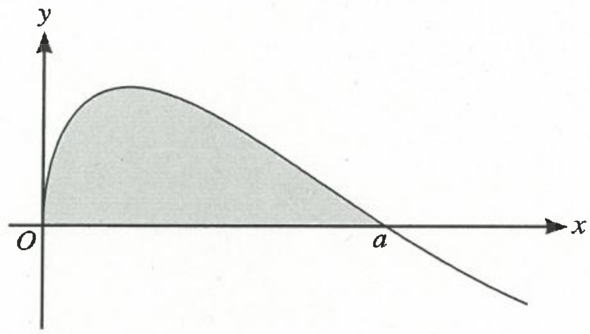
8



The diagram shows part of the curve $y = \sin\sqrt{x}$. This part of the curve intersects the x-axis at the point where $x = a$.

(a) State the exact value of a. [1]

$y=0$: $0 = \sin\sqrt{x}$

$\sin^{-1}(0) = \sqrt{x}$

$\sqrt{x} = 0$ or $\sqrt{x} = \pi$

$\times$ not the point we want from diagram

$x = \pi^2$

(b) Using the substitution $u = \sqrt{x}$, find the exact area of the shaded region in the first quadrant bounded by this part of the curve and the x-axis. [7]

$\int_0^{\pi^2} \sin\sqrt{x}\, dx$

$\sqrt{x} = u$

$x = u^2$

$\frac{dx}{du} = 2u$

$\boxed{dx = 2u\,du}$

$\int_{x=0}^{x=\pi^2} \sin u \times 2u\,du$

$u = \sqrt{\pi^2}$

$= \pi$

$u = \sqrt{0}$

$= 0$

$\int_0^{\pi} 2u\sin u\, du$

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$u = 2u \qquad v = -\cos u$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = \sin u$

$= -2u\cos u - \int_0^{\pi} -2\cos u \, du$

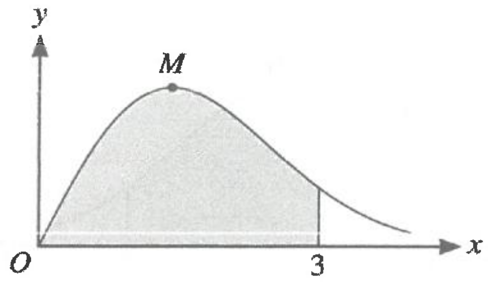
$= -2u\cos u + \int_0^{\pi} 2\cos u \, du$

$= \left[-2u\cos u + 2\sin u\right]_0^{\pi}$

$= \left[-2\pi\cos(\pi) + 2\sin(\pi)\right] - \left[0 + 0\right]$

$= \left[-2\pi \times -1 + 0\right] - \left[0\right]$

$= \underline{\underline{2\pi}}$



The diagram shows the curve $y = xe^{-\frac{1}{4}x^2}$, for $x \geqslant 0$, and its maximum point M.

(b) Using the substitution $x = \sqrt{u}$, or otherwise, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 3$. [5]

$$\int_{x=0}^{x=3} xe^{-\frac{1}{4}x^2}\,dx$$

$x = \sqrt{u}$

$x^2 = u$

$2x = \frac{du}{dx}$

$du = 2x\,dx$

$\boxed{dx = \frac{du}{2x}}$

$3 = \sqrt{u}$ $\quad$ $0 = \sqrt{u}$

$9 = u$ $\quad$ $0 = u$

$$\int_0^9 \cancel{x} e^{-\frac{1}{4}u} \frac{du}{2\cancel{x}}$$

$$= \int_0^9 \frac{1}{2} e^{-\frac{1}{4}u}\,du$$

$$= \left[-4 \times \frac{1}{2} e^{-\frac{1}{4}u}\right]_0^9$$

$$= \left[-2e^{-\frac{9}{4}}\right] - \left[-2e^{0}\right]$$

$$= -2e^{-\frac{9}{4}} + 2e^{0}$$

$$= -2e^{-\frac{9}{2}} + 2$$

9 Let $f(x) = \dfrac{1}{(9-x)\sqrt{x}}$.

(b) Using the substitution $u = \sqrt{x}$, show that $\int_0^4 f(x)\,dx = \frac{1}{3}\ln 5$. [6]

