4 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{xy}{1+x^2},$$

and $y = 2$ when $x = 0$.

Solve the differential equation, obtaining a simplified expression for y in terms of x. [7]

$$\int \frac{1}{y}\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\ln|y| = \frac{1}{2}\ln|1+x^2| + C$$

$y=2,\ x=0:$

$$\ln 2 = \frac{1}{2}\ln 1 + C$$

$$\ln 2 = 0 + C$$

$$\underline{C = \ln 2}$$

$$\rightarrow \ln|y| = \frac{1}{2}\ln|1+x^2| + \ln 2$$

$$\ln|y| = \ln|1+x^2|^{\frac{1}{2}} + \ln 2$$

$$\ln y = \ln\left(2|1+x^2|^{\frac{1}{2}}\right)$$

$$\ln y = \ln\left(2\sqrt{1+x^2}\right)$$

$$\underline{y = 2\sqrt{1+x^2}}$$

8 The variables x and y satisfy the differential equation

$$e^{4x}\frac{dy}{dx} = \cos^2 3y.$$

It is given that $y = 0$ when $x = 2$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

$$\int \frac{1}{\cos^2 3y}\,dy = \int \frac{1}{e^{4x}}\,dx$$

$$\int \sec^2 3y\,dy = \int e^{-4x}\,dx$$

$$\frac{1}{3}\tan 3y = \frac{-1}{4}e^{-4x} + C$$

$y=0,\ x=2:$

$$\frac{1}{3}\tan(0) = \frac{-1}{4}e^{-8} + C$$

$$0 = \frac{-1}{4}e^{-8} + C$$

$$C = \frac{1}{4}e^{-8}$$

$$\rightarrow \frac{1}{3}\tan 3y = \frac{-1}{4}e^{-4x} + \frac{1}{4}e^{-8}$$

$$\tan 3y = \frac{-3}{4}e^{-4x} + \frac{3}{4}e^{-8}$$

$$3y = \tan^{-1}\left(\frac{-3}{4}e^{-4x} + \frac{3}{4}e^{-8}\right)$$

$$\underline{y = \frac{1}{3}\tan^{-1}\left(\frac{-3}{4}e^{-4x} + \frac{3}{4}e^{-8}\right)}$$

4 The variables x and y satisfy the differential equation

$$(1-\cos x)\frac{dy}{dx} = y\sin x.$$

It is given that $y = 4$ when $x = \pi$.

(a) Solve the differential equation, obtaining an expression for y in terms of x. [6]

$$\frac{dy}{dx} = \frac{y\sin x}{1-\cos x}$$

$$\int \frac{1}{y}\,dy = \int \frac{\sin x}{1-\cos x}\,dx$$

$$\ln|y| = \ln|1-\cos x| + C$$

$y=4,\ x=\pi$:

$$\ln 4 = \ln|1-\cos(\pi)| + C$$

$$\ln 4 = \ln 2 + C$$

$$C = \ln 4 - \ln 2$$

$$= \ln\left(\frac{4}{2}\right)$$

$$= \ln 2$$

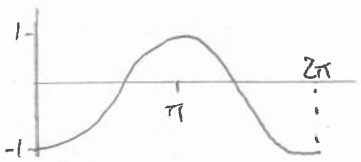
$$\ln|y| = \ln|1-\cos x| + \ln 2$$

$$\ln|y| = \ln|2(1-\cos x)|$$

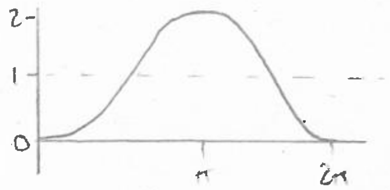
$$y = 2(1-\cos x)$$

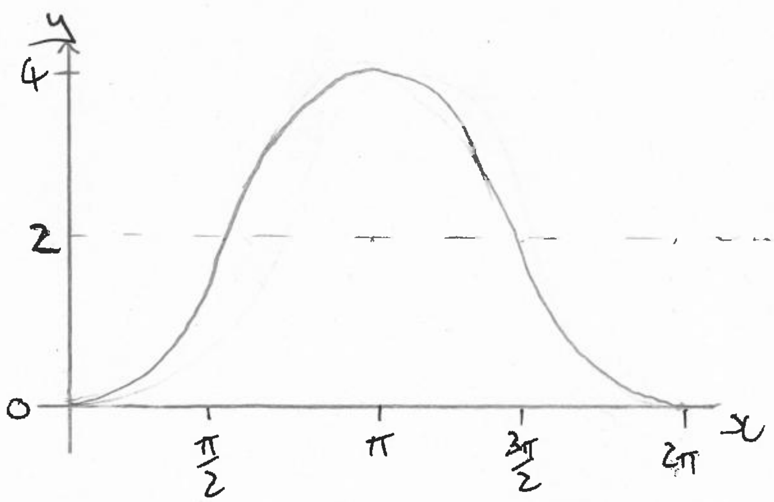
(b) Sketch the graph of y against x for $0 < x < 2\pi$. [1]

$-\cos x$;



$1-\cos x$;





9 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{3y} \sin^2 2x.$$

It is given that $y = 0$ when $x = 0$.

Solve the differential equation and find the value of y when $x = \frac{1}{2}$. [7]

$\int \frac{1}{e^{3y}} \, dy = \int \sin^2 2x \, dx$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 4x = 1 - 2\sin^2 2x$

$2\sin^2 2x = 1 - \cos 4x$

$\sin^2 2x = \frac{1}{2} - \frac{1}{2}\cos 4x$

$\int e^{-3y} \, dy = \int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right) dx$

$-\frac{1}{3}e^{-3y} = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$

$x = 0,\ y = 0:$

$-\frac{1}{3}e^{0} = 0 - 0 + C$

$-\frac{1}{3} = C$

$\rightarrow \quad -\frac{1}{3}e^{-3y} = \frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{3}$

$x = \frac{1}{2}:$

$-\frac{1}{3}e^{-3y} = \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{8}\sin(2) - \frac{1}{3}$

$-\frac{1}{3}e^{-3y} = \frac{1}{4} - \frac{1}{3} - \frac{1}{8}\sin(2)$

$-\frac{1}{3}e^{-3y} = -\frac{1}{12} - \frac{1}{8}\sin(2)$

$e^{-3y} = \frac{1}{4} + \frac{3}{8}\sin(2)$

$-3y = \ln\left(\frac{1}{4} + \frac{3}{8}\sin(2)\right)$

$y = -\frac{1}{3}\ln\left(\frac{1}{4} + \frac{3}{8}\sin(2)\right)$

$\underline{y = 0.175}$

8 A certain curve is such that its gradient at a point (x, y) is proportional to $\frac{y}{x\sqrt{x}}$. The curve passes through the points with coordinates $(1, 1)$ and $(4, e)$.

(a) By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [8]

$$\frac{dy}{dx} \propto \frac{y}{x\sqrt{x}}$$

$x^1 \times x^{\frac{1}{2}} = x^{\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{ky}{x\sqrt{x}}$$

$$\int \frac{1}{y}\,dy = \int kx^{-\frac{3}{2}}\,dx$$

$$\ln|y| = -2kx^{-\frac{1}{2}} + C$$

$$\ln|y| = \frac{-2k}{\sqrt{x}} + C$$

sub. (1,1):

$$\ln 1 = \frac{-2k}{\sqrt{1}} + C$$

$$0 = -2k + C$$

$$C = 2k$$

$$\rightarrow \ln|y| = \frac{-2k}{\sqrt{x}} + 2k$$

sub. (4,e):

$$\ln e = \frac{-2k}{\sqrt{4}} + 2k$$

$$1 = -k + 2k$$

$$1 = k$$

$$\rightarrow \ln|y| = \frac{-2}{\sqrt{x}} + 2$$

$$y = e^{-\frac{2}{\sqrt{x}} + 2}$$

(b) Describe what happens to y as x tends to infinity. [1]

as $x \to \infty$, $\frac{-2}{\sqrt{x}} \to 0$

so $y = e^{\frac{-2}{\sqrt{x}} + 2}$

0

becomes $y = e^2$

7 **(a)** Given that $y = \ln(\ln x)$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln x}.$$ [1]

