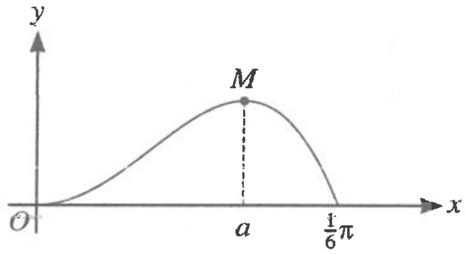
5




u v

The diagram shows the part of the curve $y = x^2 \cos 3x$ for $0 \leqslant x \leqslant \frac{1}{6}\pi$, and its maximum point $M$, where $x = a$.

(a) Show that $a$ satisfies the equation $a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$. [3]

$u = x^2 \qquad v = \cos 3x$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = -3\sin 3x$

$\frac{dy}{dx} = -3x^2 \sin 3x + 2x\cos 3x$

SP, so $\frac{dy}{dx} = 0$:

$-3x^2 \sin 3x + 2x\cos 3x = 0$

$2\cancel{x}\cos 3x = 3x^{\cancel{2}}\sin 3x$

$2\cos 3x = 3x \sin 3x$

$2 = 3x \times \frac{\sin 3x}{\cos 3x}$

$\frac{2}{3x} = \frac{\sin 3x}{\cos 3x}$

$\tan 3x = \frac{2}{3x}$

$3x = \tan^{-1}\left(\frac{2}{3x}\right)$

$x = \frac{1}{3}\tan^{-1}\left(\frac{2}{3x}\right)$

$a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$ QED

**(b)** Use an iterative formula based on the equation in **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a_n}\right)$$

$a_1 = \frac{\pi}{6}$

$a_2 = 0.3017$

$a_3 = 0.3820$

$a_4 = 0.3502$

$a_5 = 0.3624$

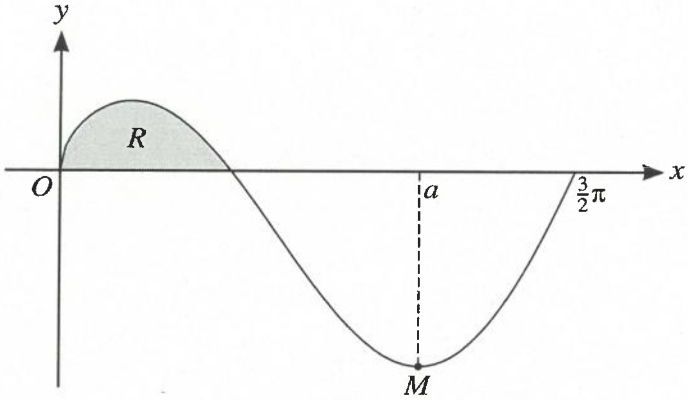
$a_6 = 0.3576$

$a_7 = 0.3595$

$a_5$, $a_6$ and $a_7$ all round to 0.36 to 2dp,

so $\underline{a = 0.36}$

10




The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point $M$, where $x = a$. The shaded region between the curve and the $x$-axis is denoted by $R$.

u v

**(a)** Show that $a$ satisfies the equation $\tan a = \dfrac{1}{2a}$. [3]

$u = x^{\frac{1}{2}}$  $v = \cos x$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$  $\frac{dv}{dx} = -\sin x$

$\frac{dy}{dx} = -x^{\frac{1}{2}}\sin x + \frac{1}{2}x^{-\frac{1}{2}}\cos x$

$\frac{dy}{dx} = 0$: $-\sqrt{x}\sin x + \frac{1}{2}\frac{\cos x}{\sqrt{x}} = 0$

$\times\sqrt{x}$  $\times\sqrt{x}$  $\times\sqrt{x}$

$-x\sin x + \frac{1}{2}\cos x = 0$

$x\sin x = \frac{1}{2}\cos x$

$\sin x = \frac{1}{2x}\cos x$

$\frac{\sin x}{\cos x} = \frac{1}{2x}$

$\tan a = \frac{1}{2a}$ QED

**(b)** The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\dfrac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to $a$.

