6 The equation $\cot \frac{1}{2}x = 3x$ has one root in the interval $0 < x < \pi$, denoted by α.

(a) Show by calculation that α lies between 0.5 and 1. [2]

Set = 0:

$\cot\frac{1}{2}x - 3x = 0$

$x = 0.5$:

$\cot\left(\frac{1}{2}\times 0.5\right) - 3(0.5) = 2.416$

$x = 1$:

$\cot\left(\frac{1}{2}\times 1\right) - 3(1) = -1.170$

Sign change (and continuous function) between $x = 0.5$ and $x = 1$ shows there's a root between 0.5 and 1.

(b) Show that, if a sequence of positive values given by the iterative formula

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$$

converges, then it converges to α. [2]

$x = \frac{1}{3}\left(x + 4\tan^{-1}\left(\frac{1}{3x}\right)\right)$

$3x = x + 4\tan^{-1}\left(\frac{1}{3x}\right)$

$2x = 4\tan^{-1}\left(\frac{1}{3x}\right)$

$\frac{x}{2} = \tan^{-1}\left(\frac{1}{3x}\right)$

$\tan\left(\frac{x}{2}\right) = \frac{1}{3x}$

$3x = \frac{1}{\tan\left(\frac{x}{2}\right)}$

$\underline{3x = \cot\left(\frac{x}{2}\right)}$ QED

(c) Use this iterative formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. **[3]**

$x_1 = 0.5000$

$x_2 = 0.9507$

$x_3 = 0.7665$

$x_4 = 0.8024$

$x_5 = 0.7924$

$x_6 = 0.7950$

$x_7 = 0.7944$

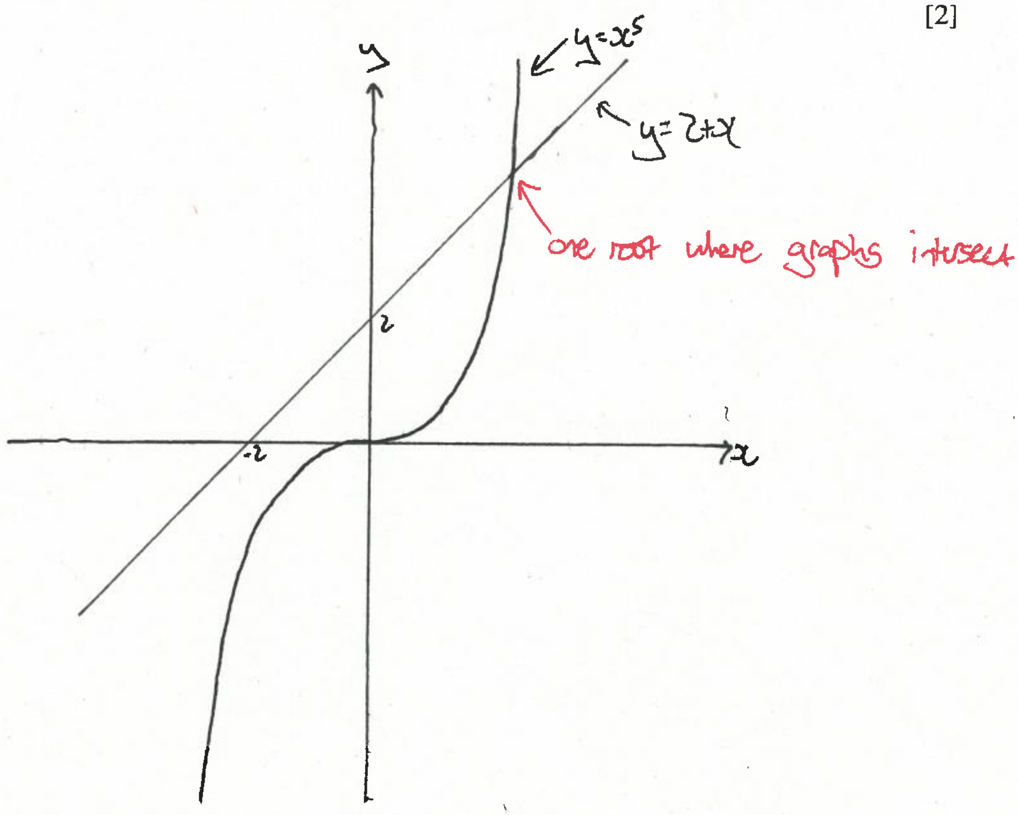
$x_8 = 0.7945$

$x_9 = 0.7945$

x_7, x_8 and x_9 all round to 0.79 to 2dp,

so $\underline{\alpha = 0.79}$

6 **(a)** By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root. [2]



