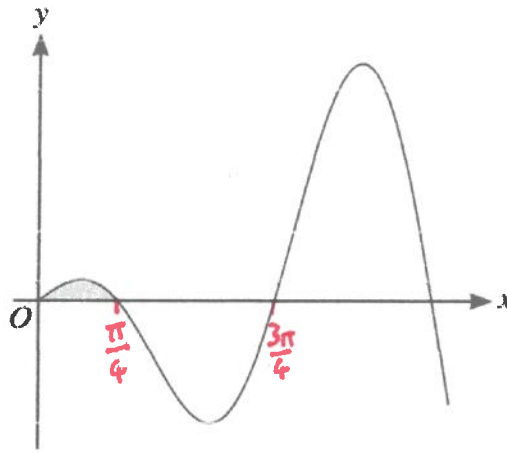




The diagram shows the curve $y = x \cos 2x$, for $x \geq 0$.

- (b) Find the exact area of the shaded region shown in the diagram, bounded by the curve and the x -axis. [5]

roots: $x \cos 2x = 0$

$x = 0$ or $\cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \dots$

$u = x$
 $\frac{du}{dx} = 1$

$v = \frac{1}{2} \sin 2x$
 $\frac{dv}{dx} = \cos 2x$

$\rightarrow x \times \frac{1}{2} \sin 2x - \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin 2x dx$

$= \left[\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{4}}$

$= \left[\frac{1}{2} \left(\frac{\pi}{4} \right) \sin \left(\frac{\pi}{2} \right) + \frac{1}{4} \cos \left(\frac{\pi}{2} \right) \right] - \left[0 + \frac{1}{4} \cos(0) \right]$

$= \left[\frac{\pi}{8} + 0 \right] - \left[0 + \frac{1}{4} \right]$

$= \frac{\pi}{8} - \frac{1}{4}$

- 4 Find the exact value of $\int_{\frac{1}{3}\pi}^{\pi} x \sin \frac{1}{2}x \, dx$.

[5]

$$u = x$$

$$\frac{du}{dx} = 1$$

$$v = -2 \cos \frac{1}{2}x$$

$$\frac{dv}{dx} = \sin \frac{1}{2}x$$

$$= -2x \cos \frac{1}{2}x - \int_{\frac{\pi}{3}}^{\pi} -2 \cos \frac{1}{2}x \, dx$$

$$= -2x \cos \frac{1}{2}x + \int_{\frac{\pi}{3}}^{\pi} 2 \cos \frac{1}{2}x \, dx$$

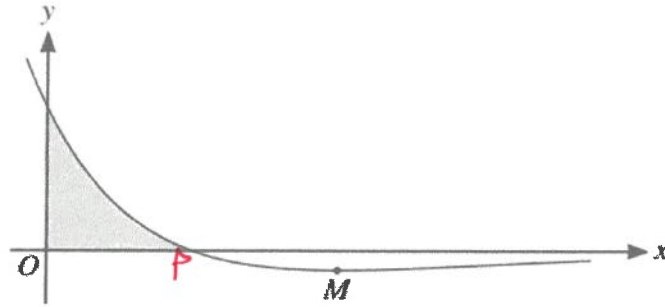
$$= \left[-2x \cos \frac{1}{2}x + 4 \sin \frac{1}{2}x \right]_{\frac{\pi}{3}}^{\pi}$$

$$= \left[-2\pi \times 0 + 4 \times 1 \right] - \left[-\frac{2\pi}{3} \times \frac{\sqrt{3}}{2} + 4 \times \frac{1}{2} \right]$$

$$= [4] - \left[-\frac{\sqrt{3}\pi}{3} + 2 \right]$$

$$= 4 - 2 + \frac{\sqrt{3}\pi}{3}$$

$$= 2 + \frac{\sqrt{3}\pi}{3}$$



The diagram shows the curve $y = (2-x)e^{-\frac{1}{2}x}$, and its minimum point M .