Use this formula to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$a_1 = 3.0000$

$a_2 = 3.3067$

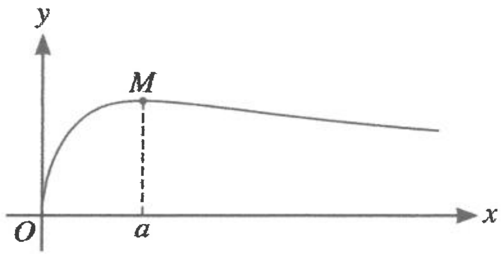
$a_3 = 3.2917$

$a_4 = 3.2923$

$a_5 = 3.2923$

$a_3$, $a_4$ and $a_5$ all round to 3.29 to 2dp, so $x = 3.29$

7




The diagram shows the curve $y = \frac{\tan^{-1} x}{\sqrt{x}}$ and its maximum point $M$ where $x = a$.

**(a)** Show that $a$ satisfies the equation

$$a = \tan\left(\frac{2a}{1+a^2}\right).$$ [4]

$$y = \frac{\tan^{-1}x}{\sqrt{x}} \quad \begin{matrix} u \\ v \end{matrix}$$

$u = \tan^{-1}x \qquad V = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{1+x^2} \qquad \frac{dV}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}} \times \frac{1}{1+x^2} - \frac{1}{2}x^{-\frac{1}{2}}\tan^{-1}x}{(x^{\frac{1}{2}})^2}$$

$$\frac{dy}{dx} = 0: \quad \frac{\frac{\sqrt{x}}{1+x^2} - \frac{1}{2} \times \frac{\tan^{-1}x}{\sqrt{x}}}{x} = 0$$

$$\frac{\sqrt{x}}{1+x^2} - \frac{\tan^{-1}x}{2\sqrt{x}} = 0$$

$$\frac{\tan^{-1}x}{2\sqrt{x}} = \frac{\sqrt{x}}{1+x^2}$$

$$\tan^{-1}x = \frac{2x}{1+x^2}$$

$$x = \tan\left(\frac{2x}{1+x^2}\right) \quad \text{QED}$$

**(b)** Verify by calculation that $a$ lies between 1.3 and 1.5. [2]

Set $=0$: $\tan\left(\frac{2a}{1+a^2}\right) - a = 0$

$a = 1.3$: $\tan\left(\frac{2\times1.3}{1+1.3^2}\right) - 1.3 = 0.148$

$a = 1.5$: $\tan\left(\frac{2\times1.5}{1+1.5^2}\right) - 1.5 = -0.178$

Given there's a sign change (and the function is continuous) between $x = 1.3$ and $x = 1.5$, there is a root between these two values.

**(c)** Use an iterative formula based on the equation in part **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \tan\left(\frac{2a_n}{1+a_n^2}\right)$$

$a_1 = 1.3000$

$a_2 = 1.4484$

$a_3 = 1.3552$

$a_4 = 1.4148$

$a_5 = 1.3769$

$a_6 = 1.4012$

$a_7 = 1.3857$

$a_8 = 1.3956$

$a_9 = 1.3893$

$a_{10} = 1.3933$

$a_{11} = 1.3907$

$a_9$, $a_{10}$ and $a_{11}$ all round to 1.39 to 2dp, so $x = 1.39$

**8** The curve with equation $y = \dfrac{x^3}{e^x - 1}$ has a stationary point at $x = p$, where $p > 0$.

**(a)** Show that $p = 3(1 - e^{-p})$. [3]

$$y = \frac{x^3}{e^x - 1} \quad \begin{matrix} u \\ v \end{matrix}$$

$u = x^3 \qquad\qquad v = e^x - 1$

$\dfrac{du}{dx} = 3x^2 \qquad\qquad \dfrac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{3x^2(e^x - 1) - x^3e^x}{(e^x - 1)^2}$$