(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part **(a)**. [2]

$$x = \frac{4x^5 + 2}{5x^4 - 1}$$

$$x(5x^4 - 1) = 4x^5 + 2$$

$$5x^5 - x = 4x^5 + 2$$

$-4x^5$ $-4x^5$

$$x^5 - x = 2$$

$$x^5 = 2 + x$$ QED

(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$x_1 = 1.5$

$x_2 = 1.33162$

$x_3 = 1.27352$

$x_4 = 1.26724$

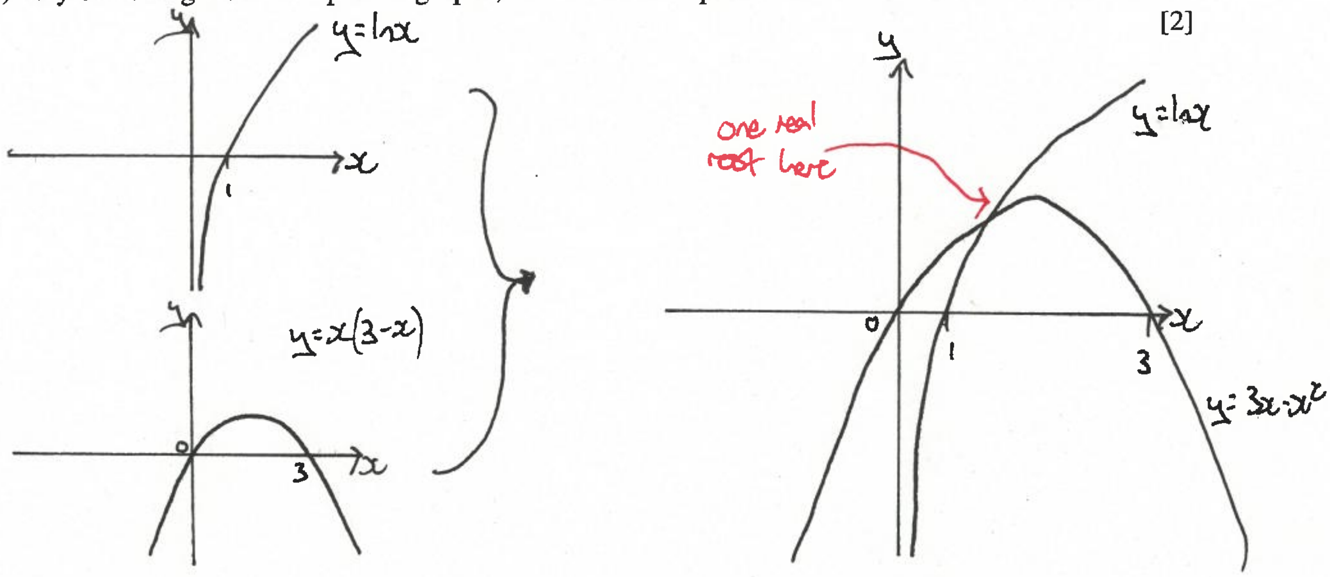
$x_5 = 1.26717$

$x_6 = 1.26717$

x_4, x_5 and x_6 all round to 1.267 to 3dp

so $x = 1.267$

5 **(a)** By sketching a suitable pair of graphs, show that the equation $\ln x = 3x - x^2$ has one real root. [2]



(b) Verify by calculation that the root lies between 2 and 2.8. [2]

Set $=0$: $\ln x + x^2 - 3x = 0$

$x=2$: $\ln 2 + 2^2 - 3(2) = -1.307$

$x=2.8$: $\ln 2.8 + 2.8^2 - 3(2.8) = 0.470$

Sign change between $x=2$ and $x=2.8$ (and given the function here is continuous) demonstrates there is a root between 2 and 2.8

(c) Use the iterative formula $x_{n+1} = \sqrt{3x_n - \ln x_n}$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 2.0000$

$x_2 = 2.3037$

$x_3 = 2.4651$

$x_4 = 2.5481$

$x_5 = 2.5902$

$x_6 = 2.6113$

$x_7 = 2.6218$

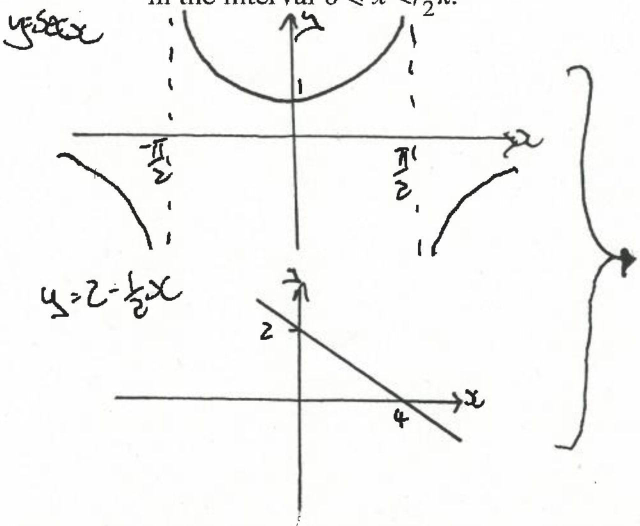
$x_8 = 2.6271$

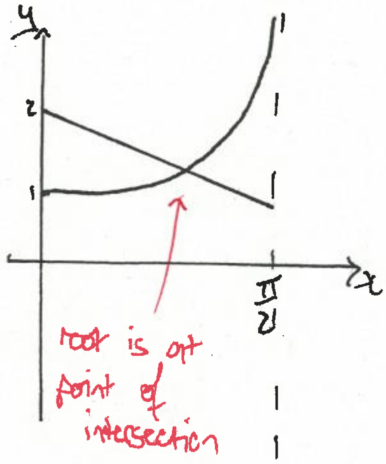
$x_9 = 2.6297$

$x_{10} = 2.6310$

x_8, x_9 and x_{10} all round to 2.63 to 2dp, so $x = 2.63$

3 **(a)** By sketching a suitable pair of graphs, show that the equation $\sec x = 2 - \frac{1}{2}x$ has exactly one root in the interval $0 \leqslant x < \frac{1}{2}\pi$. [2]





(b) Verify by calculation that this root lies between 0.8 and 1. [2]

Set $=0$: $\sec x + \frac{1}{2}x - 2 = 0$

$x = 0.8$: $\sec(0.8) + \frac{1}{2}(0.8) - 2 = -0.165$

$x = 1$: $\sec(1) + \frac{1}{2}(1) - 2 = 0.351$

Given there's a sign change between $x = 0.8$ and $x = 1$ (and the function is continuous between 0.8 and 1), this shows there's a root.