- (b) Find the area of the shaded region bounded by the curve and the axes. Give your answer in terms of e . [5]

Find point P: $(2-x)e^{-\frac{1}{2}x} = 0$

$2-x=0$ $e^{-\frac{1}{2}x}=0$

$\int_0^2 (2-x)e^{-\frac{1}{2}x} dx$ $x=2$

$u = 2-x$ $v = -2e^{-\frac{1}{2}x}$
 $\frac{du}{dx} = -1$ $\frac{dv}{dx} = e^{-\frac{1}{2}x}$

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$= -2(2-x)e^{-\frac{1}{2}x} - \int_0^2 2e^{-\frac{1}{2}x} dx$

$= [2(x-2)e^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}]_0^2$

$= [2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}]_0^2$

$= [2xe^{-\frac{1}{2}x}]_0^2$

$= (4e^{-\frac{1}{2} \times 2}) - (0)$

$= 4e^{-1}$

$= \frac{4}{e}$

- 3 Find the exact value of $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$.

[5]

$$u = x$$

$$\frac{du}{dx} = 1$$

$$v = \tan x$$

$$\frac{dv}{dx} = \sec^2 x$$

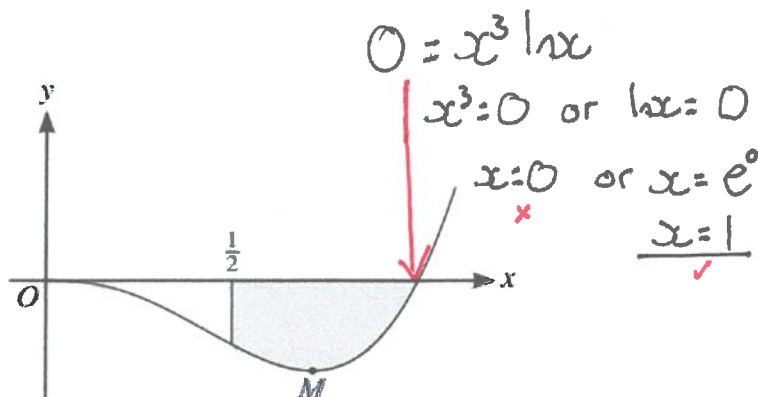
$$= x \tan x - \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= x \tan x - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$$

$$= \left[x \tan x + \ln |\cos x| \right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4} \times 1 + \ln \left(\frac{\sqrt{2}}{2} \right) \right] - [0 + \ln(1)]$$

$$= \frac{\pi}{4} + \ln \left(\frac{\sqrt{2}}{2} \right)$$

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The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M .

(b) Find the exact area of the shaded region bounded by the curve, the x -axis and the line $x = \frac{1}{2}$. [5]

$$u = \ln x \quad \frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{1}{4}x^4 \quad \frac{dv}{dx} = x^3$$

$$= \frac{1}{4}x^4 \ln x - \int_{\frac{1}{2}}^1 \frac{1}{4}x^4 \times \frac{1}{x} dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int_{\frac{1}{2}}^1 x^3 dx$$

$$= \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 \right]_{\frac{1}{2}}^1$$

$$= \left[\frac{1}{4}(1)^4 \ln(1) - \frac{1}{16}(1)^4 \right] - \left[\frac{1}{4}\left(\frac{1}{2}\right)^4 \ln\left(\frac{1}{2}\right) - \frac{1}{16}\left(\frac{1}{2}\right)^4 \right]$$

$$= \left[0 - \frac{1}{16} \right] - \left[\frac{1}{64} \ln\left(\frac{1}{2}\right) - \frac{1}{256} \right]$$

$$= -\frac{1}{16} - \frac{1}{64} \ln\left(\frac{1}{2}\right) + \frac{1}{256}$$

$$= -\frac{15}{256} - \frac{1}{64} \ln 2^{-1}$$

$$= -\frac{15}{256} + \frac{1}{64} \ln 2$$

this is negative because it is below the x -axis. We want the area, which is positive, so multiply everything by -1 :

$$\frac{15}{256} - \frac{1}{64} \ln 2$$

[Turn over]

- 9 The equation of a curve is $y = x^{-\frac{2}{3}} \ln x$ for $x > 0$. The curve has one stationary point.