$\dfrac{dy}{dx} = 0$ when $x = p$: $\dfrac{3p^2(e^p - 1) - p^3e^p}{(e^p - 1)^2} = 0$

$$3p^2(e^p - 1) - p^3e^p = 0$$

$$3p^2(e^p - 1) = p^3e^p$$

$$3(e^p - 1) = pe^p$$

$$pe^p = 3(e^p - 1)$$

$$p = \frac{3e^p - 3}{e^p}$$

$$p = \frac{3e^p}{e^p} - \frac{3}{e^p}$$

$$p = 3 - 3e^{-p}$$

$$\underline{p = 3(1 - e^{-p})} \quad \text{QED}$$

**(b)** Verify by calculation that $p$ lies between 2.5 and 3. [2]

set $=0$: $p - 3(1 - e^{-p}) = 0$

$p = 2.5$: $2.5 - 3(1 - e^{-2.5}) = -0.254$

$p = 3$: $3 - 3(1 - e^{-3}) = 0.149$

Sign change (and continuous function) between 2.5 and 3 shows there's a root between $x = 2.5$ and $x = 3$.

**(c)** Use an iterative formula based on the equation in part **(a)** to determine $p$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$p_{n+1} = 3(1 - e^{-p_n})$

$p_1 = 2.5000$

$p_2 = 2.7537$

$p_3 = 2.8089$

$p_4 = 2.8192$

$p_5 = 2.8210$

$p_6 = 2.8214$

$p_4$, $p_5$ and $p_6$ all round to 2.82 to 2dp

so $p = 2.82$

**9** Let $f(x) = \dfrac{e^{2x} + 1}{e^{2x} - 1}$, for $x > 0$.

**(a)** The equation $x = f(x)$ has one root, denoted by $a$.

Verify by calculation that $a$ lies between 1 and 1.5. [2]

$x = f(x)$: $x = \dfrac{e^{2x} + 1}{e^{2x} - 1}$

Set $= 0$: $x(e^{2x} - 1) = e^{2x} + 1$

$x(e^{2x} - 1) - e^{2x} - 1 = 0$

$x = 1$: $1(e^{2(1)} - 1) - e^{2(1)} - 1 = -2$

$x = 1.5$: $1.5(e^{2(1.5)} - 1) - e^{2(1.5)} - 1 = 7.543$

Sign change (and continuous function) between 1 and 1.5 shows there's a root between $x = 1$ and $x = 1.5$.

**(b)** Use an iterative formula based on the equation in part **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$a_{n+1} = \dfrac{e^{2a_n} + 1}{e^{2a_n} - 1}$

$a_1 = 1.0000$
$a_2 = 1.3130$
$a_3 = 1.1560$
$a_4 = 1.2199$
$a_5 = 1.1910$
$a_6 = 1.2035$
$a_7 = 1.1980$
$a_8 = 1.2004$

$a_6$, $a_7$ and $a_8$ all round to 1.20 to 2dp, so $a = 1.20$

**(c)** Find $f'(x)$. Hence find the exact value of $x$ for which $f'(x) = -8$. **[6]**

$$f(x) = \frac{e^{2x} + 1}{e^{2x} - 1} \quad \begin{matrix} u \\ v \end{matrix}$$

$u = e^{2x} + 1$ $\qquad$ $v = e^{2x} - 1$

$\frac{du}{dx} = 2e^{2x}$ $\qquad$ $\frac{dv}{dx} = 2e^{2x}$

$$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1)}{(e^{2x} - 1)^2}$$

$f'(x) = -8$: $\quad \frac{2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1)}{(e^{2x} - 1)^2} = -8$

$$2e^{2x}(e^{2x} - 1) - 2e^{2x}(e^{2x} + 1) = -8(e^{2x} - 1)^2$$

$$\cancel{2e^{4x}} - 2e^{2x} - \cancel{2e^{4x}} - 2e^{2x} = -8(e^{4x} - 2e^{2x} + 1)$$

$$-4e^{2x} = -8e^{4x} + 16e^{2x} - 8$$

$$8e^{4x} - 20e^{2x} + 8 = 0$$

$$2e^{4x} - 5e^{2x} + 2 = 0$$

let $y = e^{2x}$: $\quad 2y^2 - 5y + 2 = 0$

$$(2y - 1)(y - 2) = 0$$

$2y - 1 = 0$ or $y - 2 = 0$

$y = \frac{1}{2}$ or $y = 2$

$e^{2x} = \frac{1}{2}$ or $e^{2x} = 2$

$2x = \ln(\frac{1}{2})$ or $2x = \ln 2$

$x = \frac{1}{2}\ln(\frac{1}{2})$ $\qquad$ $\underline{\underline{x = \frac{1}{2}\ln 2}}$

negative, but $x > 0$
so reject solution.