(c) Use the iterative formula $x_{n+1} = \cos^{-1}\left(\frac{2}{4 - x_n}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 0.8000$ $\quad x_6 = 0.8761$

$x_2 = 0.8957$

$x_3 = 0.8707$

$x_4 = 0.8774$

$x_5 = 0.8756$

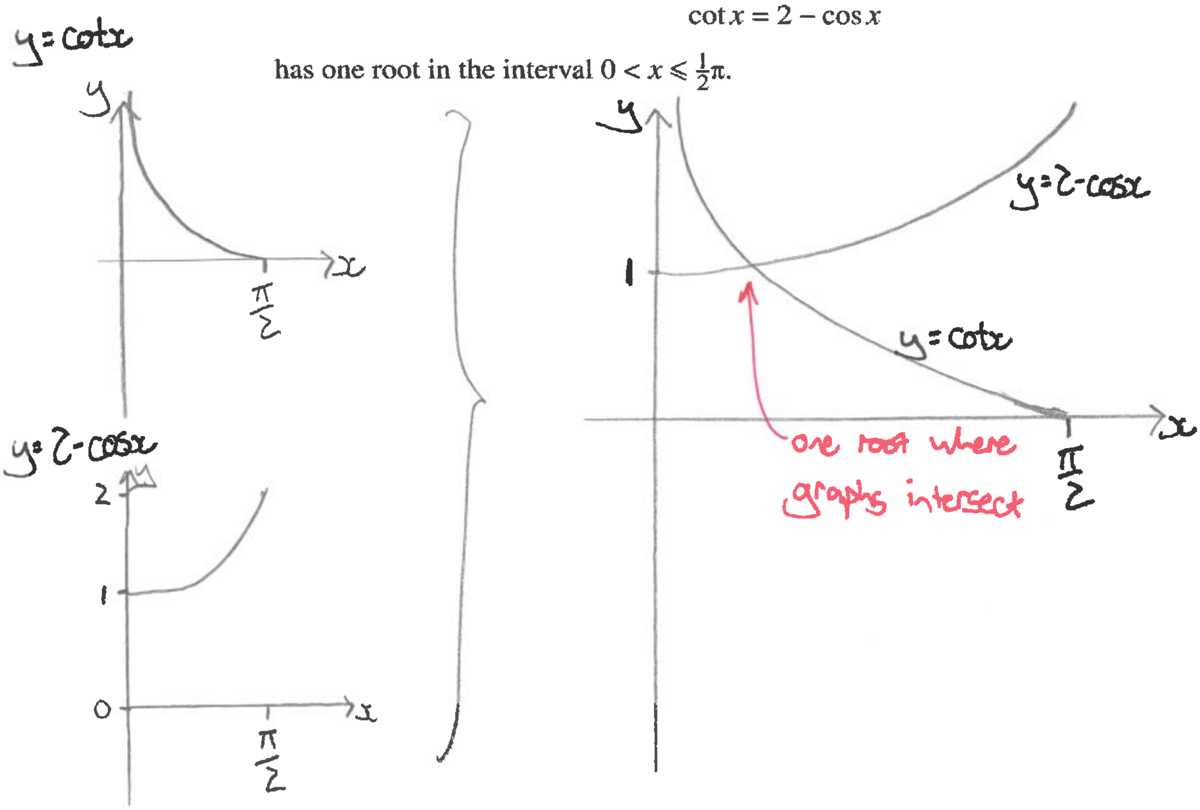
x_4, x_5 and x_6 all round to 0.88 to 2dp so

$x = 0.88$

6 **(a)** By sketching a suitable pair of graphs, show that the equation

$$\cot x = 2 - \cos x$$

has one root in the interval $0 < x \leqslant \frac{1}{2}\pi$. [2]



(b) Show by calculation that this root lies between 0.6 and 0.8. [2]

Set = 0:

$2 - \cos x - \cot x = 0$

$x = 0.6$:

$2 - \cos(0.6) - \cot(0.6) = -0.287$

$x = 0.8$:

$2 - \cos(0.8) - \cot(0.8) = 0.332$

Sign change (and continuous function) between $x = 0.6$ and $x = 0.8$ shows there's a root between 0.6 and 0.8.

