7 The variables x and y satisfy the differential equation

$$e^{2x}\frac{dy}{dx} = 4xy^2,$$

and it is given that $y = 1$ when $x = 0$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

$$\int \frac{1}{y^2}\,dy = \int \frac{4x}{e^{2x}}\,dx$$

I L A T E

$$\int y^{-2}\,dy = \int \overset{u}{4x}\,\overset{dv}{e^{-2x}}\,dx \qquad u = 4x \quad v = -\tfrac{1}{2}e^{-2x}$$

$$\frac{du}{dx} = 4 \qquad \frac{dv}{dx} = e^{-2x}$$

$$\rightarrow \; -y^{-1} = 4x \times -\tfrac{1}{2}e^{-2x} - \int -\tfrac{1}{2}e^{-2x} \times 4\,dx$$

$$-\frac{1}{y} = -2xe^{-2x} + \int 2e^{-2x}\,dx$$

$$-\frac{1}{y} = -2xe^{-2x} - e^{-2x} + c$$

$y=1,\ x=0$:

$$\frac{-1}{1} = 0 - 1 + c$$

$$-1 = -1 + c$$

$$\underline{c = 0}$$

$$\rightarrow \; \frac{-1}{y} = -2xe^{-2x} - e^{-2x}$$

$$\frac{1}{y} = 2xe^{-2x} + e^{-2x}$$

$$\frac{1}{y} = \frac{2x}{e^{2x}} + \frac{1}{e^{2x}}$$

$$\frac{1}{y} = \frac{2x+1}{e^{2x}}$$

$$\underline{y = \frac{e^{2x}}{2x+1}}$$

7 The variables x and θ satisfy the differential equation

$$\frac{x}{\tan\theta}\frac{dx}{d\theta} = x^2 + 3.$$

It is given that $x = 1$ when $\theta = 0$.

Solve the differential equation, obtaining an expression for x^2 in terms of θ. [7]

$$\int \frac{x}{x^2+3}\,dx = \int \tan\theta\,d\theta$$

$$\frac{1}{2}\int \frac{2x}{x^2+3}\,dx = \int \frac{\sin\theta}{\cos\theta}\,d\theta$$

$$\frac{1}{2}\ln|x^2+3| = -\ln|\cos\theta| + C$$

$x=1,\ \theta=0:$

$$\frac{1}{2}\ln|1+3| = -\ln|1| + C$$

$$\frac{1}{2}\ln 4 = 0 + C$$

$$\ln 4^{\frac{1}{2}} = C$$

$$\ln 2 = C$$

→ $$\frac{1}{2}\ln|x^2+3| = -\ln|\cos\theta| + \ln 2$$

$$\frac{1}{2}\ln|x^2+3| = \ln\left|\frac{2}{\cos\theta}\right|$$

$$\ln|x^2+3| = 2\ln\left|\frac{2}{\cos\theta}\right|$$

$$\ln|x^2+3| = \ln\left|\frac{2^2}{\cos^2\theta}\right|$$

$$x^2+3 = \frac{4}{\cos^2\theta}$$

$$x^2 = \frac{4}{\cos^2\theta} - 3$$

7 The variables x and y satisfy the differential equation

$$\cos 2x \frac{dy}{dx} = \frac{4\tan 2x}{\sin^2 3y},$$

where $0 \leqslant x < \frac{1}{4}\pi$. It is given that $y = 0$ when $x = \frac{1}{6}\pi$.

Solve the differential equation to obtain the value of x when $y = \frac{1}{6}\pi$. Give your answer correct to 3 decimal places. [8]

$$\int \sin^2 3y \, dy = \int \frac{4\tan 2x}{\cos 2x} \, dx$$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 6y = 1 - 2\sin^2 3y$

$2\sin^2 3y = 1 - \cos 6y$

$\sin^2 3y = \frac{1}{2} - \frac{1}{2}\cos 6y$

$$\int \left(\frac{1}{2} - \frac{1}{2}\cos 6y\right) dy = \int 4\tan 2x \sec 2x \, dx$$

integral of $\sec x \tan x$ is $\sec x$ (formula book)

$$\int \left(\frac{1}{2} - \frac{1}{2}\cos 6y\right) dy = 2\sec 2x + C$$

$$\frac{1}{2}y - \frac{1}{12}\sin 6y = 2\sec 2x + C$$

$y = 0,\ x = \frac{\pi}{6}$:

$$0 - 0 = 2\sec\left(\frac{\pi}{3}\right) + C$$

$$0 = \frac{2}{\cos\left(\frac{\pi}{3}\right)} + C$$

$$0 = 4 + C$$

$$C = -4$$

$$\frac{1}{2}y - \frac{1}{12}\sin 6y = 2\sec 2x - 4$$

cont.

