8 The first term of a progression is $\sin^2\theta$, where $0 < \theta < \frac{1}{2}\pi$. The second term of the progression is $\sin^2\theta\cos^2\theta$.

(a) Given that the progression is geometric, find the sum to infinity. [3]

u_1 : $a = \sin^2\theta$ ①

u_2 : $ar = \sin^2\theta\cos^2\theta$ ②

② ÷ ① : $r = \cos^2\theta$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{\sin^2\theta}{1-\cos^2\theta}$$

$= \sin^2\theta$

$$= \frac{\sin^2\theta}{\sin^2\theta}$$

$$= \underline{1}$$

7 A tool for putting fence posts into the ground is called a 'post-rammer'. The distances in millimetres that the post sinks into the ground on each impact of the post-rammer follow a geometric progression. The first three impacts cause the post to sink into the ground by 50 mm, 40 mm and 32 mm respectively.

(a) Verify that the 9th impact is the first in which the post sinks less than 10 mm into the ground. [3]

Sequence: $50, 40, 32 \ldots$

$r = \frac{40}{50} = 0.8$

$u_n = 50 \times 0.8^{n-1}$

8th impact: $u_8 = 50 \times 0.8^7$
$= 10.5$ mm

9th impact: $u_9 = 50 \times 0.8^8$
$= 8.39$ mm

$\rightarrow$ 9th impact is the first to be less than 10 mm.

(b) Find, to the nearest millimetre, the total depth of the post in the ground after 20 impacts. [2]

$$S_{20} = \frac{50(1-0.8^{20})}{1-0.8}$$

$$= 247\text{ mm}$$

(c) Find the greatest total depth in the ground which could theoretically be achieved. [2]

$$S_\infty = \frac{50}{1-0.8}$$

$$= 250\text{ mm}$$

4 A geometric progression is such that the third term is 1764 and the sum of the second and third terms is 3444.

Find the 50th term. [4]

u_2: $u_2 = ar$

Sum of u_2 and u_3:

$ar + 1764 = 3444$

$ar = 1680$ ①

u_3: $ar^2 = 1764$ ②

② ÷ ①: $r = \frac{21}{20}$

Sub. into ①: $a \times \frac{21}{20} = 1680$

$a = 1600$

u_{50}: $u_{50} = 1600 \times \left(\frac{21}{20}\right)^{49}$

$= 17474$ (to nearest integer)

2 The second and third terms of a geometric progression are 10 and 8 respectively.

Find the sum to infinity. [4]

$U_2: \quad ar^1 = 10 \quad (1)$

$U_3: \quad ar^2 = 8 \quad (2)$

$(2) \div (1): \quad r = \frac{8}{10}$

$r = 4/5$

sub. into (1):

$a \times \frac{4}{5} = 10$

$a = 12.5$

$S_\infty = \frac{12.5}{1 - 4/5}$

$= 62.5$

8 A woman's basic salary for her first year with a particular company is \$30 000 and at the end of the year she also gets a bonus of \$600.

(a) For her first year, express her bonus as a percentage of her basic salary. [1]

$\frac{600}{30\,000} \times 100 = 2\%$

At the end of each complete year, the woman's basic salary will increase by 3% and her bonus will increase by \$100.

↳ $r = 1.03$

(b) Express the bonus she will be paid at the end of her 24th year as a percentage of the basic salary paid during that year. [5]

Salary: $u_{24} = ar^{23}$

$= 30\,000 \times 1.03^{23}$

$= 59\,207.60$ STO

Bonus: $600 + 23 \times 100$

$= 2900$

$\frac{2900}{59\,207.60} \times 100 = 4.90\%$

9 (a) A geometric progression is such that the second term is equal to 24% of the sum to infinity.

Find the possible values of the common ratio. [3]

U_2 : $u_2 = ar^1$

S_∞ : $S_\infty = \frac{a}{1-r}$

$u_2 = 0.24(S_\infty)$

$ar = 0.24 \times \frac{a}{1-r}$

$ar(1-r) = 0.24a$

$ar - ar^2 = \frac{24a}{100}$

$100ar - 100ar^2 = 24a$

$100ar = 100ar^2 + 24a$

$100ar^2 + 24a - 100ar = 0$

$100r^2 + 24 - 100r = 0$

$100r^2 - 100r + 24 = 0$

$25r^2 - 25r + 6 = 0$

$r = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(25)(6)}}{2(25)}$

$r = \frac{3}{5}$ or $r = \frac{2}{5}$

6 The second term of a geometric progression is 54 and the sum to infinity of the progression is 243. The common ratio is greater than $\frac{1}{2}$.

