

$$2 \equiv A(2x+1) + B(2x-1)$$

$x = \frac{1}{2}$: $\quad 2 = A(2(\tfrac{1}{2})+1) + B(0)$

$$2 = 2A$$

$$A = 1$$

$x = \frac{-1}{2}$: $\quad 2 = A(0) + B(2(\tfrac{-1}{2})-1)$

$$2 = -2B$$

$$B = -1 \rightarrow f(x) = \frac{1}{2x-1} - \frac{1}{2x+1}$$

(b) Using your answer to part **(a)**, show that

$$(f(x))^2 = \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}.$$ [2]

$$(f(x))^2 = \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)\left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)$$

$$= \left(\frac{1}{2x-1}\right)^2 - 2\left(\frac{1}{2x-1}\right)\left(\frac{1}{2x+1}\right) + \left(\frac{1}{2x+1}\right)^2$$

$$= \frac{1}{(2x-1)^2} - \underbrace{\frac{2}{(2x-1)(2x+1)}}_{f(x)} + \frac{1}{(2x+1)^2}$$

$$= \frac{1}{(2x-1)^2} - \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right) + \frac{1}{(2x+1)^2}$$

$$= \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2} \quad \text{QED}$$

(c) Hence show that $\int_1^2 (f(x))^2\,dx = \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right)$. [5]

$$\int_1^2 \left(\frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}\right)dx$$

$$= \int_1^2 \left[(2x-1)^{-2} - \frac{1}{2x-1} + \frac{1}{2x+1} + (2x+1)^{-2}\right]dx$$

$$= \left[-\frac{1}{2}(2x-1)^{-1} - \frac{1}{2}\ln|2x-1| + \frac{1}{2}\ln|2x+1| - \frac{1}{2}(2x+1)^{-1}\right]_1^2$$

$$= \left[-\frac{1}{2}(2x-1)^{-1} + \frac{1}{2}\ln\left|\frac{2x+1}{2x-1}\right| - \frac{1}{2}(2x+1)^{-1}\right]_1^2$$

$$= \left[-\frac{1}{2}(2(2)-1)^{-1} + \frac{1}{2}\ln\left|\frac{2(2)+1}{2(2)-1}\right| - \frac{1}{2}(2(2)+1)^{-1}\right] - \left[-\frac{1}{2}(2(1)-1)^{-1} \ldots\right.$$

$$\left.\ldots + \frac{1}{2}\ln\left|\frac{2(1)+1}{2(1)-1}\right| - \frac{1}{2}(2(1)+1)^{-1}\right]$$

$$= \left[-\frac{1}{2}(3)^{-1} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{2}(5)^{-1}\right] - \left[-\frac{1}{2}(1)^{-1} + \frac{1}{2}\ln\left(\frac{3}{1}\right) - \frac{1}{2}(3)^{-1}\right]$$

$$= \left[-\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{2}\times\frac{1}{5}\right] - \left[-\frac{1}{2} + \frac{1}{2}\ln 3 - \frac{1}{2}\times\frac{1}{3}\right]$$

$$= \left[-\frac{1}{6} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{10}\right] - \left[-\frac{1}{2} + \frac{1}{2}\ln 3 - \frac{1}{6}\right]$$

$$= \cancel{-\frac{1}{6}} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{10} + \frac{1}{2} - \frac{1}{2}\ln 3 + \cancel{\frac{1}{6}}$$

$$= \frac{1}{2} - \frac{1}{10} + \frac{1}{2}\ln\left(\frac{5}{3}\right) - \frac{1}{2}\ln 3$$

$$= \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{3\times 3}\right)$$

$$= \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right) \quad \text{QED}$$

