

- 3 The polynomial $2x^3 + ax^2 - 11x + b$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 12. ①

Find the values of a and b .

[5]

①: $p\left(\frac{1}{2}\right) = 0$:

$$2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 - 11\left(\frac{1}{2}\right) + b = 0$$

$$2 \times \frac{1}{8} + a \times \frac{1}{4} - \frac{11}{2} + b = 0$$

$$\frac{1}{4} + \frac{a}{4} - \frac{11}{2} + b = 0$$

$$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \times 4$$

$$1 + a - 22 + 4b = 0$$

$$a + 4b = 21 \quad \text{①}$$

②: $p(-1) = 12$:

$$2(-1)^3 + a(-1)^2 - 11(-1) + b = 12$$

$$2 \times -1 + a \times 1 + 11 + b = 12$$

$$-2 + a + 11 + b = 12$$

$$a + b = 3 \quad \text{②}$$

① - ②: $3b = 18$

$$\underline{b = 6}$$

→ ②: $a + 6 = 3$

$$\underline{a = -3}$$

- 2 The polynomial $ax^3 + 5x^2 - 4x + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 2.

Find the values of a and b .

[5]

$$p(-2) = 0$$

$$a \times (-2)^3 + 5(-2)^2 - 4(-2) + b = 0$$

$$-8a + 20 + 8 + b = 0$$

$$-8a + b = -28 \quad (1)$$

$$p(-1) = 2$$

$$a(-1)^3 + 5(-1)^2 - 4(-1) + b = 2$$

$$-a + 5 + 4 + b = 2$$

$$-a + b = -7 \quad (2)$$

$$(2) - (1): \quad -a + b = -7$$

$$-8a + b = -28$$

$$7a = 21$$

$$\underline{a = 3}$$

$$\text{Sub} \rightarrow (2): \quad -3 + b = -7$$

$$\underline{b = -4}$$

- 3 The polynomial $ax^3 + x^2 + bx + 3$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 2)$ the remainder is 5.

Find the values of a and b .

[5]

$$p\left(\frac{1}{2}\right) = 0: a\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 3 = 0$$

$$a \times \frac{1}{8} + \frac{1}{4} + \frac{b}{2} + 3 = 0$$

$$\times 8 \quad \times 8 \quad \times 8 \quad \times 8 \quad \times 8$$

$$a + 2 + 4b + 24 = 0$$

$$a + 4b + 26 = 0$$

$$a + 4b = -26 \quad (1)$$

$$p(-2) = 5: a(-2)^3 + (-2)^2 + b(-2) + 3 = 5$$

$$-8a + 4 - 2b + 3 = 5$$

$$-8a - 2b + 7 = 5$$

$$-8a - 2b = -2 \quad (2)$$

$$(2) \times 2: -16a - 4b = -4 \quad (3)$$

$$(1) + (3): a + 4b = -26$$

$$+ \quad + \quad +$$

$$-16a - 4b = -4$$

$$-15a = -30$$

$$a = 2$$

$$\rightarrow (1): 2 + 4b = -26$$

$$4b = -28$$

$$b = -7$$

- 3 The polynomial $2x^3 + ax^2 + bx + 6$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $(x + 2)$ the remainder is -38 and when $p(x)$ is divided by $(2x - 1)$ the remainder is $\frac{19}{2}$.

Find the values of a and b .

①: $p(-2) = -38$:

$$2(-2)^3 + a(-2)^2 + b(-2) + 6 = -38$$

$$2 \times -8 + a \times 4 - 2b + 6 = -38$$

$$-16 + 4a - 2b + 6 = -38$$

$$4a - 2b - 10 = -38$$

$$4a - 2b = -28 \quad \text{A}$$

②: $p(\frac{1}{2}) = \frac{19}{2}$:

$$2(\frac{1}{2})^3 + a(\frac{1}{2})^2 + b(\frac{1}{2}) + 6 = \frac{19}{2}$$

$$2 \times \frac{1}{8} + a \times \frac{1}{4} + \frac{b}{2} + 6 = \frac{19}{2}$$

$$\frac{1}{4} + \frac{a}{4} + \frac{b}{2} + 6 = \frac{19}{2}$$

$$\times 4 \quad \times 4 \quad \times 4 \quad \times 4 \quad \times 4$$

$$1 + a + 2b + 24 = 38$$

$$a + 2b + 25 = 38$$

$$a + 2b = 13 \quad \text{B}$$

① + ②: $5a = -15$

$$\underline{a = -3}$$

→ ②: $-3 + 2b = 13$

$$2b = 16$$

$$\underline{b = 8}$$

- 5 The polynomial $ax^3 - 10x^2 + bx + 8$, where a and b are constants, is denoted by $p(x)$. It is given that $(x - 2)$ is a factor of both $p(x)$ and $p'(x)$.