$$\int_0^4 \frac{1}{(9-x)\sqrt{x}}\,dx$$

$\sqrt{x} = u$

$x = u^2$

$\dfrac{dx}{du} = 2u$

$\boxed{dx = 2u\,du}$

$$\int_{x=0}^{x=4} \frac{1}{(9-u^2)u} \times 2u\,du$$

$u = \sqrt{4}$

$= 2$

$u = \sqrt{0}$

$= 0$

$$\int_0^2 \frac{2u}{(9-u^2)u}\,du$$

$$= \int_0^2 \frac{2}{9-u^2}\,du$$

$$= 2\int_0^2 \frac{1}{9-u^2}\,du$$

formula book: $\dfrac{1}{a^2-x^2}$ with $a=3$

$$= 2\left[\frac{1}{6}\ln\left|\frac{3+u}{3-u}\right|\right]_0^2$$

$$= \frac{1}{3}\left[\ln\left(\frac{5}{1}\right)\right] - \frac{1}{3}\left[\ln\left(\frac{3}{3}\right)\right]$$

$$= \frac{1}{3}\ln 5 - \frac{1}{3}\ln 1$$

$= 0$

$$= \frac{1}{3}\ln 5$$ QED

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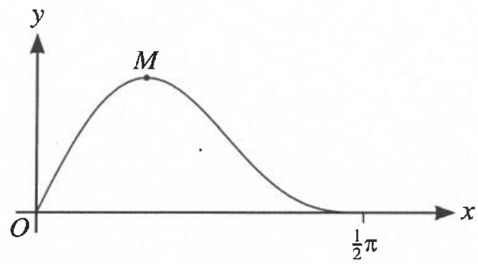


The diagram shows the curve $y = \sin 2x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Using the substitution $u = \sin x$, find the exact area of the region bounded by the curve and the x-axis. [5]

$\int_0^{\frac{\pi}{2}} \sin 2x \cos^2 x \, dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$\boxed{dx = \frac{du}{\cos x}}$

$\int_{x=0}^{x=\frac{\pi}{2}} \sin 2x \cos^2 x \times \frac{du}{\cos x}$

$u = \sin\left(\frac{\pi}{2}\right) \qquad u = \sin(0)$

$= 1 \qquad = 0$

$\int_0^1 \sin 2x \cos x \, du$

$= \int_0^1 2\sin x \cos x \times \cos x \, du$

$= \int_0^1 2\sin x \cos^2 x \, du$ $= 1 - \sin^2 x$

$= \int_0^1 2\sin x (1 - \sin^2 x) \, du$

$= \int_0^1 (2\sin x - 2\sin^3 x) \, du$

$= \int_0^1 (2u - 2u^3) \, du$

$= \left[u^2 - \frac{1}{2}u^4\right]_0^1$

$= \left[1^2 - \frac{1}{2} \times 1^4\right] - \left[0^2 - \frac{1}{2} \times 0^4\right]$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

7 (a) Use the substitution $u = \cos x$ to show that

$$\int_0^{\pi} \sin 2x\, e^{2\cos x}\, dx = \int_{-1}^{1} 2u e^{2u}\, du.$$ [4]

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\, dx$

$\boxed{dx = -\frac{du}{\sin x}}$

$\sin 2x = 2\sin x \cos x$

$\rightarrow \int_{x=0}^{x=\pi} 2\cancel{\sin x}\cos x\, e^{2\cos x} \times -\frac{du}{\cancel{\sin x}}$

$= \int_{x=0}^{x=\pi} -2\cos x\, e^{2\cos x}\, du$

$u = \cos\pi$ $\quad$ $u = \cos(0)$

$= -1$ $\quad$ $= 1$

$\int_{1}^{-1} -2ue^{2u}\, du$

Swap limits and multiply by -1:

$\int_{-1}^{1} 2ue^{2u}\, du$ QED

(b) Hence find the exact value of $\int_0^{\pi} \sin 2x\, e^{2\cos x}\, dx$. [4]

$$\int_{-1}^{1} 2ue^{2u}\, du$$

u dv

$u = 2u$ $\quad V = \frac{1}{2}e^{2u}$

$\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = e^{2u}$

$\rightarrow ue^{2u} - \int_{-1}^{1} e^{2u}\, du$

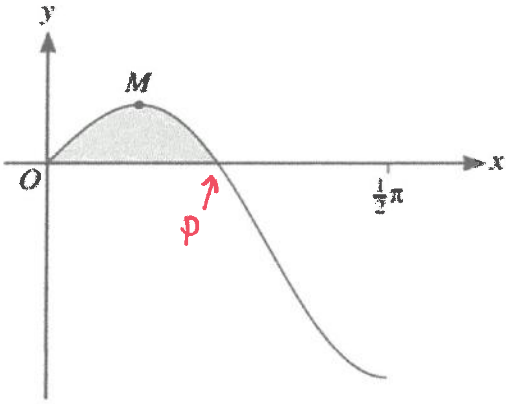
$= \left[ue^{2u} - \frac{1}{2}e^{2u}\right]_{-1}^{1}$

