1 (a) Find the first three terms in the expansion, in ascending powers of $x$, of $(1 + x)^5$. [1]

$(1+x)^5 = 1^5 + 5 \times 1^4 \times x + 10 \times 1^3 \times x^2 + \dots$

$= 1 + 5x + 10x^2 + \dots$

(b) Find the first three terms in the expansion, in ascending powers of $x$, of $(1 - 2x)^6$. [2]

$(1-2x)^6 = 1^6 + 6 \times 1^5 \times (-2x) + 15 \times 1^4 \times (-2x)^2 + \dots$

$= 1 - 12x + 15 \times 4x^2 + \dots$

$= 1 - 12x + 60x^2 + \dots$

(c) Hence find the coefficient of $x^2$ in the expansion of $(1 + x)^5(1 - 2x)^6$. [2]

$(1 + 5x + 10x^2)(1 - 12x + 60x^2)$

$x^2$ terms: $60x^2 - 60x^2 + 10x^2$

$= 10x^2$

coeff $= 10$

**2** By using a suitable substitution, solve the equation

$$(2x-3)^2 - \frac{4}{(2x-3)^2} - 3 = 0.$$ [4]

$$(2x-3)^2 - \frac{4}{(2x-3)^2} - 3 = 0$$

$\times(2x-3)^2$ $\times(2x-3)^2$ $\times(2x-3)^2$ $\times(2x-3)^2$

$$(2x-3)^4 - 4 - 3(2x-3)^2 = 0$$

$$(2x-3)^4 - 3(2x-3)^2 - 4 = 0$$

Let $Y = (2x-3)^2$ :

$$Y^2 - 3Y - 4 = 0$$

$$(Y-4)(Y+1) = 0$$

$Y - 4 = 0$ or $Y + 1 = 0$

$Y = 4$ or $Y = -1$

$(2x-3)^2 = 4$ or $(2x-3)^2 = -1$

no real sol^ns

$2x - 3 = 2$ or $2x - 3 = -2$

$2x = 5$ $\quad$ $2x = 1$

$x = \frac{5}{2}$ $\quad$ $x = \frac{1}{2}$

3 Solve the equation $\dfrac{\tan\theta + 2\sin\theta}{\tan\theta - 2\sin\theta} = 3$ for $0° < \theta < 180°$. [4]

$\tan\theta + 2\sin\theta = 3(\tan\theta - 2\sin\theta)$

$\tan\theta + 2\sin\theta = 3\tan\theta - 6\sin\theta$

$8\sin\theta - 2\tan\theta = 0$

$8\sin\theta - \dfrac{2\sin\theta}{\cos\theta} = 0$

$\times\cos\theta \qquad \times\cos\theta \qquad \times\cos\theta$

$8\sin\theta\cos\theta - 2\sin\theta = 0$

$\sin\theta(8\cos\theta - 2) = 0$

$\sin\theta = 0$ or $8\cos\theta - 2 = 0$

$\theta = 0°$ $\qquad \cos\theta = \frac{1}{4}$

$\theta = 75.5°$

outside range (and so is 180°)

$\theta = 75.5°$

**4** A line has equation $y = 3x + k$ and a curve has equation $y = x^2 + kx + 6$, where $k$ is a constant.

Find the set of values of $k$ for which the line and curve have two distinct points of intersection. [5]

solve simultaneously:

$$x^2 + kx + 6 = 3x + k$$

$$x^2 + kx - 3x + 6 - k = 0$$

$$x^2 + (k-3)x + 6 - k = 0$$

Two distinct points of intersection, so $b^2 - 4ac > 0$:

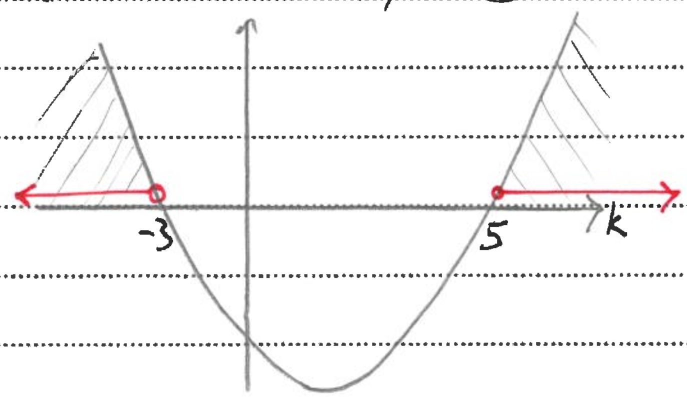
$$(k-3)^2 - 4(1)(6-k) > 0$$

$$k^2 - 6k + 9 - 24 + 4k > 0$$

$$k^2 - 2k - 15 > 0$$

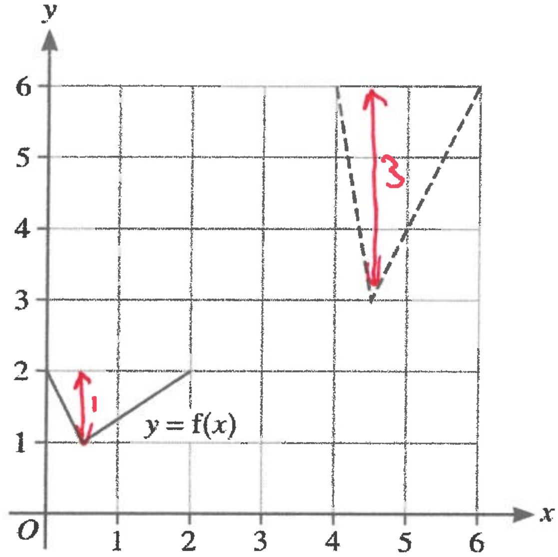
$$(k-5)(k+3) > 0$$

critical values: $k = 5$, $k = -3$




so $k < -3$, $k > 5$

5




In the diagram, the graph of $y = f(x)$ is shown with solid lines. The graph shown with broken lines is a transformation of $y = f(x)$.

**(a)** Describe fully the two single transformations of $y = f(x)$ that have been combined to give the resulting transformation. [4]

① Translation by $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

② Stretch, scale factor 3, parallel to y-axis

**(b)** State in terms of $y$, f and $x$, the equation of the graph shown with broken lines. [2]

$y = 3f(x-4)$

6 A curve is such that $\frac{dy}{dx} = \frac{6}{(3x-2)^3}$ and $A\,(1, -3)$ lies on the curve. A point is moving along the curve and at $A$ the $y$-coordinate of the point is increasing at 3 units per second.

1

**(a)** Find the rate of increase at $A$ of the $x$-coordinate of the point. [3]

$$\frac{dy}{dx} = \frac{6}{(3x-2)^3}$$

at $x=1$:

$$\frac{dy}{dx} = \frac{6}{(3(1)-2)^3}$$

$$= \frac{6}{1^3}$$

$$= 6$$

1: $\frac{dy}{dt} = 3$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$6 = 3 \times \frac{dt}{dx}$$

$$2 = \frac{dt}{dx}$$

$$\frac{1}{2} = \frac{dx}{dt}$$

6 A curve is such that $\frac{dy}{dx} = \frac{6}{(3x-2)^3}$ and $A$ (1, −3) lies on the curve. A point is moving along the curve and at $A$ the $y$-coordinate of the point is increasing at 3 units per second.

(b) Find the equation of the curve. [4]

$$y = \int 6(3x-2)^{-3}\,dx$$

$$= -3 \times \frac{1}{3}(3x-2)^{-2} + c$$

$$= -(3x-2)^{-2} + c$$

Sub. (1,−3):

$$-3 = -(3-2)^{-2} + c$$

$$-3 = -(1)^{-2} + c$$

$$-3 = -1 + c$$

$$c = -2$$

$$\rightarrow \quad y = -(3x-2)^{-2} - 2$$

**7** Functions f and g are defined as follows:

$$f : x \mapsto x^2 + 2x + 3 \text{ for } x \leqslant -1,$$
$$g : x \mapsto 2x + 1 \text{ for } x \geqslant -1.$$

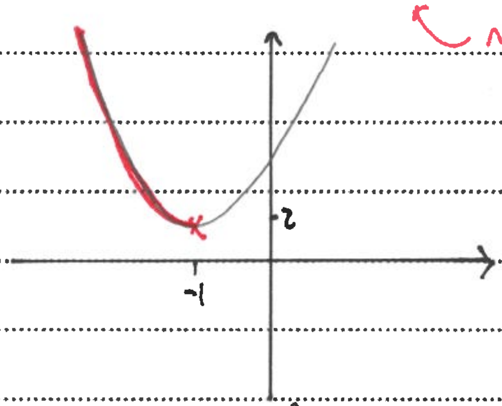
**(a)** Express $f(x)$ in the form $(x + a)^2 + b$ and state the range of f. [3]