(c) Use the iterative formula $x_{n+1} = \tan^{-1}\left(\frac{1}{2-\cos x_n}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 0.6000$

$x_2 = 0.7053$

$x_3 = 0.6792$

$x_4 = 0.6858$

$x_5 = 0.6842$

$x_6 = 0.6846$

$x_7 = 0.6845$

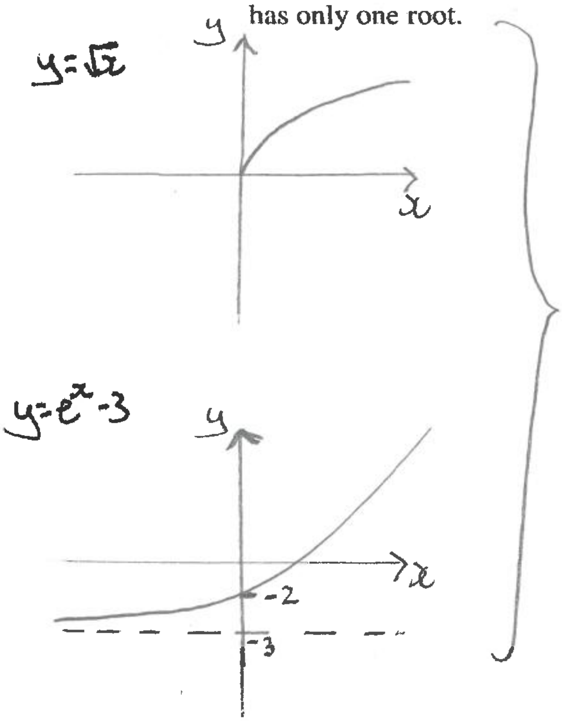
x_5, x_6 and x_7 all round to 0.68 to 2dp,

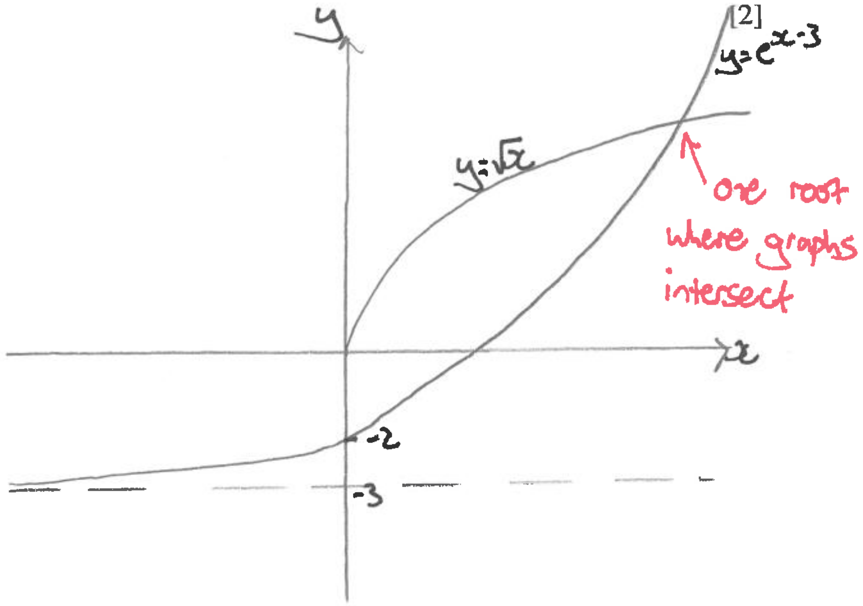
so $\underline{\underline{x = 0.68}}$

8 **(a)** By sketching a suitable pair of graphs, show that the equation

$$\sqrt{x} = e^x - 3$$

has only one root. [2]





(b) Show by calculation that this root lies between 1 and 2. [2]

Set =0:

$e^x - 3 - \sqrt{x} = 0$

$x = 1$:

$e^1 - 3 - \sqrt{1} = -1.282$

$x = 2$:

$e^2 - 3 - \sqrt{2} = 2.975$

Sign change (and continuous function) between $x=1$ and $x=2$ shows there's a root between 1 and 2.

(c) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \ln\left(3 + \sqrt{x_n}\right)$$

converges, then it converges to the root of the equation in **(a)**. [1]