11 Let $f(x) = \dfrac{5x^2 + x + 11}{(4 + x^2)(1 + x)}$.

(a) Express $f(x)$ in partial fractions. **[5]**

$$\frac{5x^2 + x + 11}{(4 + x^2)(1 + x)} \equiv \frac{Ax + B}{4 + x^2} + \frac{C}{1 + x}$$

$$5x^2 + x + 11 \equiv (Ax + B)(1 + x) + C(4 + x^2)$$

$x = -1$:

$$5(-1)^2 + -1 + 11 = 0 + C(4 + (-1)^2)$$
$$5 - 1 + 11 = 5C$$
$$15 = 5C$$
$$C = 3$$

$x = 0$:

$$0 + 0 + 11 = (0 + B)(1 + 0) + C(4 + 0)$$
$$11 = B + 4C$$
$$11 = B + 12$$
$$B = -1$$

$x = 1$:

$$5(1)^2 + 1 + 11 = (A + B)(2) + C(4 + 1)$$
$$17 = 2A + 2B + 5C$$
$$17 = 2A - 2 + 15$$
$$17 = 2A + 13$$
$$4 = 2A$$
$$A = 2$$

$$\rightarrow \frac{2x - 1}{4 + x^2} + \frac{3}{1 + x}$$

(b) Hence show that $\int_0^2 f(x)\,dx = \ln 54 - \frac{1}{8}\pi$. [5]

$$\int_0^2 \left(\frac{2x-1}{4+x^2} + \frac{3}{1+x} \right) dx$$

$\int \frac{1}{x^2+a^2}$ with $a^2 = 4$

$$= \int_0^2 \left(\frac{2x}{4+x^2} - \frac{1}{4+x^2} + \frac{3}{1+x} \right) dx$$

$$= \left[\ln|4+x^2| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + 3\ln|1+x| \right]_0^2$$

$$= \left[\ln(8) - \frac{1}{2}\tan^{-1}(1) + 3\ln(3) \right] - \left[\ln(4) - \frac{1}{2}\tan^{-1}(0) + 3\ln(1) \right]$$

$$= \ln 8 - \frac{1}{2} \times \frac{\pi}{4} + 3\ln 3 - \ln 4 + 0 - 0$$

$$= \ln 8 - \frac{\pi}{8} + \ln 3^3 - \ln 4$$

$$= \ln 8 + \ln 27 - \ln 4 - \frac{\pi}{8}$$

$$= \ln\left(\frac{8 \times 27}{4}\right) - \frac{\pi}{8}$$

$$= \ln 54 - \frac{\pi}{8}$$ QED

9 Let $f(x) = \dfrac{7x+18}{(3x+2)(x^2+4)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{7x+18}{(3x+2)(x^2+4)} \equiv \frac{A}{3x+2} + \frac{Bx+C}{x^2+4}$$

$$7x+18 \equiv A(x^2+4) + (Bx+C)(3x+2)$$

$x = \frac{-2}{3}$:

$$7\left(\tfrac{-2}{3}\right) + 18 = A\left(\left(\tfrac{-2}{3}\right)^2 + 4\right) + 0$$

$$\tfrac{-14}{3} + 18 = A\left(\tfrac{4}{9} + 4\right)$$

$$\tfrac{40}{3} = \tfrac{40}{9}A$$

$$A = 3$$

$x = 0$:

$$7(0) + 18 = A(4) + C(2)$$

$$18 = 4A + 2C$$

$$18 = 12 + 2C$$

$$2C = 6$$

$$C = 3$$

$x = 1$:

$$7(1) + 18 = A(1^2+4) + (B+C)(3+2)$$

$$25 = 5A + (B+C)\times 5$$

$$25 = 5A + 5B + 5C$$

$$25 = 15 + 5B + 15$$

$$5B = -5$$

$$B = -1$$

$$f(x) = \frac{3}{3x+2} + \frac{-x+3}{x^2+4}$$

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$. [6]

$$\int_0^2 \left(\frac{3}{3x+2} + \frac{-x+3}{x^2+4} \right) dx$$

$3 \times \int \frac{1}{x^2+a^2}$ with $a^2 = 4$

$$= \int_0^2 \left(\frac{3}{3x+2} - \frac{x}{x^2+4} + \frac{3}{x^2+4} \right) dx$$