I L A T E

**8** The constant $a$ is such that $\displaystyle\int_1^a \frac{\ln x}{\sqrt{x}}\,dx = 6$.

**(a)** Show that $a = \exp\left(\frac{1}{\sqrt{a}} + 2\right)$. [5]

[$\exp(x)$ is an alternative notation for $e^x$.]

$$\int_1^a x^{-\frac{1}{2}} \ln x \, dx = 6$$

$u = \ln x \qquad v = 2x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^{-\frac{1}{2}}$

$= 2x^{\frac{1}{2}} \ln x - \int_1^a 2x^{\frac{1}{2}} \times \frac{1}{x}\, dx$

$= 2x^{\frac{1}{2}} \ln x - 2\int_1^a x^{-\frac{1}{2}}\, dx$

$= \left[2x^{\frac{1}{2}} \ln x - 4x^{\frac{1}{2}}\right]_1^a$

$= \left[2a^{\frac{1}{2}} \ln a - 4a^{\frac{1}{2}}\right] - [0 - 4]$

$= 2\sqrt{a}\ln a - 4\sqrt{a} + 4 = 6$

$2\sqrt{a}\ln a - 4\sqrt{a} = 2$

$2\sqrt{a}\ln a = 2 + 4\sqrt{a}$

$\sqrt{a}\ln a = 1 + 2\sqrt{a}$

$\ln a = \frac{1}{\sqrt{a}} + 2$

$a = \exp\left(\frac{1}{\sqrt{a}} + 2\right)$ QED

**(b)** Verify by calculation that $a$ lies between 9 and 11. [2]

Set $=0$: $a - \exp\left(\frac{1}{\sqrt{a}} + 2\right) = 0$

$a=9$: $9 - \exp\left(\frac{1}{\sqrt{9}} + 2\right) = -1.312$

$a=11$: $11 - \exp\left(\frac{1}{\sqrt{11}} + 2\right) = 1.011$

Sign change (and continuous function) between 9 and 11 shows there's a root between $x=9$ and $x=11$.

**(c)** Use an iterative formula based on the equation in part **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$a_{n+1} = \exp\left(\frac{1}{\sqrt{a_n}} + 2\right)$

$a_1 = 9.0000$

$a_2 = 10.3123$

$a_3 = 10.0886$

$a_4 = 10.1233$

$a_5 = 10.1178$

$a_6 = 10.1187$

$a_4$, $a_5$ and $a_6$ all round to 10.12 to 2dp, therefore $\underline{\underline{x = 10.12}}$

**11** The equation of a curve is $y = \sqrt{\tan x}$, for $0 \leqslant x < \frac{1}{2}\pi$.

**(a)** Express $\frac{dy}{dx}$ in terms of $\tan x$, and verify that $\frac{dy}{dx} = 1$ when $x = \frac{1}{4}\pi$. [4]

$y = (\tan x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(\tan x)^{-\frac{1}{2}} \times \sec^2 x$ ← $\sec^2 x = 1 + \tan^2 x$

$= \frac{1}{2}(\tan x)^{-\frac{1}{2}} \times (1 + \tan^2 x)$

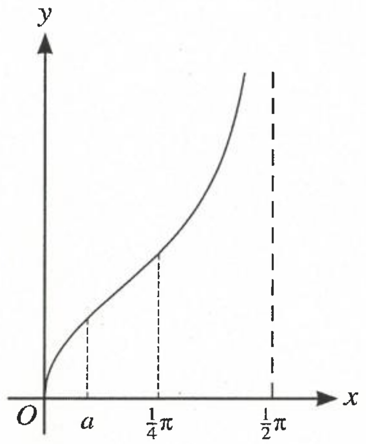
$x = \frac{\pi}{4}$ : $\frac{dy}{dx} = \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right)\right)^{-\frac{1}{2}} \times \left(1 + \tan^2\left(\frac{\pi}{4}\right)\right)$

$= \frac{1}{2}(1)^{-\frac{1}{2}} \times (1 + 1^2)$

$= \frac{1}{2} \times 1 \times 2$

$= \underline{1}$

The value of $\frac{dy}{dx}$ is also 1 at another point on the curve where $x = a$, as shown in the diagram.