(b) Show that $\int_1^8 y \, dx = 18 \ln 2 - 9$.

[5]

$$\int_1^8 x^{-\frac{2}{3}} \ln x \, dx \quad \begin{array}{l} u = \ln x \\ \frac{du}{dx} = \frac{1}{x} \end{array} \quad \begin{array}{l} v = 3x^{\frac{1}{3}} \\ \frac{dv}{dx} = x^{-\frac{2}{3}} \end{array}$$

$$= 3x^{\frac{1}{3}} \ln x - \int_1^8 3x^{\frac{1}{3}} \times \frac{1}{x} \, dx$$

$$= 3x^{\frac{1}{3}} \ln x - \int_1^8 3x^{-\frac{2}{3}} \, dx$$

$$= \left[3x^{\frac{1}{3}} \ln x - 9x^{\frac{1}{3}} \right]_1^8$$

$$= \left[3(8)^{\frac{1}{3}} \ln(8) - 9(8)^{\frac{1}{3}} \right] - \left[3(1)^{\frac{1}{3}} \ln(1) - 9(1)^{\frac{1}{3}} \right]$$

$$= [6 \ln 8 - 18] - [0 - 9]$$

$$= 6 \ln 8 - 18 + 9$$

$$= 6 \ln 8 - 9$$

$$= 6 \ln 2^3 - 9$$

$$= \underline{18 \ln 2 - 9} \quad \text{QED}$$

- 2 Find the exact value of $\int_0^1 (2-x)e^{-2x} dx$.

[5]

$$u = 2-x$$
$$\frac{du}{dx} = -1$$

$$v = -\frac{1}{2}e^{-2x}$$
$$\frac{dv}{dx} = e^{-2x}$$

$$= -\frac{1}{2}e^{-2x}(2-x) - \int_0^1 \frac{1}{2}e^{-2x} dx$$

$$= \left[-\frac{1}{2}e^{-2x}(2-x) + \frac{1}{4}e^{-2x} \right]_0^1$$

$$= \left[-\frac{1}{2}e^{-2}(2-1) + \frac{1}{4}e^{-2} \right] - \left[-\frac{1}{2}e^0(2-0) + \frac{1}{4}e^0 \right]$$

$$= \left[-\frac{1}{2}e^{-2} + \frac{1}{4}e^{-2} \right] - \left[-\frac{1}{2} \times 2 + \frac{1}{4} \right]$$

$$= \left[-\frac{1}{4}e^{-2} \right] - \left[-1 + \frac{1}{4} \right]$$

$$= -\frac{1}{4}e^{-2} + 1 - \frac{1}{4}$$

$$= \underline{\underline{-\frac{1}{4}e^{-2} + \frac{3}{4}}}$$

- 4 Find $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} x \sec^2 x \, dx$. Give your answer in a simplified exact form.

[7]

$$u = x$$

$$v = \tan x$$

$$\frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sec^2 x$$

$$= x \tan x - \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \tan x \, dx$$

$$= x \tan x - \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\sin x}{\cos x} \, dx$$

$$= \left[x \tan x + \ln |\cos x| \right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

$$= \left[\frac{\pi}{3} \times \sqrt{3} + \ln \left(\frac{1}{2} \right) \right] - \left[\frac{\pi}{6} \times \frac{\sqrt{3}}{3} + \ln \left(\frac{\sqrt{3}}{2} \right) \right]$$