$= \left[1e^{2} - \frac{1}{2}e^{2}\right] - \left[-e^{-2} - \frac{1}{2}e^{-2}\right]$

$= e^{2} - \frac{1}{2}e^{2} + e^{-2} + \frac{1}{2}e^{-2}$

$= \frac{1}{2}e^{2} + \frac{3}{2}e^{-2}$

I L A T E



The diagram shows the curve $y = \sin x \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(b) Using the substitution $u = \cos x$, find the area of the shaded region enclosed by the curve and the x-axis in the first quadrant, giving your answer in a simplified exact form. [5]

Find point P: $\sin x \cos 2x = 0$

$\sin x = 0 \qquad \cos 2x = 0$

$x = 0, \pi \qquad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\int_0^{\frac{\pi}{4}} \sin x \cos 2x \, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\boxed{dx = \frac{du}{-\sin x}}$

$\int_0^{\frac{\pi}{4}} \sin x \cos 2x \times \frac{du}{-\sin x}$

$u = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$u = \cos(0) = 1$

$= -\int_1^{\frac{1}{\sqrt{2}}} \cos 2x \, du$ $2\cos^2 x - 1$

$= -\int_1^{\frac{1}{\sqrt{2}}} (2\cos^2 x - 1) \, du$

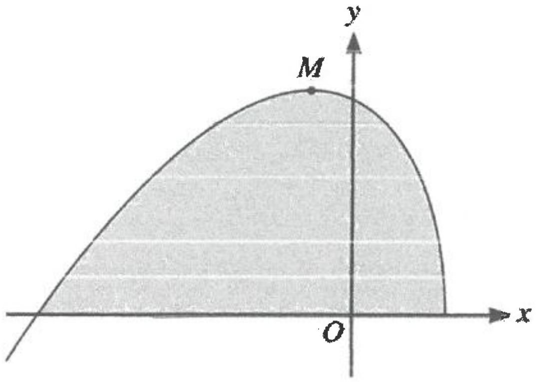
$= -\int_1^{\frac{1}{\sqrt{2}}} (2u^2 - 1) \, du = -\left[\frac{2}{3}u^3 - u\right]_1^{\frac{1}{\sqrt{2}}}$

$= -\left[\frac{2}{3} \times \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - -\left[\frac{2}{3} - 1\right]$

$= \frac{-1}{3\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{2}{3} - 1$

$= \frac{-1}{3\sqrt{2}} + \frac{3}{3\sqrt{2}} - \frac{1}{3}$

$= \frac{2}{3\sqrt{2}} - \frac{1}{3}$



The diagram shows the curve $y = (x+5)\sqrt{3-2x}$ and its maximum point M.

(b) Using the substitution $u = 3 - 2x$, find by integration the area of the shaded region bounded by the curve and the x-axis. Give your answer in the form $a\sqrt{13}$, where a is a rational number. [5]

roots: $(x+5)\sqrt{3-2x} = 0$

$x+5=0$ or $\sqrt{3-2x}=0$

$\underline{x=-5}$ $\quad 3-2x=0$

$\underline{x=\frac{3}{2}}$

$u = 3-2x$

$\frac{du}{dx} = -2$

$du = -2\,dx$

$\boxed{dx = \frac{du}{-2}}$

$u=3-2(\frac{3}{2})$
$=0$

$u=3-2(-5)$
$u=13$

$\int_{x=-5}^{x=\frac{3}{2}} (x+5)\sqrt{3-2x}\,dx$

$\int_{13}^{0} (x+5)u^{\frac{1}{2}} \times \frac{du}{-2}$

$u=3-2x$

$2x = 3-u$

$x = \frac{3-u}{2}$

$x+5 = \frac{3-u}{2}+5$

$= \frac{3-u}{2}+\frac{10}{2}$

$= \frac{13-u}{2}$

$\int_{13}^{0} \left(\frac{13-u}{2}\right)u^{\frac{1}{2}} \times \frac{du}{-2}$

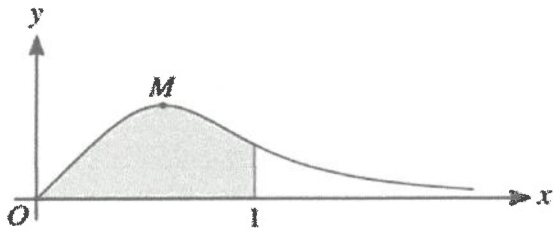
$= \frac{-1}{4}\int_{13}^{0}(13-u)u^{\frac{1}{2}}\,du \quad = \frac{-1}{4}\int_{13}^{0}(13u^{\frac{1}{2}} - u^{\frac{3}{2}})\,du$