**(b)** Show that $t^3 + t^2 + 3t - 1 = 0$, where $t = \tan a$. [4]

$\frac{1}{2}(\tan a)^{-\frac{1}{2}} \times (1 + \tan^2 a) = 1$

$\frac{1}{2\sqrt{\tan a}} \times (1 + \tan^2 a) = 1$

$\frac{1}{2\sqrt{t}}(1 + t^2) = 1$

$1 + t^2 = 2\sqrt{t}$

$(1 + t^2)^2 = (2\sqrt{t})^2$

$1 + 2t^2 + t^4 = 4t$

$t^4 + 2t^2 - 4t + 1 = 0$

from part (a): $\tan\left(\frac{\pi}{4}\right)$ is a root:

when $a = \frac{\pi}{4}$ $\quad t = \tan\left(\frac{\pi}{4}\right)$

$\underline{t = 1}$ so $(t-1)$ is a factor:

$$\begin{array}{r|l} & t^3 + t^2 + 3t - 1 \\ \hline t-1 & t^4 + 0t^3 + 2t^2 - 4t + 1 \\ & \underline{t^4 - t^3} \\ & \quad t^3 + 2t^2 \\ & \quad \underline{t^3 - t^2} \\ & \qquad 3t^2 - 4t \\ & \qquad \underline{3t^2 - 3t} \\ & \qquad\quad -t + 1 \\ & \qquad\quad \underline{-t + 1} \end{array}$$

$(t-1)(t^3 + t^2 + 3t - 1) = 0$

so $\underline{t^3 + t^2 + 3t - 1 = 0}$

QED

**(c)** Use the iterative formula

$$a_{n+1} = \tan^{-1}\left(\tfrac{1}{3}(1 - \tan^2 a_n - \tan^3 a_n)\right)$$

to determine $a$ correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

from the graph in part (b), $a$ is between $0$ and $\frac{\pi}{4}$.

$a_1 = 0.0000$

$a_2 = 0.3218$

$a_3 = 0.2767$

$a_4 = 0.2904$

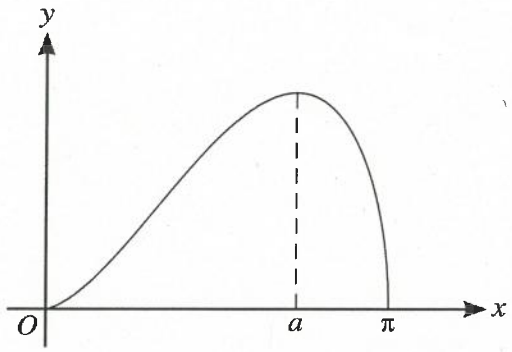
$a_5 = 0.2866$

$a_6 = 0.2877$

$a_4$, $a_5$ and $a_6$ all round to $0.29$ to 2dp, so

$\underline{a = 0.29}$

10




The curve $y = x\sqrt{\sin x}$ has one stationary point in the interval $0 < x < \pi$, where $x = a$ (see diagram).

**(a)** Show that $\tan a = -\frac{1}{2}a$. [4]

$y = \overset{u}{x}\overset{v}{\sqrt{\sin x}}$

$u = x \qquad v = (\sin x)^{\frac{1}{2}}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2}(\sin x)^{-\frac{1}{2}} \times \cos x$

$\frac{dy}{dx} = \frac{1}{2}x(\sin x)^{-\frac{1}{2}}\cos x + (\sin x)^{\frac{1}{2}}$

$\frac{dy}{dx} = 0: \quad \frac{1}{2}x\frac{\cos x}{\sqrt{\sin x}} + \sqrt{\sin x} = 0$