$y = \ln t \qquad t = \ln x$

$\frac{dy}{dt} = \frac{1}{t} \qquad \frac{dt}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{t} \times \frac{1}{x} = \frac{1}{\ln x} \times \frac{1}{x}$

$= \frac{1}{x \ln x}$

The variables x and t satisfy the differential equation

$$x \ln x + t\frac{\mathrm{d}x}{\mathrm{d}t} = 0.$$

It is given that $x = \mathrm{e}$ when $t = 2$.

(b) Solve the differential equation obtaining an expression for x in terms of t, simplifying your answer. [7]

rearrange: $\frac{dx}{dt} = \frac{-x \ln x}{t}$

use part (a) $\int \frac{1}{x \ln x} \, dx = \int \frac{-1}{t} \, dt$

$\ln(\ln x) = -\ln|t| + C$

$x = e,\ t = 2:$

$\ln(\ln e) = -\ln 2 + C$

$\ln 1 = -\ln 2 + C$

$0 = -\ln 2 + C$

$C = \ln 2$

$\rightarrow \ln(\ln x) = -\ln|t| + \ln 2$

$\ln x = e^{-\ln t + \ln 2}$

$\ln x = e^{-\ln t} \times e^{\ln 2}$

$= e^{\ln t^{-1}} \times e^{\ln 2}$

$= t^{-1} \times 2$

$= \frac{2}{t}$

$\ln x = \frac{2}{t}$

$x = e^{2/t}$

(c) Hence state what happens to the value of x as t tends to infinity. [1]

as $t \to \infty$, $\frac{2}{t} \to 0$

so $e^{2/t} \to e^{0} = \underline{1}$

7 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{y-1}{(x+1)(x+3)}.$$

It is given that $y = 2$ when $x = 0$.

Solve the differential equation, obtaining an expression for y in terms of x. [9]

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{(x+1)(x+3)}\,dx$$

RHS: $\dfrac{1}{(x+1)(x+3)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+3}$

$$1 \equiv A(x+3) + B(x+1)$$

$x=-3$: $1 = 0 - 2B$

$$\underline{B = -\tfrac{1}{2}}$$

$x=-1$:

$$1 = 2A + 0$$

$$\underline{A = \tfrac{1}{2}}$$

$$\rightarrow \int \frac{1}{y-1}\,dy = \int \left(\frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x+3}\right) dx$$

$$\int \frac{1}{y-1}\,dy = \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x+3}\,dx$$

$$\ln|y-1| = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + C$$

$$\ln|y-1| = \ln|x+1|^{\frac{1}{2}} - \ln|x+3|^{\frac{1}{2}} + C$$

$$\ln|y-1| = \ln\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}} + C$$

$$y-1 = e^{\ln\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}} + C}$$

$y-1 = e^c \times e^{\ln\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}}}$

$y-1 = Ae^{\ln\left|\frac{x+1}{x+3}\right|^{\frac{1}{2}}}$

$y-1 = A \times \sqrt{\frac{(x+1)}{(x+3)}}$

$y=2, x=0:$

$2-1 = A \times \sqrt{\frac{1}{3}}$

$1 = A \times \frac{1}{\sqrt{3}}$

$A = \sqrt{3}$

$\rightarrow y-1 = \sqrt{3} \times \sqrt{\frac{x+1}{x+3}}$

$y-1 = \sqrt{\frac{3(x+1)}{x+3}}$

$y = 1 + \sqrt{\frac{3(x+1)}{x+3}}$

7 The variables x and θ satisfy the differential equation

$$x\sin^2\theta\frac{dx}{d\theta} = \tan^2\theta - 2\cot\theta,$$

for $0 < \theta < \frac{1}{2}\pi$ and $x > 0$. It is given that $x = 2$ when $\theta = \frac{1}{4}\pi$.