(a) Find the values of a and b .

[5]

$$p(2) = 0: a(2)^3 - 10(2)^2 + b(2) + 8 = 0$$

$$8a - 40 + 2b + 8 = 0$$

$$8a + 2b - 32 = 0$$

$$8a + 2b = 32 \quad (1)$$

$$p'(x) = 3ax^2 - 20x + b$$

$$p'(2) = 0: 3a(2)^2 - 20(2) + b = 0$$

$$12a - 40 + b = 0$$

$$12a + b = 40 \quad (2)$$

$$(2) \times 2: 24a + 2b = 80 \quad (3)$$

$$(3) - (1): 16a = 48$$

$$\underline{a = 3}$$

$$\rightarrow (2): 12(3) + b = 40$$

$$36 + b = 40$$

$$\underline{b = 4}$$

(b) When a and b have these values, factorise $p(x)$ completely.

[3]

$$p(x) = 3x^3 - 10x^2 + 4x + 8$$

$$\begin{array}{r}
 3x^2 - 4x - 4 \\
 x - 2 \overline{) 3x^3 - 10x^2 + 4x + 8} \\
 \underline{3x^3 - 6x^2} \quad \downarrow \\
 -4x^2 + 4x \quad \downarrow \\
 \underline{-4x^2 + 8x} \\
 -4x + 8 \\
 \underline{-4x + 8}
 \end{array}$$

$$\rightarrow (x - 2)(3x^2 - 4x - 4)$$

factorise

$$= (x - 2)(3x + 2)(x - 2)$$

- 1 Find the quotient and remainder when $6x^4 + x^3 - x^2 + 5x - 6$ is divided by $2x^2 - x + 1$.

[3]

$$\begin{array}{r}
 3x^2 + 2x - 1 \\
 2x^2 - x + 1 \overline{) 6x^4 + x^3 - x^2 + 5x - 6} \\
 \underline{6x^4 - 3x^3 + 3x^2} \downarrow \\
 4x^3 - 4x^2 + 5x \downarrow \\
 \underline{4x^3 - 2x^2 + 2x} \\
 -2x^2 + 3x - 6 \\
 \underline{-2x^2 + x - 1} \\
 2x - 5
 \end{array}$$

So $\frac{6x^4 + x^3 - x^2 + 5x - 6}{2x^2 - x + 1} \equiv \underbrace{3x^2 + 2x - 1}_{\text{quotient}} + \frac{\underbrace{2x - 5}_{\text{remainder}}}{2x^2 - x + 1}$

- 1 Find the quotient and remainder when $2x^4 + 1$ is divided by $x^2 - x + 2$.

[3]

$$\begin{array}{r}
 2x^2 + 2x - 2 \\
 x^2 - x + 2 \overline{) 2x^4 + 0x^3 + 0x^2 + 0x + 1} \\
 \underline{2x^4 - 2x^3 + 4x^2} \downarrow \\
 2x^3 - 4x^2 + 0x \downarrow \\
 \underline{2x^3 - 2x^2 + 4x} \\
 -2x^2 - 4x + 1 \\
 \underline{-2x^2 + 2x - 4} \\
 -6x + 5
 \end{array}$$

so $\frac{2x^4 + 1}{x^2 - x + 2} = \underbrace{2x^2 + 2x - 2}_{\text{quotient}} + \frac{\underbrace{-6x + 5}_{\text{remainder}}}{x^2 - x + 2}$

- 2 Find the quotient and remainder when $2x^4 - 27$ is divided by $x^2 + x + 3$.

[3]

$$\begin{array}{r}
 2x^2 - 2x - 4 \\
 x^2 + x + 3 \overline{) 2x^4 + 0x^3 + 0x^2 + 0x - 27} \\
 \underline{2x^4 + 2x^3 + 6x^2} \downarrow \\
 -2x^3 - 6x^2 + 0x \downarrow \\
 \underline{-2x^3 - 2x^2 - 6x} \\
 -4x^2 + 6x - 27 \\
 \underline{-4x^2 - 4x - 12} \\
 10x - 15
 \end{array}$$

$$\frac{2x^4 - 27}{x^2 + x + 3} = 2x^2 - 2x - 4 + \frac{10x - 15}{x^2 + x + 3}$$

quotient

remainder

- 2 The polynomial $2x^3 - x^2 + a$, where a is a constant, is denoted by $p(x)$. It is given that $(2x + 3)$ is a factor of $p(x)$.