Find the tenth term, giving your answer in exact form. [5]

u_2: $ar = 54$ ①

S_∞: $\frac{a}{1-r} = 243$

$a = 243(1-r)$ ②

①: $a = \frac{54}{r}$

sub. into ②:

$\frac{54}{r} = 243(1-r)$

$54 = 243r(1-r)$

$54 = 243r - 243r^2$

$243r^2 - 243r + 54 = 0$

$27r^2 - 27r + 6 = 0$

$9r^2 - 9r + 2 = 0$

$r = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(9)(2)}}{2(9)}$

$\underline{r = \frac{2}{3}}$ ✓ or $\underline{r = \frac{1}{3}}$ ✗

a: $a = \frac{54}{\left(\frac{2}{3}\right)}$

$= \underline{81}$

u_{10}: ar^9

$= 81 \times \left(\frac{2}{3}\right)^9$

$= \underline{\frac{512}{243}}$

2 The first, second and third terms of a geometric progression are $2p + 6$, $-2p$ and $p + 2$ respectively, where p is positive.

Find the sum to infinity of the progression. [5]

u_1: $a = 2p + 6$ ①

u_2: $ar = -2p$ ②

u_3: $ar^2 = p + 2$ ③

② ÷ ①: $r = \frac{-2p}{2p+6}$

$= \frac{-p}{p+3}$

③ ÷ ②: $r = \frac{p+2}{-2p}$

solve simultaneously:

$\frac{-p}{p+3} = \frac{p+2}{-2p}$

$2p^2 = (p+2)(p+3)$

$2p^2 = p^2 + 5p + 6$

$p^2 - 5p - 6 = 0$

$(p-6)(p+1) = 0$

$p = 6$ ✓ or $p = -1$ ✗

Sub. into ①: $a = 2(6) + 6$

$= 18$

Sub. into ②: $18 \times r = -2(6)$

$18r = -12$

$r = \frac{-2}{3}$

$S_\infty = \frac{18}{1 - \frac{-2}{3}}$

$= \frac{18}{5/3}$

$= 10.8$

5 The fifth, sixth and seventh terms of a geometric progression are $8k$, -12 and $2k$ respectively.

Given that k is negative, find the sum to infinity of the progression. [4]

u_5 : $ar^4 = 8k$ ①

u_6 : $ar^5 = -12$ ②

u_7 : $ar^6 = 2k$ ③

② ÷ ①: $r = \frac{-12}{8k} = \frac{-3}{2k}$

③ ÷ ②: $r = \frac{2k}{-12} = \frac{k}{-6}$

Solve simultaneously:

$\frac{-3}{2k} = \frac{k}{-6}$

$18 = 2k^2$

$k^2 = 9$

$k = \pm 3$

k is negative, so $k = -3$

Sub. into r :

$r = \frac{k}{-6} = \frac{-3}{-6} = \frac{1}{2}$

Sub. into ①: $a \times \left(\frac{1}{2}\right)^4 = 8(-3)$

$a \times \frac{1}{16} = -24$

$a = -384$

$S_\infty = \frac{a}{1-r} = \frac{-384}{1-\frac{1}{2}} = -768$

3 Each year the selling price of a diamond necklace increases by 5% of the price the year before. The selling price of the necklace in the year 2000 was \$36 000.

(a) Write down an expression for the selling price of the necklace n years later and hence find the selling price in 2008. [3]

2000 : 36 000

2001 : $36\,000 \times 1.05$ (1 year later)

2002 : $36\,000 \times 1.05^2$ (2 years later)

→ price = $36\,000 \times 1.05^n$

2008 : 36000×1.05^8

$= 53\,188.40$

$= \$53\,200$ (3sf)

(b) The company that makes the necklace only sells one each year. Find the total amount of money obtained in the ten-year period starting in the year 2000. [2]

2000 : $36000 = a$

$r = 1.05$

$n = 10$

$$S_{10} = \frac{36000(1 - 1.05^{10})}{1 - 1.05}$$

$= \$452\,804.13$

$= \$453\,000$ (3sf)

8 A geometric progression has first term a, common ratio r and sum to infinity S. A second geometric progression has first term a, common ratio R and sum to infinity $2S$.