$f(x) = x^2 + 2x + 3$

$= (x+1)^2 - 1 + 3$

$= (x+1)^2 + 2$

min. pt at $(-1, 2)$




range: $f(x) \geqslant 2$

**(b)** Find an expression for $f^{-1}(x)$. [2]

$y = (x+1)^2 + 2$

$y - 2 = (x+1)^2$

$\pm\sqrt{y-2} = \underbrace{x+1}$

domain of f is $x \leqslant -1$, so $x+1$ is $\leqslant 0$ (negative), meaning we want the negative square root only:

$-\sqrt{y-2} = x+1$

$-1-\sqrt{y-2} = x$

$x = -1 - \sqrt{y-2}$

$f^{-1}(x) = -1 - \sqrt{x-2}$

**(c)** Solve the equation $gf(x) = 13$. [3]

$gf(x) = 2(x^2 + 2x + 3) + 1$

$= 2x^2 + 4x + 6 + 1$

$= 2x^2 + 4x + 7$

$gf(x) = 13$:

$2x^2 + 4x + 7 = 13$

$2x^2 + 4x - 6 = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

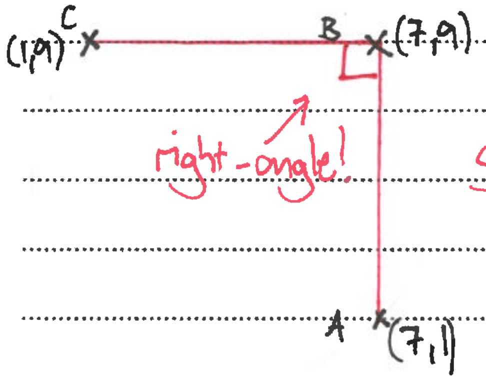
$x = -3$ or $x = 1$

But domain of f is $x \leqslant -1$, so $x = -3$ only

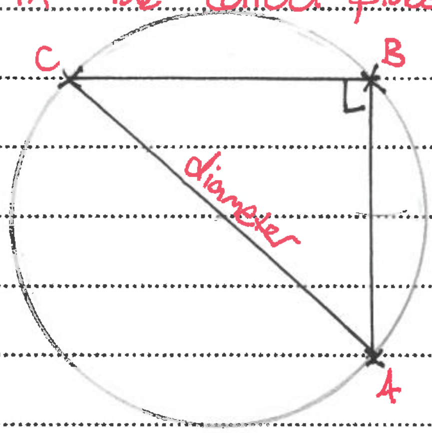
**8** The points $A\,(7,\,1)$, $B\,(7,\,9)$ and $C\,(1,\,9)$ are on the circumference of a circle.

**(a)** Find an equation of the circle. [5]

This question is simpler than it looks - draw a diagram with the points in the correct place:




SO :




→ AC is a diameter, so mid-pt = centre:

$$\left(\frac{7+1}{2}, \frac{1+9}{2}\right) = (4,5)$$

$$\rightarrow (x-4)^2 + (y-5)^2 = r^2$$

sub. any point:

sub. (7,9): $(7-4)^2 + (9-5)^2 = r^2$

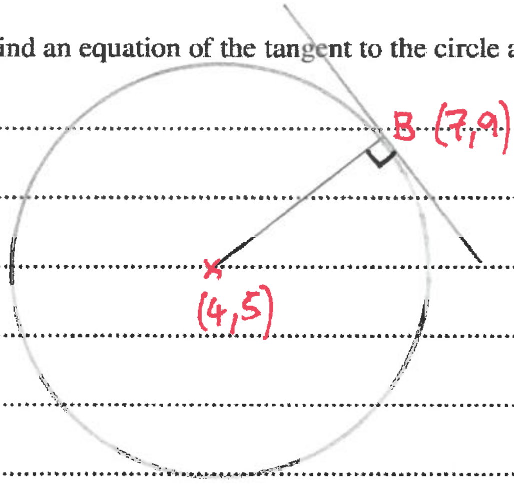
$$3^2 + 4^2 = r^2$$

$$9 + 16 = r^2$$

$$25 = r^2$$

$$\rightarrow \underline{(x-4)^2 + (y-5)^2 = 25}$$

**(b)** Find an equation of the tangent to the circle at $B$. [2]




$$\text{Gradient of radius} = \frac{9-5}{7-4} = \frac{4}{3}$$

$$\therefore \text{Gradient of tangent} = \frac{-3}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y - 9 = \frac{-3}{4}(x-7)$$

$$y - 9 = \frac{-3}{4}x + \frac{21}{4}$$

$$\underline{y = \frac{-3}{4}x + \frac{57}{4}}$$

9 The first term of a progression is $\cos\theta$, where $0 < \theta < \frac{1}{2}\pi$.

(a) For the case where the progression is geometric, the sum to infinity is $\dfrac{1}{\cos\theta}$.