$\times\sqrt{\sin x} \qquad \times\sqrt{\sin x} \qquad \times\sqrt{\sin x}$

$\frac{1}{2}x\cos x + \sin x = 0$

$\sin x = -\frac{1}{2}x\cos x$

$\frac{\sin x}{\cos x} = -\frac{1}{2}x$

$\tan x = -\frac{1}{2}x$

or $\underline{\tan a = -\frac{1}{2}a}$ QED

**(b)** Verify by calculation that $a$ lies between 2 and 2.5. [2]

Set $=0$: $\tan x + \frac{1}{2}x = 0$

$x=2$: $\tan(2) + \frac{1}{2}(2) = -1.185$

$x=2.5$: $\tan(2.5) + \frac{1}{2}(2.5) = 0.503$

Given there's a sign change (and the function is continuous) between $x=2$ and $x=2.5$, there's a root between $x=2$ and $x=2.5$.

**(c)** Show that if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula $x_{n+1} = \pi - \tan^{-1}(\frac{1}{2}x_n)$ converges, then it converges to $a$, the root of the equation in part **(a)**. [2]

$x = \pi - \tan^{-1}(\frac{1}{2}x)$

$\tan^{-1}(\frac{1}{2}x) = \pi - x$

$\frac{1}{2}x = \tan(\pi - x)$

$\frac{1}{2}x = \dfrac{\tan\pi - \tan x}{1 + \tan\pi \tan x}$

$\frac{1}{2}x = \dfrac{0 - \tan x}{1 + 0}$

$\frac{1}{2}x = -\tan x$

$\tan x = -\frac{1}{2}x$ QED

**(d)** Use the iterative formula given in part **(c)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 2.0000$

$x_2 = 2.3562$

$x_3 = 2.2746$

$x_4 = 2.2920$

$x_5 = 2.2883$

$x_6 = 2.2891$

$x_4$, $x_5$ and $x_6$ all round to 2.29 to 2dp

so $x = 2.29$

I L A T E

9 The constant $a$ is such that $\int_0^a \overset{u}{x}\,\overset{dv}{e^{-2x}}\,dx = \frac{1}{8}$.

(a) Show that $a = \frac{1}{2}\ln(4a+2)$. [5]

$u = x \qquad v = -\frac{1}{2}e^{-2x}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = e^{-2x}$

$\rightarrow -\frac{1}{2}xe^{-2x} + \int_0^a \frac{1}{2}e^{-2x}\,dx$

$= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^a$

$= \left[\frac{-1}{2}(a)e^{-2a} - \frac{1}{4}e^{-2a}\right] - \left[0 - \frac{1}{4}e^{0}\right]$

$= \frac{-a}{2}e^{-2a} - \frac{1}{4}e^{-2a} + \frac{1}{4}$

integral $= \frac{1}{8}$:

$\frac{-a}{2}e^{-2a} \times 8 - \frac{1}{4}e^{-2a} \times 8 + \frac{1}{4} \times 8 = \frac{1}{8} \times 8$

$-4ae^{-2a} - 2e^{-2a} + 2 = 1$

$-4ae^{-2a} - 2e^{-2a} + 1 = 0$

$4ae^{-2a} + 2e^{-2a} = 1$

$e^{-2a}(4a+2) = 1$

$e^{-2a} = \frac{1}{4a+2}$

$e^{2a} = 4a+2$

$2a = \ln(4a+2)$

$a = \frac{1}{2}\ln(4a+2)$ QED

**(b)** Verify by calculation that $a$ lies between 0.5 and 1. [2]

Set = 0:

$$a - \frac{1}{2}\ln(4a + 2) = 0$$

$a = 0.5$:

$$0.5 - \frac{1}{2}\ln(2 + 2) = -0.193$$

$a = 1$:

$$1 - \frac{1}{2}\ln(4 + 2) = 0.104$$

Sign change (and continuous function) between $a = 0.5$ and $a = 1$ shows there's a root between 0.5 and 1.