$y = \frac{\pi}{6}$:

$$\frac{1}{2}\left(\frac{\pi}{6}\right) - \frac{1}{12}\sin(\pi) = 2\sec 2x - 4$$

$$\frac{\pi}{12} - 0 = 2\sec 2x - 4$$

$$2\sec 2x = \frac{\pi}{12} + 4$$

$$\sec 2x = \frac{\pi}{24} + 2$$

$$\frac{1}{\cos 2x} = \frac{\pi}{24} + \frac{48}{24}$$

$$\frac{1}{\cos 2x} = \frac{\pi + 48}{24}$$

$$\cos 2x = \frac{24}{\pi + 48}$$

$$2x = 1.082$$

$$x = 0.541$$

6 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = xe^{y-x},$$

and $y = 0$ when $x = 0$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [7]

$\frac{dy}{dx} = xe^{y} \times e^{-x}$

$\int \frac{1}{e^{y}} dy = \int xe^{-x} dx$

$\int e^{-y} dy = \int \overset{u}{x} \overset{dv}{e^{-x}} dx$ $\quad u = x \quad V = -e^{-x}$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = e^{-x}$

$-e^{-y} = -xe^{-x} - \int -e^{-x} dx$

$-e^{-y} = -xe^{-x} + \int e^{-x} dx$

$-e^{-y} = -xe^{-x} - e^{-x} + C$

$y = 0, x = 0$:

$-1 = 0 - 1 + C$

$C = 0$

$\rightarrow -e^{-y} = -xe^{-x} - e^{-x}$

$e^{-y} = xe^{-x} + e^{-x}$

$e^{-y} = (x+1)e^{-x}$

$-y = \ln[(x+1)e^{-x}]$

$y = -\ln[(x+1)e^{-x}]$

I L A T E

(b) Find the value of y when $x = 1$, giving your answer in the form $a - \ln b$, where a and b are integers. [1]

$x=1$: $y = -\ln[2e^{-x}]$

$y = -(\ln 2e^{-x})$
$= -(\ln 2 + \ln e^{-x})$
$= -(\ln 2 - 1)$

$= -\ln 2 + 1$
$= 1 - \ln 2$

7 A curve is such that the gradient at a general point with coordinates (x, y) is proportional to $\frac{y}{\sqrt{x+1}}$. The curve passes through the points with coordinates $(0, 1)$ and $(3, \mathrm{e})$.

By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [7]

$$\frac{dy}{dx} \propto \frac{y}{\sqrt{x+1}}$$

$$\frac{dy}{dx} = \frac{ky}{\sqrt{x+1}}$$

$$\int \frac{1}{y}\,dy = \int \frac{k}{\sqrt{x+1}}\,dx$$

use reverse chain rule

$$\int \frac{1}{y}\,dy = \int k(x+1)^{-\frac{1}{2}}\,dx$$

$$\ln|y| = 2k(x+1)^{\frac{1}{2}} + C$$

$(0,1)$:

$$\ln 1 = 2k(1)^{\frac{1}{2}} + C$$

$$0 = 2k + C$$

$$\underline{C = -2k}$$

$$\rightarrow \ln|y| = 2k(x+1)^{\frac{1}{2}} - 2k$$

$(3,e)$:

$$\ln e = 2k(4)^{\frac{1}{2}} - 2k$$

$$1 = 4k - 2k$$

$$1 = 2k$$

$$\underline{k = \tfrac{1}{2}}$$

$$\rightarrow \ln|y| = (x+1)^{\frac{1}{2}} - 1$$

$$\underline{y = e^{(x+1)^{\frac{1}{2}} - 1}}$$

6 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{1 + 4y^2}{e^x}.$$

It is given that $y = 0$ when $x = 1$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [7]

$$\frac{dy}{dx} = \frac{1 + 4y^2}{e^x}$$

$$\int \frac{1 \div 4}{\underset{\div 4}{1} + \underset{\div 4}{4y^2}}\, dy = \int \frac{1}{e^x}\, dx$$