$$= \left[\ln|3x+2| - \frac{1}{2}\ln|x^2+4| + 3 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) \right]_0^2$$

$$= \left[\ln(8) - \frac{1}{2}\ln(8) + \frac{3}{2}\tan^{-1}(1) \right] - \left[\ln(2) - \frac{1}{2}\ln(4) + \frac{3}{2}\tan^{-1}(0) \right]$$

$$= \ln 8 - \frac{1}{2}\ln 8 + \frac{3}{2} \times \frac{\pi}{4} - \ln 2 + \frac{1}{2}\ln 4 - 0$$

$$= \ln 8 - \ln 2 + \frac{1}{2}\ln 4 - \frac{1}{2}\ln 8 + \frac{3\pi}{8}$$

$$= \ln\left(\frac{8}{2}\right) + \frac{1}{2}\ln\left(\frac{4}{8}\right) + \frac{3\pi}{8}$$

$$= \ln 4 + \frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{3\pi}{8}$$

$$= \ln 4 + \frac{1}{2}\ln 2^{-1} + \frac{3\pi}{8}$$

$$= \ln 4 - \frac{1}{2}\ln 2 + \frac{3\pi}{8}$$

$$= \ln 4 - \ln\sqrt{2} + \frac{3\pi}{8}$$

$$= \ln\left(\frac{4}{\sqrt{2}}\right) + \frac{3\pi}{8}$$

5 (a) Find the quotient and remainder when $2x^3 - x^2 + 6x + 3$ is divided by $x^2 + 3$. [3]

$$
\begin{array}{r|l}
 & 2x \quad -1 \\
\hline
x^2 + 0x + 3 & 2x^3 - x^2 + 6x + 3 \\
 & \underline{2x^3 + 0x^2 + 6x} \\
 & \quad -x^2 + 0x + 3 \\
 & \quad \underline{-x^2 + 0x - 3} \\
 & \qquad\qquad 6
\end{array}
$$

quotient

remainder

$$= 2x - 1 + \frac{6}{x^2 + 3}$$

(b) Using your answer to part **(a)**, find the exact value of $\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}\,dx$. [5]

from (a) : $\int_1^3 \left(2x - 1 + \frac{6}{x^2+3}\right) dx$

$= \int_1^3 (2x-1)\,dx + 6\int_1^3 \frac{1}{x^2+3}\,dx$ $\leftarrow \int \frac{1}{x^2+a^2}$ with $a^2 = 3$ so $a = \sqrt{3}$

$= \left[x^2 - x\right]_1^3 + 6\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^3$

$= \left[3^2 - 3\right] - \left[1^2 - 1\right] + 6\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{3}{\sqrt{3}}\right)\right] - 6\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$

$= \left[9 - 3\right] - \left[0\right] + 6\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{3}\right] - 6\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{6}\right]$

$= 6 + 6 \times \frac{\pi}{3\sqrt{3}} - 6 \times \frac{\pi}{6\sqrt{3}}$

$= 6 + \frac{2\pi}{\sqrt{3}} - \frac{\pi}{\sqrt{3}}$

$= \underline{6 + \frac{\pi}{\sqrt{3}}}$ $\left(\text{or } 6 + \frac{\sqrt{3}\pi}{3}\right)$

8 Let $f(x) = \dfrac{x^2 + 9x}{(3x-1)(x^2+3)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{x^2+9x}{(3x-1)(x^2+3)} \equiv \frac{A}{3x-1} + \frac{Bx+C}{x^2+3}$$

$$x^2 + 9x \equiv A(x^2+3) + (Bx+C)(3x-1)$$

$\underline{x=\frac{1}{3}:}$ $\left(\frac{1}{3}\right)^2 + 9\left(\frac{1}{3}\right) = A\left(\left(\frac{1}{3}\right)^2+3\right) + (Bx+C)(0)$