$x = \ln(3 + \sqrt{x})$

$e^x = 3 + \sqrt{x}$

$\sqrt{x} = e^x - 3$ QED

(d) Use the iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 1.0000$

$x_2 = 1.3863$

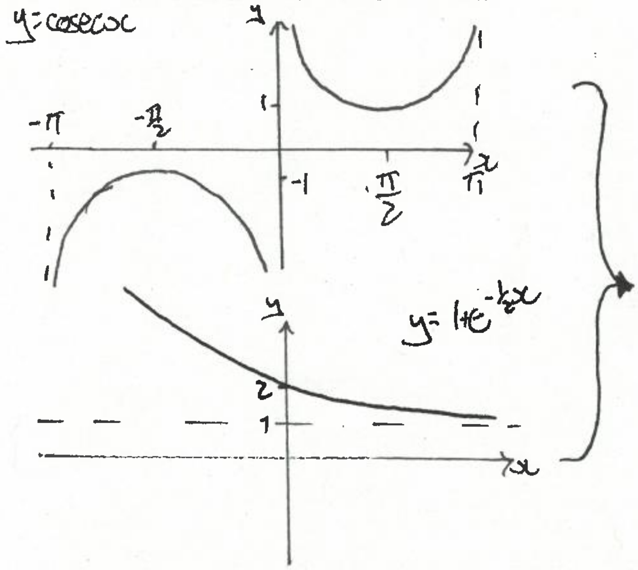
$x_3 = 1.4297$

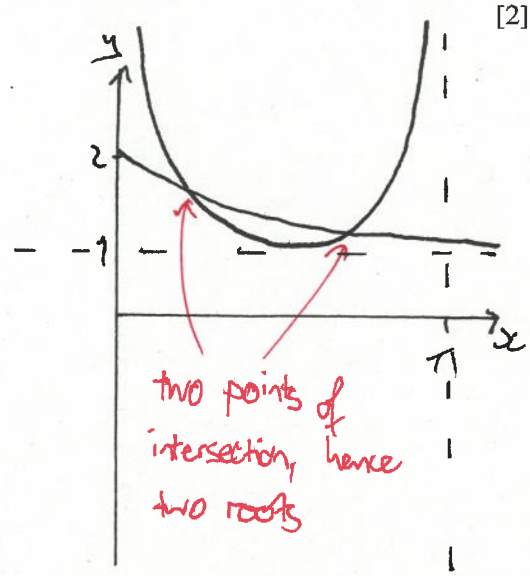
$x_4 = 1.4341$

$x_5 = 1.4345$

x_3, x_4 and x_5 all round to 1.43 to 2dp, so
$x = 1.43$

5 **(a)** By sketching a suitable pair of graphs, show that the equation $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$ has exactly two roots in the interval $0 < x < \pi$. [2]





(b) The sequence of values given by the iterative formula

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right),$$

with initial value $x_1 = 2$, converges to one of these roots.

Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 2.0000$

$x_2 = 2.3217$

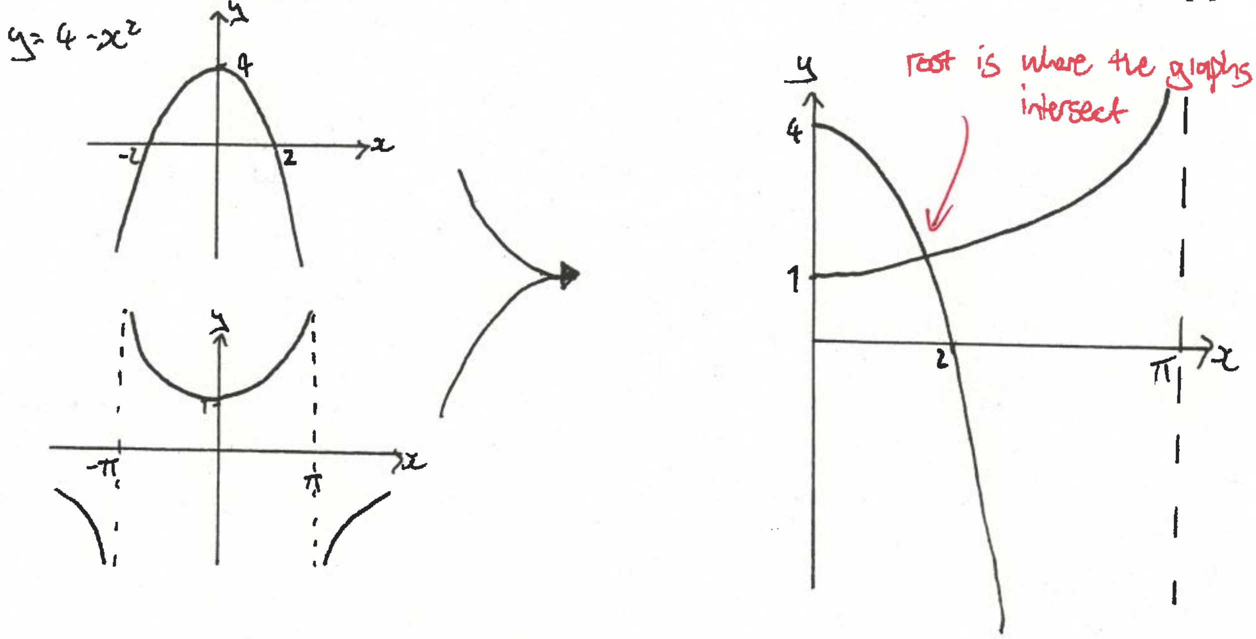
$x_3 = 2.2760$

$x_4 = 2.2824$

$x_5 = 2.2815$

x_3, x_4 and x_5 all round to 2.28 to 2dp, therefore $\underline{x = 2.28}$

7 (a) By sketching a suitable pair of graphs, show that the equation $4 - x^2 = \sec \frac{1}{2}x$ has exactly one root in the interval $0 \leqslant x < \pi$. [2]



(b) Verify by calculation that this root lies between 1 and 2. [2]

set = 0: $4 - x^2 - \sec(\frac{1}{2}x) = 0$

$\underline{x=1:}$ $4 - 1^2 - \sec(\frac{1}{2}) = 1.861$

$\underline{x=2:}$ $4 - 2^2 - \sec(1) = -1.851$

Given there's a sign change between $x=1$ and $x=2$, and the function is continuous between 1 and 2, this shows there's a root.