$\int \frac{1}{y^2+a^2}$ with

$a^2 = \frac{1}{4}$

so $a = \frac{1}{2}$

$$\int \frac{\frac{1}{4}}{\frac{1}{4} + y^2}\, dy = \int e^{-x}\, dx$$

$$\frac{1}{4}\int \frac{1}{\frac{1}{4} + y^2}\, dy = \int e^{-x}\, dx$$

$$\frac{1}{4} \times 2\tan^{-1}(2y) = -e^{-x} + C$$

$$\frac{1}{2}\tan^{-1}(2y) = -e^{-x} + C$$

Sub. $y=0, x=1$:

$$\frac{1}{2}\tan^{-1}(0) = -e^{-1} + C$$

$$0 = -e^{-1} + C$$

$$C = e^{-1}$$

$$\rightarrow \quad \frac{1}{2}\tan^{-1}(2y) = -e^{-x} + e^{-1}$$

$$\tan^{-1}(2y) = -2e^{-x} + 2e^{-1}$$

$$2y = \tan(2e^{-1} - 2e^{-x})$$

$$y = \frac{1}{2}\tan(2e^{-1} - 2e^{-x})$$

(b) State what happens to the value of y as x tends to infinity. [1]

as $x \to \infty$, $2e^{-x} \to 0$

so $y = \frac{1}{2}\tan(2e^{-1} - 2e^{-x})$

$= 0$

becomes $y = \frac{1}{2}\tan(2e^{-1})$

8 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{y^2+4}{x(y+4)}$$

for $x > 0$. It is given that $x = 4$ when $y = 2\sqrt{3}$.

Solve the differential equation to obtain the value of x when $y = 2$. [8]

$$\int \frac{y+4}{y^2+4}\,dy = \int \frac{1}{x}\,dx$$

$$\int \frac{y}{y^2+4}\,dy + \int \frac{4}{y^2+4}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\int \frac{2y}{y^2+4}\,dy + 4\int \frac{1}{y^2+4}\,dy = \int \frac{1}{x}\,dx$$

$\int \frac{1}{y^2+a^2}$ with $a^2=4$ so $a=2$

$$\frac{1}{2}\ln|y^2+4| + 4\times\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = \ln|x| + C$$

$x=4,\ y=2\sqrt{3}$:

$$\frac{1}{2}\ln|12+4| + 2\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \ln|4| + C$$

$$\frac{1}{2}\ln(16) + 2\tan^{-1}(\sqrt{3}) = \ln(4) + C$$

$$\ln 16^{\frac{1}{2}} + 2\times\frac{\pi}{3} = \ln 4 + C$$

$$\ln 4 + \frac{2\pi}{3} = \ln 4 + C$$

$$C = \frac{2\pi}{3}$$

Cont...

$\rightarrow \frac{1}{2}\ln|y^2+4| + 2\tan^{-1}\left(\frac{y}{2}\right) = \ln|x| + \frac{2\pi}{3}$

$y=2$:

$\frac{1}{2}\ln|4+4| + 2\tan^{-1}\left(\frac{2}{2}\right) = \ln|x| + \frac{2\pi}{3}$

$\frac{1}{2}\ln 8 + 2\times\frac{\pi}{4} = \ln x + \frac{2\pi}{3}$

$\frac{1}{2}\ln 8 + \frac{\pi}{2} - \frac{2\pi}{3} = \ln x$

$\ln x = \ln 8^{\frac{1}{2}} - \frac{\pi}{6}$

$x = e^{\ln 8^{\frac{1}{2}} - \frac{\pi}{6}}$

$= e^{\ln\sqrt{8}} \times e^{-\frac{\pi}{6}}$

$= \sqrt{8}\, e^{-\frac{\pi}{6}}$

9 The variables x and y satisfy the differential equation

$$(x+1)(3x+1)\frac{dy}{dx} = y,$$

and it is given that $y = 1$ when $x = 1$.