$= \frac{-1}{4}\left[\frac{26}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_{13}^{0} = \frac{-1}{4}[0-0] - \frac{-1}{4}\left[\frac{26}{3}(13)^{\frac{3}{2}} - \frac{2}{5}(13)^{\frac{5}{2}}\right]$

$= \frac{1}{4}\left[\frac{26}{3}\times 13\sqrt{13} - \frac{2}{5}\times 13^2\sqrt{13}\right] = \frac{1}{4}\left[\frac{338}{3}\sqrt{13} - \frac{338}{5}\sqrt{13}\right]$

$= \frac{1}{4}\times\frac{676}{15}\sqrt{13}$

$= \underline{\frac{169}{15}\sqrt{13}}$



The diagram shows the curve $y = \dfrac{x}{1+3x^4}$, for $x \geqslant 0$, and its maximum point M.

(b) Using the substitution $u = \sqrt{3}x^2$, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 1$. [5]

$$\int_0^1 \frac{x}{1+3x^4}\,dx$$

$$u = \sqrt{3}x^2$$
$$\frac{du}{dx} = 2\sqrt{3}x$$
$$\boxed{dx = \frac{du}{2\sqrt{3}x}}$$

$$\int_{x=0}^{x=1} \frac{x}{1+3x^4} \cdot \frac{du}{2\sqrt{3}x}$$

$$= \int_{x=0}^{x=1} \frac{1}{2\sqrt{3}} \cdot \frac{1}{1+u^2}\,du$$

$u = \sqrt{3} \times 1^2$
$= \sqrt{3}$

$u = \sqrt{3} \times 0^2$
$= 0$

$\dfrac{1}{x^2+a^2}$ with $a = 1$

$$= \frac{1}{2\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2\sqrt{3}}\left[\tan^{-1} u\right]_0^{\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}\left[\tan^{-1}\sqrt{3}\right] - \frac{1}{2\sqrt{3}}\left[\tan^{-1}(0)\right]$$

$$= \frac{1}{2\sqrt{3}}\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{6\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{18}$$

6 Let $I = \int_0^3 \frac{27}{(9+x^2)^2}\,dx$.

(a) Using the substitution $x = 3\tan\theta$, show that $I = \int_0^{\frac{1}{4}\pi} \cos^2\theta\,d\theta$. [4]

$$\int_0^3 \frac{27}{(9+x^2)^2}\,dx$$

$x = 3\tan\theta$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$\boxed{dx = 3\sec^2\theta\,d\theta}$

$$\int_0^3 \frac{27}{(9+x^2)^2} \times 3\sec^2\theta\,d\theta$$

$3 = 3\tan\theta$ $\qquad$ $0 = 3\tan\theta$

$\tan\theta = 1$ $\qquad$ $\tan\theta = 0$

$\theta = \frac{\pi}{4}$ $\qquad$ $\theta = 0$

$$\int_0^{\frac{\pi}{4}} \frac{27\times 3}{(9+(3\tan\theta)^2)^2} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{(9+9\tan^2\theta)^2} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{[9(\underbrace{1+\tan^2\theta}_{=\sec^2\theta})]^2} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{(9\sec^2\theta)^2} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{81}{81\sec^4\theta} \times \sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$$

QED

(b) Hence find the exact value of I. [4]

$\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$ $\quad$ $2\cos^2\theta - 1 = \cos 2\theta$

$2\cos^2\theta = 1 + \cos 2\theta$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$\int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{\pi}{8} + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right] - \left[0 + 0\right]$