**(c)** Use an iterative formula based on the equation in **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \frac{1}{2}\ln(4a_n + 2)$$

$a_1 = 0.5000$

$a_2 = 0.6931$

$a_3 = 0.7814$

$a_4 = 0.8171$

$a_5 = 0.8309$

$a_6 = 0.8361$

$a_7 = 0.8380$

$a_8 = 0.8388$

$a_6$, $a_7$ and $a_8$ all round to 0.84 to 2dp, so $a = 0.84$

I L A T E

**10** The constant $a$ is such that $\int_1^a x^2 \ln x \, dx = 4$.

**(a)** Show that $a = \left(\frac{35}{3\ln a - 1}\right)^{\frac{1}{3}}$. [5]

$$\int_1^a x^2 \ln x \, dx = 4$$

$u = \ln x \qquad v = \frac{1}{3}x^3$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = x^2$

$$= \frac{1}{3}x^3 \ln x - \int_1^a \frac{1}{3}x^3 \times \frac{1}{x} \, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int_1^a x^2 \, dx$$

$$= \left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right]_1^a$$

$$= \left[\frac{1}{3}a^3 \ln a - \frac{1}{9}a^3\right] - \left[\frac{1}{3} \times 1^3 \times \ln(1) - \frac{1}{9} \times 1^3\right]$$

$$= \frac{1}{3}a^3 \ln a - \frac{1}{9}a^3 + \frac{1}{9} = 4$$

$\times 9 \quad \times 9 \quad \times 9 \quad \times 9$

$$3a^3 \ln a - a^3 + 1 = 36$$

$$3a^3 \ln a - a^3 = 35$$

$$a^3(3\ln a - 1) = 35$$

$$a^3 = \frac{35}{3\ln a - 1}$$

$$a = \left(\frac{35}{3\ln a - 1}\right)^{\frac{1}{3}}$$

QED

**(b)** Verify by calculation that $a$ lies between 2.4 and 2.8. [2]

Set $=0$: $a - \left(\frac{35}{3\ln a - 1}\right)^{\frac{1}{3}} = 0$

$a = 2.4$: $2.4 - \left(\frac{35}{3\ln(2.4) - 1}\right)^{\frac{1}{3}} = -0.382$

$a = 2.8$: $2.8 - \left(\frac{35}{3\ln(2.8) - 1}\right)^{\frac{1}{3}} = 0.241$

Sign change (and continuous function) between $x = 2.4$ and $2.8$ indicates ~~root~~ between $x = 2.4$ and $x = 2.8$.

**(c)** Use an iterative formula based on the equation in part **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \left(\frac{35}{3\ln a_n - 1}\right)^{\frac{1}{3}}$$

$a_1 = 2.4000$ $a_{10} = 2.6377$

$a_2 = 2.7815$

$a_3 = 2.5671$ $a_8$, $a_9$ and $a_{10}$ all round to

$a_4 = 2.6751$ $2.64$ to 2dp so $x = 2.64$.

$a_5 = 2.6174$

$a_6 = 2.6473$

$a_7 = 2.6315$

$a_8 = 2.6398$

$a_9 = 2.6354$

7 The equation of a curve is $y = \dfrac{x}{\cos^2 x}$, for $0 \leqslant x < \frac{1}{2}\pi$. At the point where $x = a$, the tangent to the curve has gradient equal to 12.

**(a)** Show that $a = \cos^{-1}\left(\sqrt[3]{\dfrac{\cos a + 2a\sin a}{12}}\right)$. [3]

$$y = \frac{\overset{u}{x}}{\underset{v}{\cos^2 x}}$$

$u = x \qquad v = \cos^2 x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2\cos x \times -\sin x$

$= -2\sin x\cos x$

$$\frac{dy}{dx} = \frac{1 \times \cos^2 x + 2x\sin x\cos x}{\cos^4 x}$$

at $x = a$, $\frac{dy}{dx} = 12$: $\dfrac{\cos^2 a + 2a\sin a\cos a}{\cos^4 a} = 12$