Solve the differential equation and find the exact value of y when $x = 3$, giving your answer in a simplified form. [9]

$$\int \frac{1}{y}\,dy = \int \frac{1}{(x+1)(3x+1)}\,dx$$

RHS: $\frac{1}{(x+1)(3x+1)} \equiv \frac{A}{x+1} + \frac{B}{3x+1}$

$$1 \equiv A(3x+1) + B(x+1)$$

$x = -\frac{1}{3}$: $1 = 0 + \frac{2}{3}B$

$$\underline{B = \frac{3}{2}}$$

$x = -1$: $1 = -2A + 0$

$$\underline{A = \frac{-1}{2}}$$

$$\int \frac{1}{y}\,dy = \int \frac{-\frac{1}{2}}{x+1}\,dx + \int \frac{\frac{3}{2}}{3x+1}\,dx$$

$$\int \frac{1}{y}\,dy = \frac{-1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{3}{3x+1}\,dx$$

$$\ln|y| = \frac{-1}{2}\ln|x+1| + \frac{1}{2}\ln|3x+1| + C$$

$$\ln|y| = \frac{1}{2}\ln|3x+1| - \frac{1}{2}\ln|x+1| + C$$

$$\ln|y| = \ln|3x+1|^{\frac{1}{2}} - \ln|x+1|^{\frac{1}{2}} + C$$

$$\ln|y| = \ln\left|\frac{3x+1}{x+1}\right|^{\frac{1}{2}} + C$$

$$y = e^{\ln\left|\frac{3x+1}{x+1}\right|^{\frac{1}{2}} + C}$$

$$y = e^{c} \times e^{\ln\left|\frac{3x+1}{x+1}\right|^{\frac{1}{2}}}$$

$$y = Ae^{\ln\left|\frac{3x+1}{x+1}\right|^{\frac{1}{2}}}$$

$$y = A \times \sqrt{\frac{3x+1}{x+1}}$$

$y=1,\ x=1:$

$$1 = A \times \sqrt{\frac{4}{2}}$$

$$1 = A\sqrt{2}$$

$$A = \frac{1}{\sqrt{2}}$$

$$\rightarrow \quad y = \frac{1}{\sqrt{2}} \times \sqrt{\frac{3x+1}{x+1}}$$

$$y = \sqrt{\frac{3x+1}{2(x+1)}}$$

when $x=3$:

$$y = \sqrt{\frac{9+1}{2\times 4}}$$

$$= \sqrt{\frac{10}{8}}$$

$$= \sqrt{\frac{5}{4}}$$

$$= \frac{1}{2}\sqrt{5}$$

7 The variables x and t satisfy the differential equation

$$e^{3t}\frac{dx}{dt} = \cos^2 2x,$$

for $t \geqslant 0$. It is given that $x = 0$ when $t = 0$.

(a) Solve the differential equation and obtain an expression for x in terms of t. [7]

$\int \frac{1}{\cos^2 2x}\,dx = \int \frac{1}{e^{3t}}\,dt$

$\int \sec^2 2x\,dx = \int e^{-3t}\,dt$

$\frac{1}{2}\tan 2x = -\frac{1}{3}e^{-3t} + C$

$x=0,\ t=0:$

$0 = \frac{-1}{3}(1) + C$

$C = \frac{1}{3}$

$\rightarrow \frac{1}{2}\tan 2x = \frac{-1}{3}e^{-3t} + \frac{1}{3}$

$\tan 2x = \frac{-2}{3}e^{-3t} + \frac{2}{3}$

$= \frac{2}{3} - \frac{2}{3}e^{-3t}$

$= \frac{2}{3}(1 - e^{-3t})$

$2x = \tan^{-1}\left(\frac{2}{3}(1-e^{-3t})\right)$

$x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}(1-e^{-3t})\right)$

(b) State what happens to the value of x when t tends to infinity. [1]

as $t \to \infty$, $e^{-3t} \to 0$

so $x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}\left(1 - e^{-3t}\right)\right)$

$= 0$

becomes $x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}\right)$

$x = 0.294$ (3sf)

10 The variables x and t satisfy the differential equation $\frac{dx}{dt} = x^2(1+2x)$, and $x = 1$ when $t = 0$.

Using partial fractions, solve the differential equation, obtaining an expression for t in terms of x. [11]

$$\int \frac{1}{x^2(1+2x)}\,dx = \int 1\,dt$$

LHS: $\frac{1}{x^2(1+2x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1+2x}$

$1 \equiv Ax(1+2x) + B(1+2x) + Cx^2$

$x = -\frac{1}{2}$: $1 = 0 + 0 + \frac{1}{4}C$

$\underline{C = 4}$

$x = 0$: $1 = 0 + B + 0$

$\underline{B = 1}$

$x = 1$: $1 = A \times 1 \times 3 + 3B + C$

$1 = 3A + 3B + C$ $B=1$, $C=4$

$1 = 3A + 3 + 4$

$3A = -6$

$\underline{A = -2}$

$\rightarrow \int \frac{-2}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{4}{1+2x}\,dx = \int 1\,dt$