(c) Use the iterative formula $x_{n+1} = \sqrt{4 - \sec \frac{1}{2}x_n}$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 1.0000$

$x_2 = 1.6913$

$x_3 = 1.5787$

$x_4 = 1.6063$

$x_5 = 1.6000$

$x_6 = 1.6014$

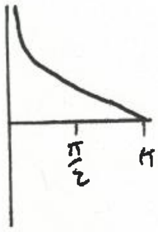
$x_7 = 1.6011$

x_5, x_6 and x_7 all round to 1.60 to 2dp so $\underline{x = 1.60}$

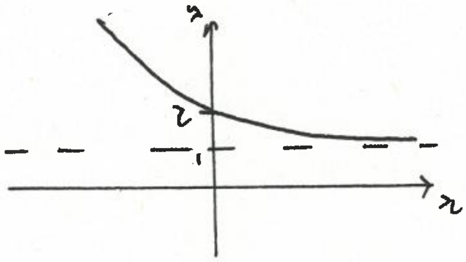
6 **(a)** By sketching a suitable pair of graphs, show that the equation $\cot \frac{1}{2}x = 1 + e^{-x}$ has exactly one root in the interval $0 < x \leqslant \pi$. [2]

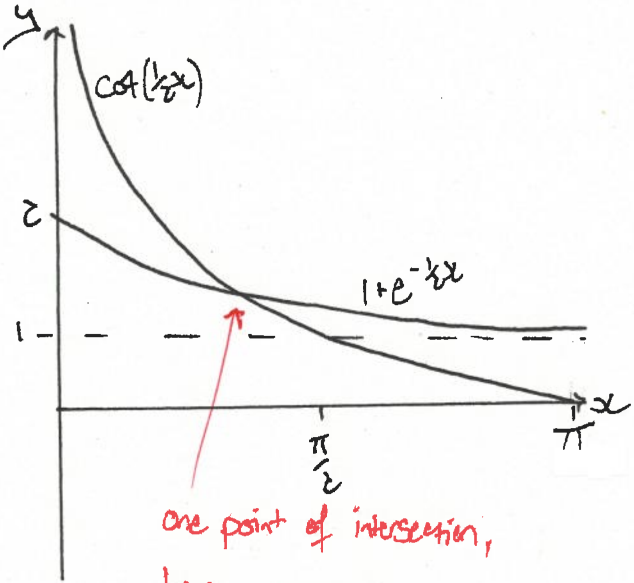
$y = \cot x$:

$y = \cot(\frac{1}{2}x)$:



$y = 1 + e^{-\frac{1}{2}x}$:





one point of intersection, hence one root

(b) Verify by calculation that this root lies between 1 and 1.5. [2]

Set $= 0$: $\cot(\frac{1}{2}x) - 1 - e^{-x} = 0$

$x = 1$: $\cot(\frac{1}{2}) - 1 - e^{-1} =$

$\frac{1}{\tan(\frac{1}{2})} - 1 - e^{-1} = 0.463$

$x = 1.5$: $\frac{1}{\tan(0.75)} - 1 - e^{-1.5} = -0.150$

Given there's a sign change (and the function is continuous) between 1 and 1.5, there must be a root between these values

(c) Use the iterative formula $x_{n+1} = 2\tan^{-1}\left(\dfrac{1}{1+e^{-x_n}}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 1.0000$

$x_2 = 1.2625$

$x_3 = 1.3242$

$x_4 = 1.3371$

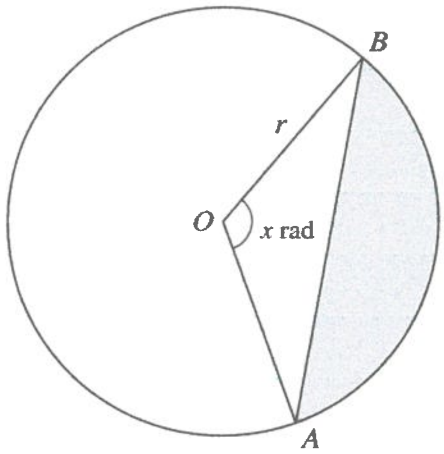
$x_5 = 1.3397$

$x_6 = 1.3402$

x_4, x_5 and x_6 all round to 1.34 to 2dp, therefore

$\underline{x = 1.34}$

7



The diagram shows a circle with centre O and radius r. The angle of the **minor** sector AOB of the circle is x radians. The area of the **major** sector of the circle is 3 times the area of the shaded region.

(a) Show that $x = \frac{3}{4}\sin x + \frac{1}{2}\pi$. [4]

Area of shaded region = ◁ − ◁

$= \frac{1}{2}r^2\theta - \frac{1}{2}ab\sin C$

$= \frac{1}{2}r^2x - \frac{1}{2}r^2\sin x$

Area of major sector:

Angle of major sector $= 2\pi - x$

Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}r^2(2\pi - x)$

Area of major sector $= 3 \times$ Shaded region

$\frac{1}{2}\cancel{r^2}(2\pi - x) = 3 \times (\frac{1}{2}\cancel{r^2}x - \frac{1}{2}\cancel{r^2}\sin x)$

$\frac{1}{2}(2\pi - x) = \frac{3}{2}x - \frac{3}{2}\sin x$

$2\pi - x = 3x - 3\sin x$

$2\pi + 3\sin x = 4x$

$4x = 2\pi + 3\sin x$

$x = \frac{\pi}{2} + \frac{3}{4}\sin x$ QED

(b) Show by calculation that the root of the equation in **(a)** lies between 2 and 2.5. [2]

set $=0$: $\frac{3}{4}\sin x + \frac{\pi}{2} - x = 0$

$x=2$: $\frac{3}{4}\sin 2 + \frac{\pi}{2} - 2 = 0.253$

$x=2.5$: $\frac{3}{4}\sin 2.5 + \frac{\pi}{2} - 2.5 = -0.480$

Sign change (and continuous function) between $x=2$ and $x=2.5$ shows there's a root between 2 and 2.5