$$= \frac{\sqrt{3}\pi}{3} + \ln \left(\frac{1}{2} \right) - \frac{\sqrt{3}\pi}{18} - \ln \left(\frac{\sqrt{3}}{2} \right)$$

$$= \frac{6\sqrt{3}\pi}{18} - \frac{\sqrt{3}\pi}{18} + \ln \left(\frac{1}{2} \right) - \ln \left(\frac{\sqrt{3}}{2} \right)$$

$$= \frac{5\sqrt{3}\pi}{18} + \ln \left(\frac{\frac{1}{2} \times 2}{\frac{\sqrt{3}}{2} \times 2} \right)$$

$$= \frac{5\sqrt{3}\pi}{18} + \ln \left(\frac{1}{\sqrt{3}} \right)$$

$$= \frac{5\sqrt{3}\pi}{18} + \ln 3^{-\frac{1}{2}}$$

$$= \frac{5\sqrt{3}\pi}{18} - \frac{1}{2} \ln 3$$

(b) By using integration by parts, show that for all $a > 1$, $\int_1^a \frac{\ln x}{x^4} dx < \frac{1}{9}$.

[6]

$$\int x^{-4} \ln x \, dx$$

$$u = \ln x$$

$$v = -\frac{1}{3}x^{-3}$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^{-4}$$

$$= -\frac{1}{3}x^{-3} \times \ln x - \int_1^a -\frac{1}{3}x^{-3} \times \frac{1}{x} \, dx$$

$$= -\frac{1}{3}x^{-3} \ln x + \int_1^a \frac{1}{3}x^{-4} \, dx$$

$$= \left[-\frac{1}{3}x^{-3} \ln x - \frac{1}{9}x^{-3} \right]_1^a$$

$$= \left[-\frac{1}{3}a^{-3} \ln a - \frac{1}{9}a^{-3} \right] - \left[-\frac{1}{3} \times \ln 1 - \frac{1}{9} \right]$$

$$= \left[-\frac{1}{3a^3} \ln a - \frac{1}{9a^3} \right] - \left[-\frac{1}{9} \right]$$

$$= -\frac{1}{3a^3} \ln a - \frac{1}{9a^3} + \frac{1}{9}$$

$$= \frac{1}{9} - \frac{1}{3a^3} \ln a - \frac{1}{9a^3}$$

if $a > 1$, both of these terms are negative,

so get subtracted from $\frac{1}{9}$,

therefore for all $a > 1$, the integral $< \frac{1}{9}$.

3 Find the exact value of

$$\int_1^4 x^{\frac{3}{2}} \ln x \, dx.$$

[5]

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{2}{5} x^{\frac{5}{2}}$$

$$\frac{dv}{dx} = x^{\frac{3}{2}}$$

$$= \frac{2}{5} x^{\frac{5}{2}} \ln x - \int_1^4 \frac{2}{5} x^{\frac{5}{2}} \times \frac{1}{x} \, dx$$

$$= \frac{2}{5} x^{\frac{5}{2}} \ln x - \frac{2}{5} \int_1^4 x^{\frac{3}{2}} \, dx$$

$$= \left[\frac{2}{5} x^{\frac{5}{2}} \ln x - \frac{2}{5} \times \frac{2}{5} x^{\frac{5}{2}} \right]_1^4$$

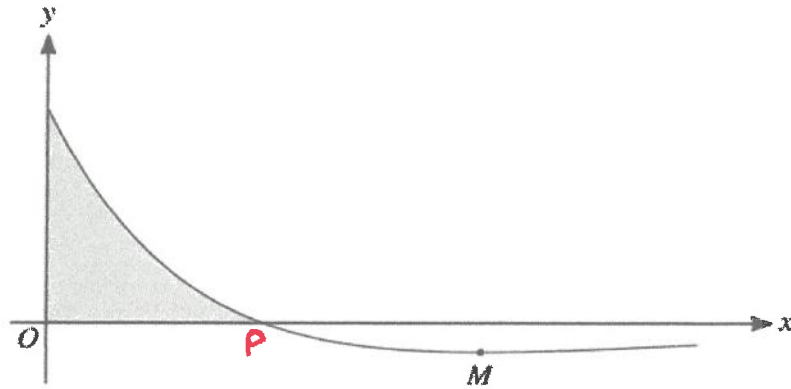
$$= \left[\frac{2}{5} x^{\frac{5}{2}} \ln x - \frac{4}{25} x^{\frac{5}{2}} \right]_1^4$$