(a) Show that $r = 2R - 1$. [3]

first progression: S_∞ :

$$S = \frac{a}{1-r} \quad ①$$

second progression: S_∞ :

$$2S = \frac{a}{1-R} \quad ②$$

$$① \times 2: \quad 2S = \frac{2a}{1-r}$$

Solve simultaneously with ② :

$$\frac{a}{1-R} = \frac{2a}{1-r}$$

$$(1-r)\cancel{a} = (1-R)2\cancel{a}$$

$$1-r = (1-R)2$$

$$1-r = 2-2R$$

$$-r = 1-2R$$

$$r = -1+2R$$

$$\underline{r = 2R-1}$$ QED

It is now given that the 3rd term of the first progression is equal to the 2nd term of the second progression.

(b) Express S in terms of a. [4]

U_3 of first progression:

$$u_3 = ar^2 \quad (1)$$

U_2 of second progression:

$$u_2 = aR \quad (2)$$

(1) = (2):

$$ar^2 = aR$$

$r = 2R - 1$ (from part (a)):

$$\cancel{a}(2R-1)^2 = \cancel{a}R$$

$$(2R-1)^2 = R$$

$$4R^2 - 4R + 1 = R$$

$$4R^2 - 5R + 1 = 0$$

$$(4R-1)(R-1) = 0$$

$$R = \frac{1}{4} \quad \text{or} \quad R = 1$$

Sub. into $2S = \frac{a}{1-R}$ from part (a):

$R = \frac{1}{4}$: $2S = \frac{a}{1-\frac{1}{4}}$

$$2S = \frac{a^{\times 4}}{\frac{3}{4}_{\times 4}}$$

$$2S = \frac{4a}{3}$$

$$S = \frac{2a}{3} \checkmark$$

$R = 1$: $2S = \frac{a}{1-1}$

$$= \frac{a}{0} \times$$

9 The second term of a geometric progression is 16 and the sum to infinity is 100.

(a) Find the two possible values of the first term. [4]

U_2: $ar^1 = 16$ (1)

S_∞: $\dfrac{a}{1-r} = 100$

$a = 100(1-r)$ (2)

Sub. (2) into (1):

$100(1-r) \times r = 16$

$100r - 100r^2 = 16$

$100r^2 - 100r + 16 = 0$

$25r^2 - 25r + 4 = 0$

$r = \dfrac{-(-25) \pm \sqrt{(-25)^2 - 4(25)(4)}}{2(25)}$

$r = \dfrac{4}{5}$ or $r = \dfrac{1}{5}$

Sub. into (1):

$r = \frac{4}{5}$: $a \times \left(\frac{4}{5}\right) = 16$ $\quad$ $r = \frac{1}{5}$: $a \times \left(\frac{1}{5}\right) = 16$

$a = 20$ $\quad$ $a = 80$

(b) Show that the nth term of one of the two possible geometric progressions is equal to 4^{n-2} multiplied by the nth term of the other geometric progression. [4]

progression with $a = 80$ and $r = \frac{1}{5}$:

$$u_n = 80 \times \left(\frac{1}{5}\right)^{n-1}$$

$$= 80 \times \left(\frac{1}{5}\right)^n \times \left(\frac{1}{5}\right)^{-1}$$

$$= 80 \times \left(\frac{1}{5}\right)^n \times 5$$

$$= \underline{400 \times \left(\frac{1}{5}\right)^n}$$

progression with $a = 20$ and $r = \frac{4}{5}$:

$$u_n = 20 \times \left(\frac{4}{5}\right)^{n-1}$$

$$= 20 \times \left(\frac{4}{5}\right)^n \times \left(\frac{4}{5}\right)^{-1}$$

$$= 20 \times \left(\frac{4}{5}\right)^n \times \frac{5}{4}$$

$$= 25 \times \left(\frac{4}{5}\right)^n$$

$$= 25 \times 4^n \times \left(\frac{1}{5}\right)^n$$

$$= \frac{400}{16} \times 4^n \times \left(\frac{1}{5}\right)^n$$

$$= \frac{400}{4^2} \times 4^n \times \left(\frac{1}{5}\right)^n$$

$$= 400 \times 4^{-2} \times 4^n \times \left(\frac{1}{5}\right)^n$$

$$= 400 \times 4^{n-2} \times \left(\frac{1}{5}\right)^n$$

$$= \underline{4^{n-2} \times \left(400 \times \left(\frac{1}{5}\right)^n\right)}$$

QED