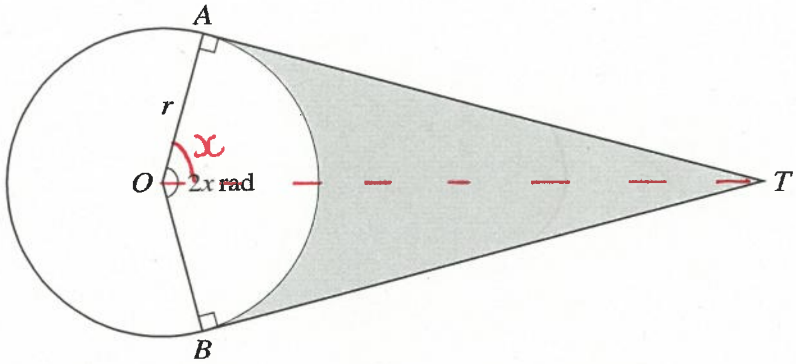
(c) Use an iterative formula based on the equation in **(a)** to calculate this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_{n+1} = \frac{3}{4}\sin x_n + \frac{\pi}{2}$

$x_1 = 2.0000$
$x_2 = 2.2528$
$x_3 = 2.1530$
$x_4 = 2.1972$
$x_5 = 2.1784$
$x_6 = 2.1866$
$x_7 = 2.1830$
$x_8 = 2.1846$
$x_9 = 2.1839$

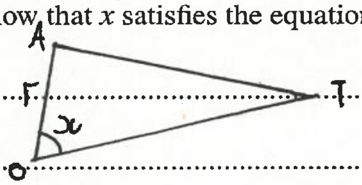
x_7, x_8 and x_9 all round to 2.18 to 2dp, so $\underline{x = 2.18}$

6



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The area of the shaded region is equal to the area of the circle.

(a) Show that x satisfies the equation $\tan x = \pi + x$. [3]



$\tan x = \frac{AT}{r}$

$\rightarrow AT = r\tan x$

Area of $\triangle ATO = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times r\tan x \times r$

$= \frac{1}{2}r^2 \tan x$

$\triangle ATO + \triangle BTO = \underline{r^2 \tan x}$

Area of minor sector $AOB = \frac{1}{2}r^2 2x$

$= \underline{r^2 x}$

Shaded Area $= r^2\tan x - r^2 x$

Shaded Area = Area of Circle

$\cancel{r^2}\tan x - \cancel{r^2}x = \pi \cancel{r^2}$

$\tan x - x = \pi$

$\underline{\tan x = \pi + x}$ QED

(b) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.4. [2]

Set $=0$: $\tan x - \pi - x = 0$

$x = 1$: $\tan(1) - \pi - 1 = -2.584$

$x = 1.4$: $\tan(1.4) - \pi - 1.4 = 1.256$

Given there's a sign change (and the function is continuous) between $x = 1$ and $x = 1.4$, there's a root between 1 and 1.4.

(c) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi + x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 1.0000$ $x_5 = 1.3518$

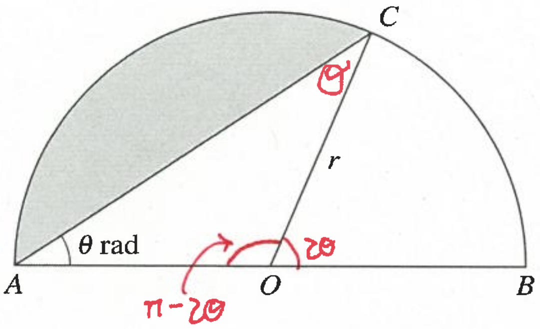
$x_2 = 1.3339$

$x_3 = 1.3510$

$x_4 = 1.3518$

x_3, x_4 and x_5 all round to 1.35 to 2dp, therefore $\underline{x = 1.35}$

9



The diagram shows a semicircle with diameter AB, centre O and radius r. The shaded region is the minor segment on the chord AC and its area is one third of the area of the semicircle. The angle CAB is θ radians.

(a) Show that $\theta = \frac{1}{3}(\pi - 1.5\sin 2\theta)$. [4]

$ACO = \theta$ (isosceles triangle)

$AOC = \pi - \theta - \theta = \pi - 2\theta$ (angles in a triangle)

$COB = \pi - (\pi - 2\theta) = 2\theta$ (angles on straight line)

area sector $AOC = \frac{1}{2}r^2(\pi - 2\theta)$

area triangle $AOC = \frac{1}{2} \times r \times r \times \sin(\pi - 2\theta)$ $\sin(A-B) = \sin A\cos B - \cos A \sin B$

$= \frac{1}{2}r^2[\sin\pi\cos 2\theta - \cos\pi\sin 2\theta]$

$= \frac{1}{2}r^2[0 - -\sin 2\theta]$

$= \frac{1}{2}r^2\sin 2\theta$

area of shaded segment $= \frac{1}{2}r^2(\pi - 2\theta) - \frac{1}{2}r^2\sin 2\theta$

area segment $= \frac{1}{3} \times$ area of semicircle

$\frac{1}{2}r^2(\pi - 2\theta) - \frac{1}{2}r^2\sin 2\theta = \frac{1}{3} \times \frac{\pi r^2}{2}$