(i) Show that the second term is $\cos\theta\sin^2\theta$. [3]

$u_1$: $a = \cos\theta$

$S_\infty$: $S_\infty = \dfrac{a}{1-r}$

$= \dfrac{\cos\theta}{1-r}$

$S_\infty = \dfrac{1}{\cos\theta}$: $\dfrac{\cos\theta}{1-r} = \dfrac{1}{\cos\theta}$

$\cos^2\theta = 1 - r$

$r = 1 - \cos^2\theta$

$r = \sin^2\theta$

$u_2$: $u_2 = ar$

$= \cos\theta\sin^2\theta$ QED

(ii) Find the sum of the first 12 terms when $\theta = \frac{1}{3}\pi$, giving your answer correct to 4 significant figures. [2]

$u_1$: $a = \cos\left(\frac{\pi}{3}\right)$

$= \frac{1}{2}$

$r$: $r = \sin^2\left(\frac{\pi}{3}\right)$

$= \frac{3}{4}$

$S_{12}$: $S_{12} = \dfrac{\frac{1}{2}\left(1 - \left(\frac{3}{4}\right)^{12}\right)}{1 - \frac{3}{4}}$

$= 1.937$

**(b)** For the case where the progression is arithmetic, the first two terms are again $\cos\theta$ and $\cos\theta\sin^2\theta$ respectively.

Find the 85th term when $\theta = \frac{1}{3}\pi$. [4]

$u_1$: $u_1 = \cos\left(\frac{\pi}{3}\right)$
$= \frac{1}{2}$

$u_2$: $u_2 = \cos\left(\frac{\pi}{3}\right) \times \sin^2\left(\frac{\pi}{3}\right)$
$= \frac{1}{2} \times \frac{3}{4}$
$= \frac{3}{8}$

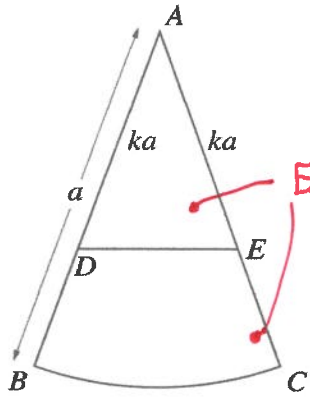
Sequence: $\frac{1}{2}, \frac{3}{8}, \ldots\ldots$

common difference: $d = \frac{3}{8} - \frac{1}{2}$
$= -\frac{1}{8}$

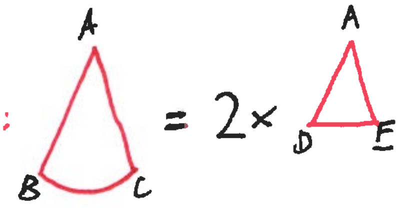
$a = \frac{1}{2}$, $d = \frac{-1}{8}$

$u_{85}$: $u_{85} = \frac{1}{2} + 84\left(\frac{-1}{8}\right)$
$= -10$

**10**




Equal area :




The diagram shows a sector $ABC$ which is part of a circle of radius $a$. The points $D$ and $E$ lie on $AB$ and $AC$ respectively and are such that $AD = AE = ka$, where $k < 1$. The line $DE$ divides the sector into two regions which are equal in area.

**(a)** For the case where angle $BAC = \frac{1}{6}\pi$ radians, find $k$ correct to 4 significant figures. [5]

Area of sector ABC: $\text{Area} = \frac{1}{2} \times a^2 \times \frac{\pi}{6}$

$= \frac{\pi a^2}{12}$

Area of triangle ADE: $\text{Area} = \frac{1}{2} \times ka \times ka \times \sin\left(\frac{\pi}{6}\right)$

$= \frac{1}{4}k^2a^2$

Sector ABC = 2× Triangle ADE:

$$\frac{\pi a^2}{12} = 2 \times \frac{k^2a^2}{4}$$

$$\frac{\pi a^2}{12} = \frac{k^2a^2}{2}$$

$$2\pi a^2 = 12k^2a^2$$

$$\pi \cancel{a^2} = 6k^2\cancel{a^2}$$

$$\pi = 6k^2$$

$$k^2 = \frac{\pi}{6}$$

$$k = \pm\sqrt{\frac{\pi}{6}}$$

$$= 0.7236$$

(can't be negative because it's a length and a is positive)

**(b)** For the general case in which angle $BAC = \theta$ radians, where $0 < \theta < \frac{1}{2}\pi$, it is given that $\frac{\theta}{\sin\theta} > 1$.