(a) Show that $\frac{d}{d\theta}(\cot^2\theta) = -\frac{2\cot\theta}{\sin^2\theta}$.

(You may assume without proof that the derivative of $\cot\theta$ with respect to θ is $-\operatorname{cosec}^2\theta$.) [1]

$\frac{d}{d\theta}(\cot\theta)^2 = 2(\cot\theta)^1 \times -\operatorname{cosec}^2\theta$ (chain rule)

$= -2\cot\theta \times \frac{1}{\sin^2\theta}$

$= \frac{-2\cot\theta}{\sin^2\theta}$ QED

(b) Solve the differential equation and find the value of x when $\theta = \frac{1}{6}\pi$. [7]

$\int x\,dx = \int \frac{1}{\sin^2\theta}(\tan^2\theta - 2\cot\theta)\,d\theta$

$\int x\,dx = \int\left(\frac{1}{\sin^2\theta} \times \frac{\sin^2\theta}{\cos^2\theta} - \frac{2\cot\theta}{\sin^2\theta}\right)d\theta$

$\int x\,dx = \int\frac{1}{\cos^2\theta}\,d\theta - \int\frac{2\cot\theta}{\sin^2\theta}\,d\theta$

$\int x\,dx = \int\sec^2\theta\,d\theta - \int\frac{2\cot\theta}{\sin^2\theta}\,d\theta$ use part (a)

$\frac{1}{2}x^2 = \tan\theta + \cot^2\theta + C$

$x=2,\ \theta=\frac{\pi}{4}:$

$$\frac{1}{2}\times 2^2 = \tan\left(\frac{\pi}{4}\right) + \cot^2\left(\frac{\pi}{4}\right) + C$$

$$2 = 1 + 1 + C$$

$$\underline{C = 0}$$

$$\rightarrow \quad \underline{\frac{1}{2}x^2 = \tan\theta + \cot^2\theta}$$

When $\theta=\frac{\pi}{6}:$

$$\frac{1}{2}x^2 = \tan\left(\frac{\pi}{6}\right) + \cot^2\left(\frac{\pi}{6}\right)$$

$$\frac{1}{2}x^2 = \frac{1}{\sqrt{3}} + (\sqrt{3})^2$$

$$\frac{1}{2}x^2 = \frac{1}{\sqrt{3}} + 3$$

$$x^2 = \frac{2}{\sqrt{3}} + 6$$

$$x = \sqrt{\frac{2}{\sqrt{3}} + 6}$$

$$= \underline{2.67} \quad (3sf)$$

8 The coordinates (x, y) of a general point of a curve satisfy the differential equation

$$x\frac{dy}{dx} = (1 - 2x^2)y,$$

for $x > 0$. It is given that $y = 1$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [6]

$$\int \frac{1}{y}\,dy = \int \frac{1 - 2x^2}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx - \int \frac{2x^2}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx - \int 2x\,dx$$

$$\ln|y| = \ln|x| - x^2 + C$$

$y = 1, x = 1$:

$$\ln 1 = \ln 1 - 1 + C$$

$$0 = 0 - 1 + C$$

$$\underline{C = 1}$$

$$\rightarrow \ln|y| = \ln|x| - x^2 + 1$$

$$y = e^{\ln|x| - x^2 + 1}$$

$$y = e^{\ln|x|} \times e^{1 - x^2}$$

$$\underline{y = xe^{1 - x^2}}$$

8 **(a)** The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{4+9y^2}{e^{2x+1}}.$$