$-2\int \frac{1}{x}\,dx + \int x^{-2}\,dx + 2\int \frac{2}{1+2x}\,dx = \int 1\,dt$

$-2\ln|x| - x^{-1} + 2\ln|1+2x| = t + C$

$2\ln|1+2x| - 2\ln|x| - \frac{1}{x} = t + C$

$\ln|1+2x|^2 - \ln|x|^2 - \frac{1}{x} = t + C$

$\ln\left|\frac{1+2x}{x}\right|^2 - \frac{1}{x} = t + C$

$x = 1, t = 0$:

$\ln|3|^2 - 1 = 0 + C$

$\ln 9 - 1 = C$

$\underline{t = \ln\left|\frac{1+2x}{x}\right|^2 - \frac{1}{x} - \ln 9 + 1}$

8 At time t days after the start of observations, the number of insects in a population is N. The variation in the number of insects is modelled by a differential equation of the form $\frac{dN}{dt} = kN^{\frac{3}{2}}\cos 0.02t$, where k is a constant and N is a continuous variable. It is given that when $t = 0$, $N = 100$.

(a) Solve the differential equation, obtaining a relation between N, k and t. [5]

$$\frac{dN}{dt} = kN^{\frac{3}{2}}\cos 0.02t$$

$$\int N^{-\frac{3}{2}}\,dN = \int k\cos 0.02t\,dt$$

$$-2N^{-\frac{1}{2}} = \frac{1}{0.02}k\sin 0.02t + C$$

$$-2N^{-\frac{1}{2}} = 50k\sin 0.02t + C$$

$t=0,\ N=100:$

$$-2(100)^{-\frac{1}{2}} = 0 + C$$

$$\frac{-2}{10} = C$$

$$C = \frac{-1}{5}$$

$$\rightarrow \frac{-2}{\sqrt{N}} = 50k\sin 0.02t - \frac{1}{5}$$

(b) Given also that $N = 625$ when $t = 50$, find the value of k. [2]

$$\frac{-2}{\sqrt{625}} = 50k\sin 1 - \frac{1}{5}$$

$$\frac{-2}{25} = 50k\sin 1 - \frac{1}{5}$$

$$50k\sin 1 = \frac{3}{25}$$

$$k = \frac{3}{1250\sin 1}$$

$$\underline{k = 0.00285} \ (3sf)$$

(c) Obtain an expression for N in terms of t, and find the greatest value of N predicted by this model. [2]

$$\frac{-2}{\sqrt{N}} = 0.143\sin 0.02t - 0.2$$

$$-2 = (0.143\sin 0.02t - 0.2) \times \sqrt{N}$$

$$\sqrt{N} = \frac{-2}{0.143\sin 0.02t - 0.2}$$

$$N = \left(\frac{-2}{0.143\sin 0.02t - 0.2}\right)^2$$

$$N = 4(0.143\sin 0.02t - 0.2)^{-2}$$

greatest value of N is when $(0.143\sin 0.02t - 0.2)$ is smallest possible.

occurs when $\sin 0.02t = 1$: $N = 4(0.143 - 0.2)^{-2}$

$$= 4(-0.0574)^{-2}$$

$$= \underline{1214} \ (\text{nearest integer})$$

[Turn over

10 A gardener is filling an ornamental pool with water, using a hose that delivers 30 litres of water per minute. Initially the pool is empty. At time t minutes after filling begins the volume of water in the pool is V litres. The pool has a small leak and loses water at a rate of $0.01V$ litres per minute.

The differential equation satisfied by V and t is of the form $\frac{dV}{dt} = a - bV$.

(a) Write down the values of the constants a and b. [1]

$\frac{dV}{dt} = 30 - 0.01V$

$a = 30 \quad b = 0.01$

(b) Solve the differential equation and find the value of t when $V = 1000$. [6]

$\int \frac{1}{30 - 0.01V}\,dV = \int 1\,dt$

$-100\ln|30 - 0.01V| = t + C$

* $t = 0,\ V = 0$:

$-100\ln 30 = 0 + C$

$C = -100\ln 30$

$\rightarrow t = 100\ln 30 - 100\ln|30 - 0.01V|$

$V = 1000$:

$t = 100\ln 30 - 100\ln|30 - 10|$

$= 100\ln 30 - 100\ln 20$

$= 40.5$ (3sf)