(a) Find the value of a .

[2]

$$p\left(\frac{-3}{2}\right) = 0: 2\left(\frac{-3}{2}\right)^3 - \left(\frac{-3}{2}\right)^2 + a = 0$$

$$2\left(\frac{-27}{8}\right) - \left(\frac{9}{4}\right) + a = 0$$

$$\frac{-27}{4} - \frac{9}{4} + a = 0$$

$$\frac{-36}{4} + a = 0$$

$$-9 + a = 0$$

$$a = 9$$

(b) When a has this value, solve the inequality $p(x) < 0$.

[4]

$$p(x) < 0: 2x^3 - x^2 + 9 < 0$$

factorise:

$$\begin{array}{r} x^2 - 2x + 3 \\ 2x + 3 \overline{) 2x^3 - x^2 + 0x + 9} \\ \underline{2x^3 + 3x^2} \downarrow \\ \phantom{2x + 3 \overline{) }} -4x^2 + 0x \downarrow \\ \phantom{2x + 3 \overline{) }} \underline{-4x^2 - 6x} \downarrow \\ \phantom{2x + 3 \overline{) }} 6x + 9 \\ \phantom{2x + 3 \overline{) }} \underline{6x + 9} \\ \phantom{2x + 3 \overline{) }} 0 \end{array}$$

$$\rightarrow (2x + 3)(x^2 - 2x + 3) < 0$$

$$\begin{aligned} \uparrow b^2 - 4ac &: (-2)^2 - 4(1)(3) \\ &= 4 - 12 \\ &= -8 \end{aligned}$$

$$b^2 - 4ac < 0 \text{ so no real roots}$$

$$\text{critical value: } x = \frac{-3}{2}$$

$$\text{try: } x < \frac{-3}{2} \quad x > \frac{-3}{2}$$

$$2(-2)^3 - (-2)^2 + 9 = -11 \quad 2(0)^3 - (0)^2 + 9 = 9$$

$$-11 < 0 \quad 9 > 0$$

- 3 The polynomial $2x^4 + ax^3 + bx - 1$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $x^2 - x + 1$ the remainder is $3x + 2$.

Find the values of a and b .

[5]

$$\begin{array}{r}
 2x^2 + (a+2)x + a \\
 x^2 - x + 1 \overline{) 2x^4 + ax^3 + 0x^2 + bx - 1} \\
 \underline{2x^4 - 2x^3 + 2x^2} \\
 (a+2)x^3 - 2x^2 + bx \\
 \underline{(a+2)x^3 - (a+2)x^2 + (a+2)x} \\
 ax^2 + (b-a-2)x - 1 \\
 \underline{ax^2 - ax + a} \\
 (b-2)x - 1-a
 \end{array}$$

$$\rightarrow (b-2)x - 1 - a = 3x + 2$$

compare coeffs: $b-2=3$ $-1-a=2$

$$b=5$$

$$-1 = 2+a$$

$$a = -3$$

Alternative Method:

$$\frac{2x^4 + ax^3 + bx - 1}{x^2 - x + 1} \equiv \alpha x^2 + \beta x + \gamma + \frac{3x+2}{x^2-x+1}$$

$$2x^4 + ax^3 + bx - 1 \equiv (\alpha x^2 + \beta x + \gamma)(x^2 - x + 1) + 3x + 2$$

$$2x^4 + ax^3 + bx - 1 \equiv \alpha x^4 - \alpha x^3 + \alpha x^2 + \beta x^3 - \beta x^2 + \beta x + \gamma x^2 - \gamma x + \gamma + 3x + 2$$

$$\equiv \alpha x^4 + (\beta - \alpha)x^3 + (\alpha - \beta + \gamma)x^2 + (3 + \beta - \gamma)x + \gamma + 2$$

compare coeffs:

$$x^4: 2 = \alpha \quad \text{const: } -1 = \gamma + 2 \quad x^2: 0 = \alpha - \beta + \gamma$$

$$\gamma = -3$$

$$0 = 2 - \beta - 3$$

$$\beta = -1$$

$$x^3: a = \beta - \alpha$$

$$= -1 - 2$$

$$a = -3$$

$$x: b = 3 + \beta - \gamma$$

$$= 3 - 1 - (-3)$$

$$= 3 - 1 + 3$$

$$\rightarrow b = 5$$