It is given that $y = 0$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

$$\int \frac{1 \div 9}{4 + 9y^2 \div 9}\,dy = \int \frac{1}{e^{2x+1}}\,dx$$

$$\int \frac{\frac{1}{9}}{\frac{4}{9} + y^2}\,dy = \int e^{-2x-1}\,dx$$

$$\frac{1}{9}\int \frac{1}{\frac{4}{9} + y^2}\,dy = \int e^{-2x-1}\,dx$$

$\int \frac{1}{y^2+a^2}$ with $a^2 = \frac{4}{9}$ so $a = \frac{2}{3}$

$$\frac{1}{9} \times \frac{3}{2} \tan^{-1}\left(\frac{3}{2}y\right) = \frac{-1}{2}e^{-2x-1} + C$$

$y=0, x=1$:

$$\frac{1}{6}\tan^{-1}(0) = \frac{-1}{2}e^{-3} + C$$

$$0 = \frac{-1}{2}e^{-3} + C$$

$$C = \frac{1}{2}e^{-3}$$

$$\rightarrow \frac{1}{6}\tan^{-1}\left(\frac{3}{2}y\right) = \frac{-1}{2}e^{-2x-1} + \frac{1}{2}e^{-3}$$

$\times 6$ $\quad \times 6$ $\quad \times 6$

$$\tan^{-1}\left(\frac{3y}{2}\right) = -3e^{-2x-1} + 3e^{-3}$$

$$\frac{3y}{2} = \tan\left(-3e^{-2x-1} + 3e^{-3}\right)$$

cont..

$$3y = 2\tan\left(-3e^{-2x-1} + 3e^{-3}\right)$$

$$y = \frac{2}{3}\tan\left(-3e^{-2x-1} + 3e^{-3}\right)$$

(b) State what happens to the value of y as x tends to infinity. Give your answer in an exact form. [1]

as $x \to \infty$, $-3e^{-2x-1} \to 0$

so $y \to \frac{2}{3}\tan\left(3e^{-3}\right)$

11 The variables x and y satisfy the differential equation

$$x^2\frac{dy}{dx} + y^2 + y = 0.$$

It is given that $x = 1$ when $y = 1$.

(a) Solve the differential equation to obtain an expression for y in terms of x. [8]

$$x^2 \frac{dy}{dx} = -y^2 - y$$

$$\int \frac{1}{-y^2 - y}\, dy = \int \frac{1}{x^2}\, dx$$

$$\int \frac{-1}{y^2 + y}\, dy = \int x^{-2}\, dx$$

LHS:

$$\frac{-1}{y^2+y} \equiv \frac{-1}{y(y+1)} \equiv \frac{A}{y} + \frac{B}{y+1}$$

$$-1 \equiv A(y+1) + By$$

$y = -1$:

$$-1 = 0 - B$$

$$-1 = -B$$

$$B = 1$$

$y = 0$:

$$-1 = A + 0$$

$$A = -1$$

$$\rightarrow \int \frac{-1}{y}\, dy + \int \frac{1}{y+1}\, dy = \int x^{-2}\, dx$$

$$-\ln|y| + \ln|y+1| = -x^{-1} + C$$

$$\ln\left|\frac{y+1}{y}\right| = -x^{-1} + C$$

$x=1,\ y=1:$

$$\ln\left|\frac{2}{1}\right| = -1 + C$$

$$\ln 2 + 1 = C$$

$$\rightarrow \ln\left|\frac{y+1}{y}\right| = \frac{-1}{x} + \ln 2 + 1$$

$$\frac{y+1}{y} = e^{-\frac{1}{x} + \ln 2 + 1}$$

$$\frac{y+1}{y} = e^{-\frac{1}{x}+1} \times e^{\ln 2}$$

$$\frac{y+1}{y} = 2e^{-\frac{1}{x}+1}$$

$$y+1 = 2ye^{-\frac{1}{x}+1}$$

$$y+1 = 2ye^{1-\frac{1}{x}}$$

$$y - 2ye^{1-\frac{1}{x}} = -1$$

$$y\left(1 - 2e^{1-\frac{1}{x}}\right) = -1$$

$$y = \frac{-1}{1-2e^{1-\frac{1}{x}}} \quad \begin{matrix} \times -1 \\ \times -1 \end{matrix}$$