$= \underline{\underline{\frac{\pi}{8} + \frac{1}{4}}}$

4 Using the substitution $u = \sqrt{x}$, find the exact value of

$$\int_3^{\infty} \frac{1}{(x+1)\sqrt{x}} \, dx.$$ [6]

$$\int_3^{\infty} \frac{1}{(x+1)\sqrt{x}} dx$$

$\sqrt{x} = u$

$x = u^2$

$\frac{dx}{du} = 2u$

$\boxed{dx = 2u\,du}$

$$\int_3^{\infty} \frac{1}{(x+1)\sqrt{x}} \times 2u\,du$$

$u = \sqrt{\infty}$ $\quad u = \sqrt{3}$

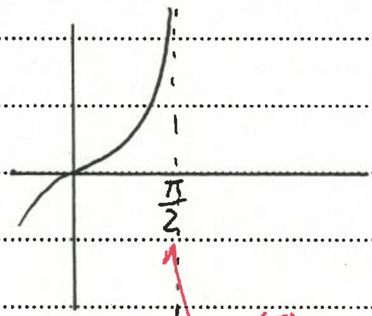
$= \infty$

$$\int_{\sqrt{3}}^{\infty} \frac{1}{(u^2+1)u} \times 2u\,du$$

$$= \int_{\sqrt{3}}^{\infty} \frac{2u}{(u^2+1)u} du$$

$$= \int_{\sqrt{3}}^{\infty} \frac{2}{1+u^2} du$$

$$= 2\int_{\sqrt{3}}^{\infty} \frac{1}{1+u^2} du$$



$\tan\left(\frac{\pi}{2}\right) = \infty$

so $\frac{\pi}{2} = \tan^{-1}(\infty)$

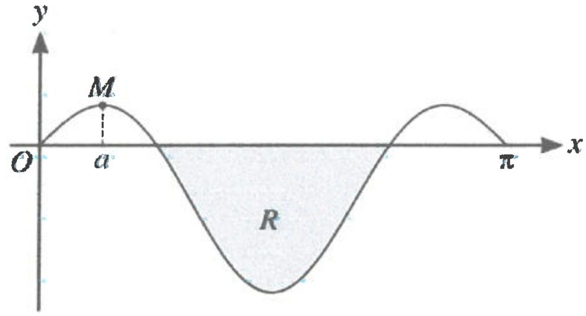
$$= 2\left[\tan^{-1}(u)\right]_{\sqrt{3}}^{\infty}$$

$$= 2\left[\tan^{-1}(\infty)\right] - 2\left[\tan^{-1}(\sqrt{3})\right]$$

$$= 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{3}\right)$$

$$= \pi - \frac{2\pi}{3}$$

$$= \frac{\pi}{3}$$



The diagram shows the curve $y = \sin x \cos 2x$, for $0 \leqslant x \leqslant \pi$, and a maximum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(b) Find the exact area of the region R, giving your answer in simplified form. [4]

roots: $\sin x \cos 2x = 0$

$\sin x = 0$ or $\cos 2x = 0$

$x = 0, \pi$ $\quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

x x $\quad x = \pi/4, 3\pi/4$ ✓ ✓

$\int \sin x \cos 2x \, dx$

$(2\cos^2 x - 1)$

$= \int \sin x (2\cos^2 x - 1) \, dx$

$= \int 2\cos^2 x \sin x \, dx - \int \sin x \, dx$ integrate normally

use substitution $u = \cos x$

$= \int_{x=\frac{\pi}{4}}^{x=\frac{3\pi}{4}} 2\cos^2 x \sin x \, dx$ $\quad u = \cos x$

$\frac{du}{dx} = -\sin x$

$\boxed{dx = \frac{du}{-\sin x}}$

$u = \cos\left(\frac{3\pi}{4}\right) = \frac{-\sqrt{2}}{2}$ $\quad u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\int_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} 2u^2 \times \sin x \times \frac{du}{-\sin x} - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \, dx$

$= \int_{\sqrt{2}/2}^{-\frac{\sqrt{2}}{2}} -2u^2 \, du - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \, dx$

$= \left[-\frac{2}{3}u^3\right]_{\frac{\sqrt{2}}{2}}^{-\frac{\sqrt{2}}{2}} - \left[-\cos x\right]_{\pi/4}^{\frac{3\pi}{4}}$

$\left[\frac{-2}{3}\left(\frac{-\sqrt{2}}{2}\right)^3\right] - \left[\frac{-2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] - \left[-\cos\left(\frac{3\pi}{4}\right)\right] + \left[-\cos\left(\frac{\pi}{4}\right)\right]$

$= \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

$= -\frac{2\sqrt{2}}{3}$ $\quad$ Area is positive,

so $= \frac{2\sqrt{2}}{3}$

  [Turn over