(c) Obtain an expression for V in terms of t and hence state what happens to V as t becomes large. [2]

$t = 100\ln 30 - 100\ln|30 - 0.01V|$

$100\ln|30 - 0.01V| = 100\ln 30 - t$

$\ln|30 - 0.01V| = \ln 30 - 0.01t$

$30 - 0.01V = e^{\ln 30 - 0.01t}$

$30 - 0.01V = e^{\ln 30} \times e^{-0.01t}$

$30 - 0.01V = 30e^{-0.01t}$

$0.01V = 30 - 30e^{-0.01t}$

$V = 3000 - 3000e^{-0.01t}$

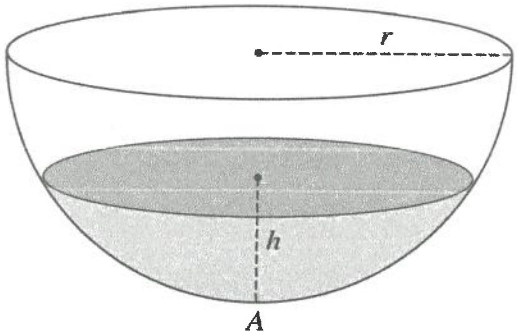
$= 3000(1 - e^{-0.01t})$

As $t \to \infty$, $e^{-0.01t} \to 0$

so this becomes: $V = 3000(1-0)$

$= 3000$

10



A tank containing water is in the form of a hemisphere. The axis is vertical, the lowest point is A and the radius is r, as shown in the diagram. The depth of water at time t is h. At time $t = 0$ the tank is full and the depth of the water is r. At this instant a tap at A is opened and water begins to flow out at a rate proportional to $\sqrt{h}$. The tank becomes empty at time $t = 14$.

*

The volume of water in the tank is V when the depth is h. It is given that $V = \frac{1}{3}\pi(3rh^2 - h^3)$.

(a) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}},$$

where B is a positive constant. [4]

$V = \frac{1}{3}\pi(3rh^2 - h^3)$

$= \pi r h^2 - \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = 2\pi r h - \pi h^2$

water flowing out, so volume decreasing

* $\frac{dV}{dt} \propto -\sqrt{h}$

$\frac{dV}{dt} = -k\sqrt{h}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{2\pi r h - \pi h^2} \times -k\sqrt{h}$

$= -\frac{kh^{\frac{1}{2}}}{2\pi r h - \pi h^2}$

$= -\frac{k}{2\pi r h^{\frac{1}{2}} - \pi h^{\frac{3}{2}}}$ $= -\frac{k/\pi}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}}$

QED (where $B = \frac{k}{\pi}$)

(b) Solve the differential equation and obtain an expression for t in terms of h and r. [8]

$\frac{dh}{dt} = -\frac{B}{2rh^{1/2} - h^{3/2}}$

$\int (2rh^{\frac{1}{2}} - h^{\frac{3}{2}})\,dh = -\int B\,dt$

$\frac{2}{3} \times 2rh^{3/2} - \frac{2}{5}h^{5/2} = -Bt + C$

$\frac{4}{3}rh^{3/2} - \frac{2}{5}h^{5/2} = -Bt + C$

At $t = 14$, $V = 0$ so $h = 0$:

$0 - 0 = -14B + C$

$C = 14B$

$\rightarrow \frac{4}{3}rh^{3/2} - \frac{2}{5}h^{5/2} = -Bt + 14B$

At $t = 0$, $r = h$:

$\frac{4}{3}h \times h^{3/2} - \frac{2}{5}h^{5/2} = -Bt + 14B$

$\frac{4}{3}h^{5/2} - \frac{2}{5}h^{5/2} = -Bt + 14B$ $\leftarrow = 0$

$\frac{14}{15}h^{5/2} = 14B$

$B = \frac{1}{15}h^{5/2}$

$\rightarrow \frac{4}{3}rh^{3/2} - \frac{2}{5}h^{5/2} = \frac{-1}{15}h^{5/2}t + \frac{14}{15}h^{5/2}$

$\frac{1}{15}h^{5/2}t = \frac{2}{5}h^{5/2} - \frac{4}{3}rh^{3/2} + \frac{14}{15}h^{5/2}$

$h^{5/2}t = 6h^{5/2} - 20rh^{3/2} + 14h^{5/2}$

$t = 6 - 20rh^{-1} + 14$

$t = 20 - 20rh^{-1}$