$$y = \frac{1}{2e^{1-\frac{1}{x}} - 1}$$

(b) State what happens to the value of y when x tends to infinity. Give your answer in an exact form. [1]

as $x \to \infty$, $e^{1-\frac{1}{x}} \to e^1$

so $y \to \dfrac{1}{2e^1 - 1}$

$$= \frac{1}{2e-1}$$

8 In a certain chemical reaction the amount, x grams, of a substance is increasing. The differential equation satisfied by x and t, the time in seconds since the reaction began, is

$$\frac{dx}{dt} = kxe^{-0.1t},$$

where k is a positive constant. It is given that $x = 20$ at the start of the reaction.

(a) Solve the differential equation, obtaining a relation between x, t and k. [5]

$$\int \frac{1}{x}\,dx = \int ke^{-0.1t}\,dt$$

$$\ln|x| = -10ke^{-0.1t} + C$$

$x = 20,\ t = 0:$

$$\ln 20 = -10k + C$$

$$C = \ln 20 + 10k$$

$$\ln x = -10ke^{-0.1t} + 10k + \ln 20$$

(b) Given that $x = 40$ when $t = 10$, find the value of k and find the value approached by x as t becomes large. [3]

$x=40,\ t=10$: $\ln 40 = -10ke^{-1} + 10k + \ln 20$

$\ln 40 - \ln 20 = k(-10e^{-1} + 10)$

$\ln\left(\frac{40}{20}\right) = k(10 - 10e^{-1})$

$\ln 2 = k(10 - 10e^{-1})$

$k = \dfrac{\ln 2}{10 - 10e^{-1}}$

As $t \to \infty$, $e^{-0.1t} \to 0$

so $\ln x = -10ke^{-0.1t} + 10k + \ln 20$

becomes: $\ln x = 10k + \ln 20$

$\ln x = \dfrac{10\ln 2}{10 - 10e^{-1}} + \ln 20$

$x = \exp\left(\dfrac{10\ln 2}{10-10e^{-1}} + \ln 20\right)$

$x = 59.9$ (3sf)

10 A large plantation of area 20 km^2 is becoming infected with a plant disease. At time t years the area infected is x km^2 and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected. *

When $t = 0$, $x = 1$ and $\frac{dx}{dt} = 1$.

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{19x}{20 - x}.$$ [2]

Area infected $= x$ Area not infected $= 20 - x$

* $\frac{dx}{dt} \propto \frac{x}{20-x}$

$\frac{dx}{dt} = \frac{kx}{20-x}$ $t=0, x=1, \frac{dx}{dt}=1:$

$1 = \frac{k}{20-1}$

$k = 19$

$\rightarrow \frac{dx}{dt} = \frac{19x}{20-x}$ QED

(b) Solve the differential equation and show that when $t = 1$ the value of x satisfies the equation $x = e^{0.9+0.05x}$. [5]

$\int \frac{20-x}{19x}\,dx = \int 1\,dt$

$\int \left(\frac{20}{19x} - \frac{x}{19x}\right)dx = \int 1\,dt$

$\frac{20}{19}\int \frac{1}{x}\,dx - \int \frac{1}{19}\,dx = \int 1\,dt$

$\frac{20}{19}\ln|x| - \frac{1}{19}x = t + C$

$20\ln|x| - x = 19t + 19C$

$t=0, x=1:$

$20\ln 1 - 1 = 0 + 19C$

$-1 = 19C$

$\rightarrow 20\ln|x| - x = 19t - 1$

$t=1:$

$20\ln|x| - x = 19 - 1$

$20\ln|x| - x = 18$

$20\ln|x| = 18 + x$

$\ln|x| = 0.9 + 0.05x$

$x = e^{0.9+0.05x}$

QED

(c) Use an iterative formula based on the equation in part (b), with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_{n+1} = e^{0.9 + 0.05x_n}$