$$\frac{28}{9} = \frac{28}{9}A$$

$$\underline{A = 1}$$

$\underline{x=0:}$ $0 = 3A + C(-1)$

$$0 = 3(1) - C$$

$$\underline{C = 3}$$

$\underline{x=1:}$ $(1)^2 + 9(1) = A(4) + (B+C)(2)$

$$10 = 1\times 4 + (B+3)\times 2$$

$$10 = 4 + 2B + 6$$

$$10 = 10 + 2B$$

$$2B = 0$$

$$\underline{B = 0}$$

$$\rightarrow f(x) = \frac{1}{3x-1} + \frac{3}{x^2+3}$$

(b) Hence find $\int_1^3 f(x)\,dx$, giving your answer in a simplified exact form. [5]

$$\int_1^3 \left(\frac{1}{3x-1} + \frac{3}{x^2+3} \right) dx$$

$$= \int_1^3 \frac{1}{3x-1}\,dx + 3\int_1^3 \frac{1}{x^2+3}\,dx$$

$\int \frac{1}{x^2+a^2}$ with $a^2 = 3$

so $a = \sqrt{3}$

$$= \left[\frac{1}{3}\ln|3x-1|\right]_1^3 + 3\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^3$$

$$= \left[\frac{1}{3}\ln 8\right] - \left[\frac{1}{3}\ln 2\right] + 3\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)\right] - 3\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{3}\ln 8 - \frac{1}{3}\ln 2 + 3\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{3}\right] - 3\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{6}\right]$$

$$= \frac{1}{3}\ln 4 + 3 \times \frac{\pi}{3\sqrt{3}} - 3 \times \frac{\pi}{6\sqrt{3}}$$

$$= \frac{1}{3}\ln 4 + \frac{\pi}{\sqrt{3}} - \frac{\pi}{2\sqrt{3}}$$

$$= \frac{1}{3}\ln 4 + \frac{2\pi}{2\sqrt{3}} - \frac{\pi}{2\sqrt{3}}$$

$$= \frac{1}{3}\ln 4 + \frac{\pi}{2\sqrt{3}}$$

8 **(a)** Find the quotient and remainder when $8x^3 + 4x^2 + 2x + 7$ is divided by $4x^2 + 1$. [3]

$$
\begin{array}{r|l}
 & 2x + 1 \\
4x^2 + 0x + 1 & 8x^3 + 4x^2 + 2x + 7 \\
 & \underline{8x^3 + 0x^2 + 2x} \downarrow \\
 & \quad\quad 4x^2 + 0x + 7 \\
 & \quad\quad \underline{4x^2 + 0x + 1} \\
 & \quad\quad\quad\quad\quad\quad 6
\end{array}
$$

quotient remainder

$= 2x + 1 + \dfrac{6}{4x^2 + 1}$

(b) Hence find the exact value of $\int_0^{\frac{1}{2}} \frac{8x^3 + 4x^2 + 2x + 7}{4x^2 + 1}\,dx$. [5]

from (a): $\int_0^{\frac{1}{2}} \left(2x + 1 + \frac{6}{4x^2+1}\right) dx$

$= \int_0^{\frac{1}{2}} (2x+1)\,dx + 6\int_0^{\frac{1}{2}} \frac{1 \div 4}{4x^2+1 \div 4}\,dx$

$= \int_0^{\frac{1}{2}} (2x+1)\,dx + 6\int_0^{\frac{1}{2}} \frac{\frac{1}{4}}{x^2+\frac{1}{4}}\,dx$

$= \int_0^{\frac{1}{2}} (2x+1)\,dx + \frac{6}{4}\int_0^{\frac{1}{2}} \frac{1}{x^2+\frac{1}{4}}\,dx$ $\leftarrow \int \frac{1}{x^2+a^2}$ with $a^2 = \frac{1}{4}$

so $a = \frac{1}{2}$

$= \left[x^2 + x\right]_0^{\frac{1}{2}} + \frac{3}{2}\left[\frac{1}{(\frac{1}{2})}\tan^{-1}\left(\frac{x}{(\frac{1}{2})}\right)\right]_0^{\frac{1}{2}}$