Find the set of possible values of $k$. [3]

Area of sector ABC: $\text{Area} = \frac{1}{2} \times a^2 \times \theta$

$= \frac{a^2\theta}{2}$

Area of triangle ADE: $\text{Area} = \frac{1}{2} \times ka \times ka \times \sin\theta$

$= \frac{k^2a^2\sin\theta}{2}$

Sector ABC = 2x Triangle ADE:

$$\frac{a^2\theta}{2} = 2 \times \frac{k^2a^2\sin\theta}{2}$$

$$\frac{a^2\theta}{2} = k^2a^2\sin\theta$$

$$\cancel{a^2}\theta = 2k^2\cancel{a^2}\sin\theta$$

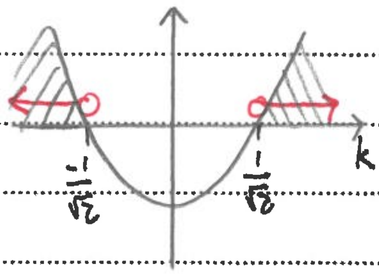
$$\theta = 2k^2\sin\theta$$

$$\frac{\theta}{\sin\theta} = 2k^2$$

$\frac{\theta}{\sin\theta} > 1$, so: $2k^2 > 1$

$k^2 > \frac{1}{2}$ or: $\left(k + \frac{1}{\sqrt{2}}\right)\left(k - \frac{1}{\sqrt{2}}\right) > 0$

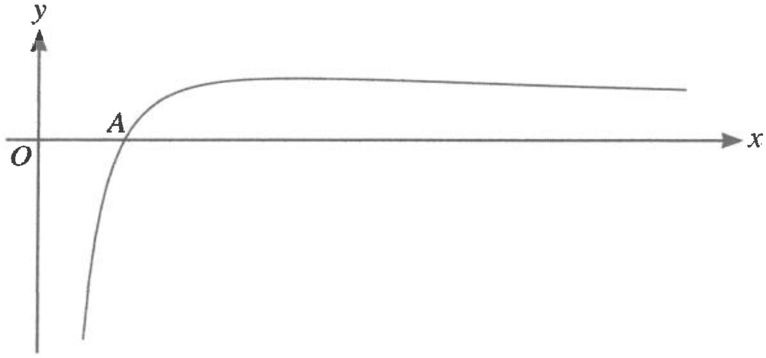
Critical Values: $k = -\frac{1}{\sqrt{2}}$, $k = \frac{1}{\sqrt{2}}$




k must be positive, so $k > \frac{1}{\sqrt{2}}$ only, but also $k < 1$ (given in question) so:

$$\frac{1}{\sqrt{2}} < k < 1$$

11




The diagram shows the curve with equation $y = 9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right)$. The curve crosses the $x$-axis at the point $A$.

**(a)** Find the $x$-coordinate of $A$. [2]

at A, y=0: $9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right) = 0$

$x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}} = 0$

$x^{-\frac{1}{2}} = 4x^{-\frac{3}{2}}$

$\frac{1}{x^{\frac{1}{2}}} = \frac{4}{x^{3/2}}$

$x^{3/2} = 4x^{\frac{1}{2}}$

$x^{3/2} - 4x^{\frac{1}{2}} = 0$

$x^{\frac{1}{2}}(x-4) = 0$

$x^{\frac{1}{2}} = 0$ or $x - 4 = 0$

$x = 0$ $\quad$ $x = 4$

not correct point (from diagram)