$\times 6$ $\times 6$ $\times 6$

$3r^2(\pi - 2\theta) - 3r^2\sin 2\theta = \pi r^2$

$3(\pi - 2\theta) - 3\sin 2\theta = \pi$

$3\pi - 6\theta - 3\sin 2\theta = \pi$

$6\theta = 2\pi - 3\sin 2\theta$

$\theta = \frac{1}{6}(2\pi - 3\sin 2\theta)$ $\theta = \frac{1}{3}(\pi - 1.5\sin 2\theta)$ QED

(b) Verify by calculation that $0.5 < \theta < 0.7$. [2]

set $=0$: $\theta - \frac{1}{3}(\pi - 1.5\sin 2\theta) = 0$

$\theta = 0.5$: $0.5 - \frac{1}{3}(\pi - 1.5\sin(2(0.5))) = -0.126$

$\theta = 0.7$: $0.7 - \frac{1}{3}(\pi - 1.5\sin(2(0.7))) = 0.146$

Sign change (and continuous function) between 0.5 and 0.7 shows there's a root between $\theta = 0.5$ and $\theta = 0.7$.

(c) Use an iterative formula based on the equation in part **(a)** to determine θ correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$$\theta_{n+1} = \frac{1}{3}(\pi - 1.5\sin 2\theta_n)$$

$\theta_1 = 0.50000$

$\theta_2 = 0.62646$

$\theta_3 = 0.57225$

$\theta_4 = 0.59195$

$\theta_5 = 0.58416$

$\theta_6 = 0.58715$

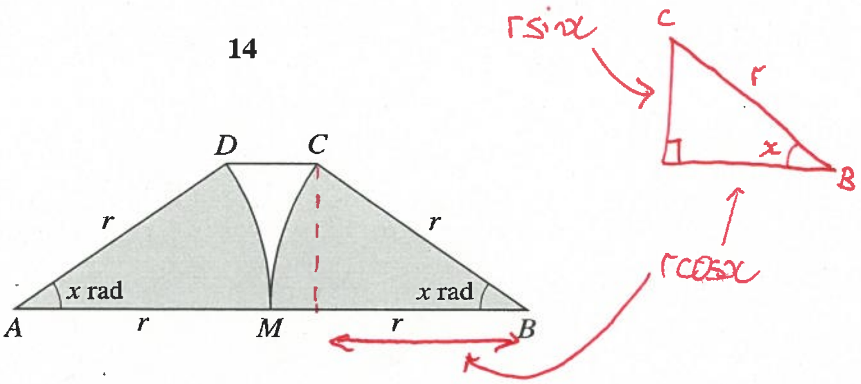
$\theta_7 = 0.58600$

$\theta_8 = 0.58644$

$\theta_9 = 0.58626$

θ_7, θ_8 and θ_9 all round to 0.586 to 3dp, so $\theta = 0.586$

10



The diagram shows a trapezium $ABCD$ in which $AD = BC = r$ and $AB = 2r$. The acute angles BAD and ABC are both equal to x radians. Circular arcs of radius r with centres A and B meet at M, the midpoint of AB.

(a) Given that the sum of the areas of the shaded sectors is 90% of the area of the trapezium, show that x satisfies the equation $x = 0.9(2 - \cos x)\sin x$. [3]

Trapezium: $DC = 2r - 2r\cos x$

Area $= \frac{1}{2}(2r + 2r - 2r\cos x) \times r\sin x$

$= \frac{1}{2}(4r - 2r\cos x) \times r\sin x$

$= (2r - r\cos x) \times r\sin x$

$= r^2 \sin x(2 - \cos x)$

Area of one sector $= \frac{1}{2}r^2 x$

" " both sectors $= r^2 x$

Area of sectors $= 0.9 \times$ Area of Trapezium

$\cancel{r^2} x = 0.9\cancel{r^2}\sin x(2 - \cos x)$

$x = 0.9\sin x(2 - \cos x)$ QED

(b) Verify by calculation that x lies between 0.5 and 0.7. [2]

Set = 0: $x - 0.9\sin x(2 - \cos x) = 0$

$x = 0.5$: $0.5 - 0.9\sin 0.5(2 - \cos 0.5) = 0.0157$

$x = 0.7$: $0.7 - 0.9\sin 0.7(2 - \cos 0.7) = -0.0161$

Given there's a sign change (and the function is continuous) between 0.5 and 0.7, there's a root between $x = 0.5$ and $x = 0.7$.

(c) Show that if a sequence of values in the interval $0 < x < \frac{1}{2}\pi$ given by the iterative formula

$$x_{n+1} = \cos^{-1}\left(2 - \frac{x_n}{0.9\sin x_n}\right)$$

converges, then it converges to the root of the equation in part **(a)**. [2]

$x = \cos^{-1}\left(2 - \frac{x}{0.9\sin x}\right)$

$\cos x = 2 - \frac{x}{0.9\sin x}$

$0.9\sin x\cos x = 0.9\times 2\sin x - x$

$x = 0.9\times 2\sin x - 0.9\sin x\cos x$

$x = 0.9\sin x(2-\cos x)$ QED

(d) Use this iterative formula to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 0.5000$

$x_2 = 0.5713$

$x_3 = 0.5987$

$x_4 = 0.6100$

$x_5 = 0.6149$

$x_6 = 0.6169$

$x_7 = 0.6178$

$x_8 = 0.6181$

x_6, x_7 and x_8 all round to 0.62 to 2dp so

$x = 0.62$