$x_1 = 2.0000$

$x_2 = 2.7183$

$x_3 = 2.8177$

$x_4 = 2.8317$

$x_5 = 2.8337$

$x_6 = 2.8340$

x_4, x_5 and x_6 all round to 2.83 to 2dp, so

$\underline{\underline{x = 2.83}}$

(d) Calculate the value of t at which the entire plantation becomes infected. [1]

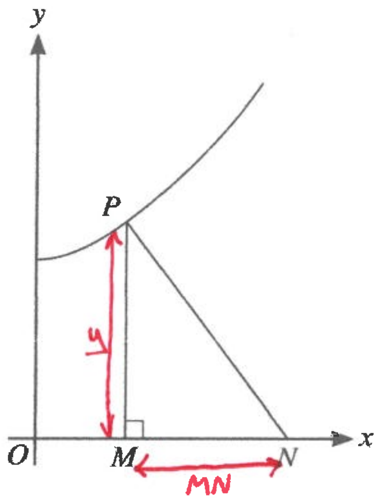
$20\ln|x| - x = 19t - 1$

$x = 20$: $20\ln 20 - 20 + 1 = 19t$

$40.9.. = 19t$

$\underline{\underline{t = 2.15 \text{ years}}}$ (3sf)

7



For the curve shown in the diagram, the normal to the curve at the point P with coordinates (x, y) meets the x-axis at N. The point M is the foot of the perpendicular from P to the x-axis.

The curve is such that for all values of x in the interval $0 \leqslant x < \frac{1}{2}\pi$, the area of triangle PMN is equal to $\tan x$. *

(a) **(i)** Show that $\dfrac{MN}{y} = \dfrac{dy}{dx}$. [1]

gradient of normal at $P = \dfrac{\Delta y}{\Delta x} = \dfrac{-y}{MN}$

gradient of curve at $P = \dfrac{-1}{\left(\frac{-y}{MN}\right)}$ (gradient of normal $= \frac{-1}{m}$)

$\dfrac{dy}{dx} = \dfrac{-1}{\frac{-y}{MN}} \rightarrow \dfrac{dy}{dx} = \dfrac{MN}{y}$

(ii) Hence show that x and y satisfy the differential equation $\frac{1}{2}y^2\dfrac{dy}{dx} = \tan x$. [2]

Area of triangle $PMN = \frac{1}{2} \times MN \times y$

from (a): $MN = y\dfrac{dy}{dx}$

$\rightarrow$ Area $= \frac{1}{2}y^2\dfrac{dy}{dx}$

* $\tan x = \frac{1}{2}y^2\dfrac{dy}{dx}$

(b) Given that $y = 1$ when $x = 0$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

$$\int \tan x \, dx = \int \frac{1}{2} y^2 \, dy$$

$$\int \frac{\sin x}{\cos x} \, dx = \frac{1}{2} \int y^2 \, dy$$

$$-\ln|\cos x| = \frac{1}{6} y^3 + C$$

$y=1, x=0$:

$$-\ln 1 = \frac{1}{6} + C$$

$$0 = \frac{1}{6} + C$$

$$C = -\frac{1}{6}$$

$$\rightarrow -\ln|\cos x| = \frac{1}{6} y^3 - \frac{1}{6}$$

$$\frac{1}{6} y^3 = \frac{1}{6} - \ln|\cos x|$$

$$y^3 = 1 - 6\ln|\cos x|$$

cos x is positive when $0 \leqslant x \leqslant \frac{\pi}{2}$

$$y^3 = 1 - 6\ln\cos x$$

$$y = \sqrt[3]{1 - 6\ln\cos x}$$