**(b)** Find the equation of the tangent to the curve at $A$. [4]

$y = 9x^{-\frac{1}{2}} - 36x^{-\frac{3}{2}}$

$\frac{dy}{dx} = \frac{-9}{2}x^{-\frac{3}{2}} + 54x^{-\frac{5}{2}}$

sb. x=4: $\frac{-9}{2}(4)^{-\frac{3}{2}} + 54(4)^{-\frac{5}{2}}$

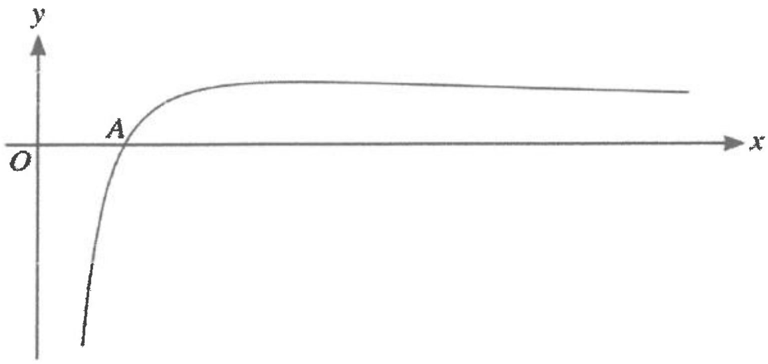
$= \frac{-9}{2} \times \frac{1}{8} + 54 \times \frac{1}{32}$

$= \frac{9}{8}$

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{9}{8}(x-4)$

$y = \frac{9}{8}x - \frac{9}{2}$




The diagram shows the curve with equation $y = 9(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}})$. The curve crosses the $x$-axis at the point $A$.

**(c)** Find the $x$-coordinate of the maximum point of the curve. [2]

$$y = 9(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}})$$
$$y = 9x^{-\frac{1}{2}} - 36x^{-\frac{3}{2}}$$
$$\frac{dy}{dx} = \frac{-9}{2}x^{-\frac{3}{2}} + 54x^{-\frac{5}{2}}$$

SP: $\frac{dy}{dx} = 0$: $\frac{-9}{2x^{3/2}} + \frac{54}{x^{5/2}} = 0$

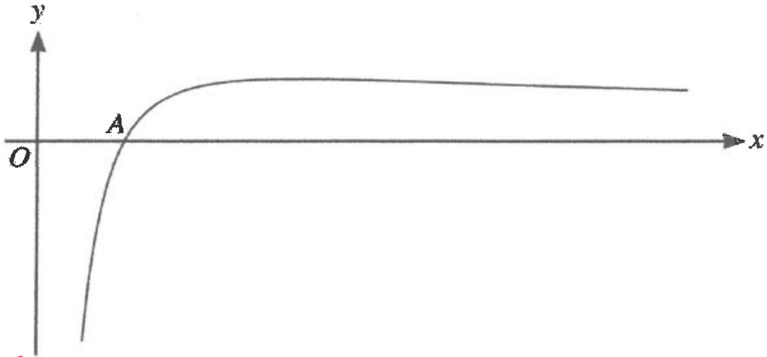
$$\frac{54}{x^{5/2}} = \frac{9}{2x^{3/2}}$$
$$108x^{3/2} = 9x^{5/2}$$
$$12x^{3/2} = x^{5/2}$$
$$12x^{3/2} - x^{5/2} = 0$$
$$x^{3/2}(12 - x) = 0$$

$x^{3/2} = 0$ or $12 - x = 0$

$x = 0$ ✗ $\quad\quad$ $\underline{x = 12}$

can see from diagram this isn't the point we want

11




from part (a) $A = (4,0)$

The diagram shows the curve with equation $y = 9(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}})$. The curve crosses the $x$-axis at the point $A$.

**(d)** Find the area of the region bounded by the curve, the $x$-axis and the line $x = 9$. [4]

$$\int_4^9 9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right) dx$$

$$9\int_4^9 \left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right) dx$$

$$9\left[2x^{\frac{1}{2}} + 8x^{-\frac{1}{2}}\right]_4^9$$

$$= 9\left[\left(2(9)^{\frac{1}{2}} + 8(9)^{-\frac{1}{2}}\right) - \left(2(4)^{\frac{1}{2}} + 8(4)^{-\frac{1}{2}}\right)\right]$$

$$= 9\left[\left(2\times 3 + 8\times\tfrac{1}{3}\right) - \left(2\times 2 + 8\times\tfrac{1}{2}\right)\right]$$

$$= 9\left[\left(6 + \tfrac{8}{3}\right) - (4 + 4)\right]$$

$$= 9\left[6 + \tfrac{8}{3} - 8\right]$$

$$= 9\left[\tfrac{2}{3}\right]$$

$$= \underline{6}$$