$\cos^2 a + 2a\sin a\cos a = 12\cos^4 a$

$\cos a + 2a\sin a = 12\cos^3 a$

$\dfrac{\cos a + 2a\sin a}{12} = \cos^3 a$

$\cos a = \sqrt[3]{\dfrac{\cos a + 2a\sin a}{12}}$

$a = \cos^{-1}\left(\sqrt[3]{\dfrac{\cos a + 2a\sin a}{12}}\right)$

QED

**(b)** Verify by calculation that $a$ lies between 0.9 and 1. [2]

Set $=0$: $a - \cos^{-1}\left(\sqrt[3]{\frac{\cos a + 2a\sin a}{12}}\right) = 0$

$a = 0.9$: $0.9 - \cos^{-1}\left(\sqrt[3]{\frac{\cos(0.9) + 2(0.9)\sin(0.9)}{12}}\right) = -0.0846$

$a = 1$: $1 - \cos^{-1}\left(\sqrt[3]{\frac{\cos(1) + 2(1)\sin(1)}{12}}\right) = 0.0358$

sign change (and continuous function) between 0.9 and 1
shows there's a root between $x = 0.9$ and $x = 1$.

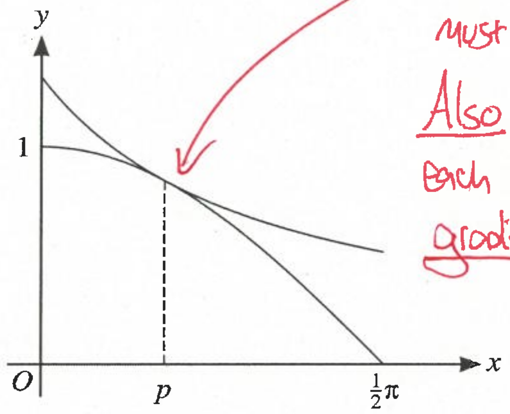
**(c)** Use an iterative formula based on the equation in part **(a)** to determine $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$$a_{n+1} = \cos^{-1}\left(\sqrt[3]{\frac{\cos(a_n) + 2a_n\sin(a_n)}{12}}\right)$$

$a_1 = 0.9000$
$a_2 = 0.9846$
$a_3 = 0.9673$
$a_4 = 0.9708$
$a_5 = 0.9701$

$a_3$, $a_4$ and $a_5$ all round to 0.97 to 2dp
so $a = 0.97$

9




Lines intersect, so equations must be equal.
Also, lines are tangential to each other, meaning their gradients must be equal.

The diagram shows the curves ① $y = \cos x$ and ② $y = \dfrac{k}{1+x}$, where $k$ is a constant, for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = p$.

**(a)** Show that $p$ satisfies the equation $\tan p = \dfrac{1}{1+p}$. [5]

equations equal: $\cos x = \dfrac{k}{1+x}$ (A)

gradients: ①: $\dfrac{dy}{dx} = -\sin x$ ②: $y = k(1+x)^{-1}$

$\dfrac{dy}{dx} = -k(1+x)^{-2}$

gradients equal: $-\sin x = \dfrac{-k}{(1+x)^2}$

$\sin x = \dfrac{k}{(1+x)^2}$ (B)

(B) ÷ (A): $\dfrac{\sin x}{\cos x} = \dfrac{\frac{k}{(1+x)^2}}{\frac{k}{(1+x)}}$ $\times(1+x)^2$ $\times(1+x)^2$

$\tan x = \dfrac{k}{k(1+x)}$

$\tan x = \dfrac{1}{1+x}$

$\tan p = \dfrac{1}{1+p}$ QED

**(b)** Use the iterative formula $p_{n+1} = \tan^{-1}\left(\frac{1}{1+p_n}\right)$ to determine the value of $p$ correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$p_1 = 0.00000$

$p_2 = 0.78540$

$p_3 = 0.51056$

$p_4 = 0.58477$

$p_5 = 0.56291$

$p_6 = 0.56919$

$p_7 = 0.56737$

$p_8 = 0.56790$

$p_9 = 0.56775$

$p_{10} = 0.56779$

$p_8$, $p_9$ and $p_{10}$ all round to 0.568 to 3dp

so $x = 0.568$.

**(c)** Hence find the value of $k$ correct to 2 decimal places. [2]

Sub into A: $\cos x = \frac{k}{1+x}$

$\cos(0.568) = \frac{k}{1+0.568}$

$\cos(0.568) = \frac{k}{1.568}$

$k = 1.568\cos(0.568)$

$= \underline{1.32}$