$$= \left[\frac{2}{5} (4)^{\frac{5}{2}} \ln 4 - \frac{4}{25} (4)^{\frac{5}{2}} \right] - \left[\frac{2}{5} \ln(1) - \frac{4}{25} (1) \right]$$

$$= \left[\frac{2}{5} \times 32 \ln 4 - \frac{4}{25} \times 32 \right] - \left[0 - \frac{4}{25} \right]$$

$$= \frac{64}{5} \ln 4 - \frac{128}{25} + \frac{4}{25}$$

$$= \underline{\underline{\frac{64}{5} \ln 4 - \frac{124}{25}}}$$



The diagram shows part of the curve $y = (3-x)e^{-\frac{1}{3}x}$ for $x \geq 0$, and its minimum point M .

- (b) Find the area of the shaded region bounded by the curve and the axes, giving your answer in terms of e . [5]

Find point P: $(3-x)e^{-\frac{1}{3}x} = 0$

$3-x = 0$ or $e^{-\frac{1}{3}x} = 0$

$x = 3$ ✓

$\int_0^3 (3-x)e^{-\frac{1}{3}x} dx$

$u = 3-x$

$v = -3e^{-\frac{1}{3}x}$

$\frac{du}{dx} = -1$

$\frac{dv}{dx} = e^{-\frac{1}{3}x}$

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$= -3(3-x)e^{-\frac{1}{3}x} - \int_0^3 3e^{-\frac{1}{3}x} dx$

$= 3(x-3)e^{-\frac{1}{3}x} - \int_0^3 3e^{-\frac{1}{3}x} dx$

$= \left[3(x-3)e^{-\frac{1}{3}x} + 9e^{-\frac{1}{3}x} \right]_0^3$

$= \left[3xe^{-\frac{1}{3}x} - 9e^{-\frac{1}{3}x} + 9e^{-\frac{1}{3}x} \right]_0^3$

$= \left[3xe^{-\frac{1}{3}x} \right]_0^3$

$= \left[9e^{-\frac{1}{3} \times 3} \right] - [0]$

$= 9e^{-1} = \frac{9}{e}$

- 4 Using integration by parts, find the exact value of $\int_0^2 \tan^{-1}\left(\frac{1}{2}x\right) dx$.

[5]

$$u = \tan^{-1}\left(\frac{1}{2}x\right) \quad v = x$$

$$\frac{du}{dx} = \frac{1}{2} \times \frac{1}{1 + \left(\frac{1}{2}x\right)^2} \quad \frac{dv}{dx} = 1$$

$$\Rightarrow x \tan^{-1}\left(\frac{1}{2}x\right) - \frac{1}{2} \int_0^2 \frac{x}{1 + \left(\frac{1}{2}x\right)^2} dx$$

$$x \tan^{-1}\left(\frac{1}{2}x\right) - \frac{1}{2} \int_0^2 \frac{x}{1 + \frac{x^2}{4}} dx$$

$$x \tan^{-1}\left(\frac{1}{2}x\right) - \frac{1}{2} \int_0^2 \frac{4x}{4 + x^2} dx$$

$$x \tan^{-1}\left(\frac{1}{2}x\right) - \int_0^2 \frac{2x}{4 + x^2} dx$$

$$= \left[x \tan^{-1}\left(\frac{1}{2}x\right) - \ln|4 + x^2| \right]_0^2$$

$$= \left[2 \tan^{-1}\left(\frac{1}{2} \times 2\right) - \ln|4 + 4| \right] - \left[0 - \ln|4| \right]$$