$= \left[x^2 + x\right]_0^{\frac{1}{2}} + \frac{3}{2}\left[2\tan^{-1}(2x)\right]_0^{\frac{1}{2}}$

$= \left[\left(\frac{1}{2}\right)^2 + \frac{1}{2}\right] - [0] + \frac{3}{2}\left[2\tan^{-1}(1)\right] - \frac{3}{2}\left[2\tan^{-1}(0)\right]$

$= \frac{1}{4} + \frac{1}{2} + \frac{3}{2} \times 2 \times \frac{\pi}{4} - 0$

$= \underline{\frac{3}{4} + \frac{3\pi}{4}}$

10 Let $f(x) = \dfrac{4 - x + x^2}{(1+x)(2+x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{4 - x + x^2}{(1+x)(2+x^2)} \equiv \frac{A}{1+x} + \frac{Bx + C}{2+x^2}$$

$$4 - x + x^2 \equiv A(2+x^2) + (Bx+C)(1+x)$$

$x=-1$:

$$4 - (-1) + (-1)^2 = A(2 + (-1)^2) + 0$$

$$6 = 3A$$

$$A = 2$$

$x=0$:

$$4 - 0 + 0^2 = A(2) + C(1)$$

$$4 = 2A + C$$

$$4 = 2(2) + C$$

$$4 = 4 + C$$

$$C = 0$$

$x=1$:

$$4 - 1 + 1^2 = A(2+1^2) + (B(1) + C)(1+1)$$

$$4 = 3A + B(2)$$

$$4 = 3 \times 2 + 2B$$

$$4 = 6 + 2B$$

$$2B = -2$$

$$B = -1$$

$$f(x) = \frac{2}{1+x} - \frac{x}{2+x^2}$$

(b) Find the exact value of $\int_0^4 f(x)\,dx$. Give your answer as a single logarithm. [5]

$\int_0^4 \left(\frac{2}{1+x} - \frac{x}{2+x^2}\right) dx$

$= \left[2\ln|1+x| - \frac{1}{2}\ln|2+x^2|\right]_0^4$

$= \left[2\ln 5 - \frac{1}{2}\ln 18\right] - \left[2\ln 1 - \frac{1}{2}\ln 2\right]$

$= 2\ln 5 - \frac{1}{2}\ln 18 - 0 + \frac{1}{2}\ln 2$

$= 2\ln 5 + \frac{1}{2}\ln 2 - \frac{1}{2}\ln 18$

$= 2\ln 5 + \frac{1}{2}\ln\left(\frac{2}{18}\right)$

$= \ln 5^2 + \frac{1}{2}\ln\left(\frac{1}{9}\right)$

$= \ln 25 + \frac{1}{2}\ln 9^{-1}$

$= \ln 25 - \frac{1}{2}\ln 9$

$= \ln 25 - \ln 9^{\frac{1}{2}}$

$= \ln 25 - \ln 3$

$= \ln\left(\frac{25}{3}\right)$

11 Let $f(x) = \dfrac{5 - x + 6x^2}{(3 - x)(1 + 3x^2)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{5 - x + 6x^2}{(3-x)(1+3x^2)} \equiv \frac{A}{3-x} + \frac{Bx + C}{1 + 3x^2}$$

$$5 - x + 6x^2 \equiv A(1+3x^2) + (Bx+C)(3-x)$$

$x=3$:

$$5 - 3 + 6(3)^2 = A(1+3(3)^2) + 0$$

$$5 - 3 + 54 = 28A$$

$$56 = 28A$$

$$A = 2$$

$x=0$:

$$5 - 0 + 0 = A(1) + (C)(3)$$

$$5 = A + 3C$$

$$5 = 2 + 3C$$

$$3C = 3$$

$$C = 1$$

$x=1$:

$$5 - 1 + 6(1)^2 = A(1+3) + (B+C)(3-1)$$

$$10 = 4A + (B+C)\times 2$$

$$10 = 4A + 2B + 2C$$

$$10 = 8 + 2B + 2$$

$$10 = 10 + 2B$$

$$2B = 0$$

$$B = 0$$

$$f(x) = \frac{2}{3-x} + \frac{1}{1+3x^2}$$

(b) Find the exact value of $\int_0^1 f(x)\,dx$, simplifying your answer. [5]

$$\int_0^1 \left(\frac{2}{3-x} + \frac{1}{1+3x^2}\right) dx$$

$$= \int_0^1 \frac{2}{3-x}\,dx + \int_0^1 \frac{1 \div 3}{1+3x^2 \div 3}\,dx$$

$$= \int_0^1 \frac{2}{3-x}\,dx + \int_0^1 \frac{\frac{1}{3}}{\frac{1}{3}+x^2}\,dx$$

$$= \int_0^1 \frac{2}{3-x}\,dx + \frac{1}{3}\int_0^1 \frac{1}{\frac{1}{3}+x^2}\,dx$$

$\int \frac{1}{x^2+a^2}$ with $a^2 = \frac{1}{3}$ so $a = \frac{1}{\sqrt{3}}$

$$= \Big[-2\ln|3-x|\Big]_0^1 + \frac{1}{3}\Big[\sqrt{3}\tan^{-1}(\sqrt{3}x)\Big]_0^1$$

$$= \Big[-2\ln(2)\Big] - \Big[-2\ln(3)\Big] + \frac{1}{3}\Big[\sqrt{3}\tan^{-1}(\sqrt{3})\Big] - \frac{1}{3}\Big[\sqrt{3}\tan^{-1}(0)\Big]$$

$$= -2\ln 2 + 2\ln 3 + \frac{\sqrt{3}}{3} \times \frac{\pi}{3} - 0$$

$$= 2\ln 3 - 2\ln 2 + \frac{\pi\sqrt{3}}{9}$$

$$= \ln 3^2 - \ln 2^2 + \frac{\pi\sqrt{3}}{9}$$

$$= \ln 9 - \ln 4 + \frac{\pi\sqrt{3}}{9}$$

$$= \ln\left(\frac{9}{4}\right) + \frac{\pi\sqrt{3}}{9}$$

5 Find the exact value of $\displaystyle\int_0^6 \frac{x(x+1)}{x^2+4}\,dx$. [6]

$$\frac{x(x+1)}{x^2+4} = \frac{x^2+x}{x^2+4}$$ ← improper fraction, so use algebraic division:

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2+0x+4 & x^2+x+0 \\ & x^2+0x+4 \\ \hline & \quad x-4 \end{array}$$

$$\rightarrow 1 + \frac{x-4}{x^2+4}$$

$$\rightarrow \int_0^6 \left(1 + \frac{x}{x^2+4} - \frac{4}{x^2+4}\right) dx$$

$\displaystyle\int \frac{1}{x^2+a^2}$ with $a^2=4$ so $a=2$

$$= \int_0^6 1\,dx + \frac{1}{2}\int_0^6 \frac{2x}{x^2+4}dx - 4\int_0^6 \frac{1}{x^2+4}dx$$

$$= \left[x + \frac{1}{2}\ln|x^2+4| - 4\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^6$$

$$= \left[6 + \frac{1}{2}\ln(40) - 2\tan^{-1}(3)\right] - \left[0 + \frac{1}{2}\ln(4) - 2\tan^{-1}(0)\right]$$

$$= 6 + \frac{1}{2}\ln 40 - 2\tan^{-1}(3) - \frac{1}{2}\ln 4 - 0$$

$$= 6 + \ln 40^{\frac{1}{2}} - \ln 4^{\frac{1}{2}} - 2\tan^{-1}(3)$$

$$= 6 + \ln\left(\frac{40}{4}\right)^{\frac{1}{2}} - 2\tan^{-1}(3)$$

$$= 6 + \ln 10^{\frac{1}{2}} - 2\tan^{-1}(3)$$

$$= 6 + \frac{1}{2}\ln 10 - 2\tan^{-1}(3)$$