$$= \left[2 \tan^{-1}(1) - \ln 8 \right] - \left[-\ln 4 \right]$$

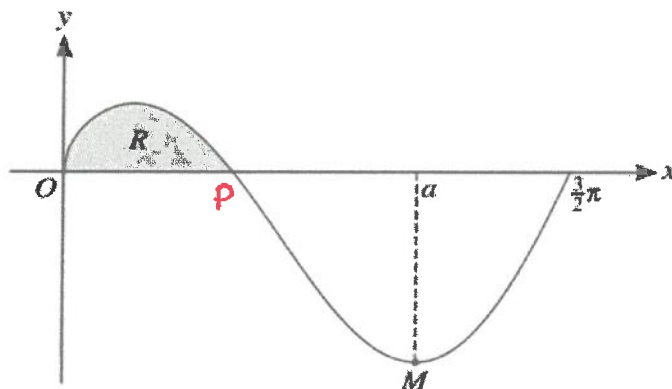
$$= 2 \times \frac{\pi}{4} - \ln 8 + \ln 4$$

$$= \frac{\pi}{2} - \ln 8 + \ln 4 \rightarrow \frac{\pi}{2} + \ln\left(\frac{4}{8}\right)$$

$$= \frac{\pi}{2} - \ln 8 + \ln 4 = \frac{\pi}{2} + \ln\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2} + \ln 2^{-1} = \frac{\pi}{2} - \ln 2$$

$$= \frac{\pi}{2} - \ln 2$$



The diagram shows the curve $y = \sqrt{x} \cos x$, for $0 \leq x \leq \frac{3}{2}\pi$, and its minimum point M , where $x = a$. The shaded region between the curve and the x -axis is denoted by R .

- (c) Find the volume of the solid obtained when the region R is rotated completely about the x -axis. Give your answer in terms of π . [6]

$$\text{Volume} = \pi \int y^2 dx$$

Point P: $\sqrt{x} \cos x = 0$

$$= \pi \int_0^{\frac{\pi}{2}} (\sqrt{x} \cos x)^2 dx$$

$\sqrt{x} = 0$ $\cos x = 0$

$x = 0$ $x = \frac{\pi}{2}$

$$= \pi \int_0^{\frac{\pi}{2}} x \cos^2 x dx$$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{1}{2} \cos 2x + \frac{1}{2} = \cos^2 x$

$$= \pi \int_0^{\frac{\pi}{2}} x \left(\frac{1}{2} \cos 2x + \frac{1}{2} \right) dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} x \right) dx + \pi \int_0^{\frac{\pi}{2}} \frac{1}{2} x \cos 2x dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} x dx + \frac{\pi}{2} \int_0^{\frac{\pi}{2}} x \cos 2x dx$$

$u = x$

$\frac{du}{dx} = 1$

$v = \frac{1}{2} \sin 2x$

$\frac{dv}{dx} = \cos 2x$

$$= \frac{\pi}{2} \left[\frac{x^2}{2} \right]_0^{\frac{\pi}{2}} + \frac{\pi}{2} \left[\frac{x}{2} \sin 2x - \int \frac{1}{2} \sin 2x dx \right]$$

$$= \frac{\pi}{2} \left[\frac{x^2}{2} \right]_0^{\frac{\pi}{2}} + \frac{\pi}{2} \left[\frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} \left(\frac{\pi^2}{8} \right) - \frac{\pi}{2} (0) + \frac{\pi}{2} \left(0 - \frac{1}{4} \right) - \frac{\pi}{2} \left(0 + \frac{1}{4} \right)$$

$$= \frac{\pi^3}{16} - \frac{\pi}{8} - \frac{\pi}{8}$$

$$= \frac{\pi^3}{16} - \frac{\pi}{4}$$