4 The random variable X takes the values 1, 2, 3, 4 only. The probability that X takes the value x is $kx(5-x)$, where k is a constant.

(a) Draw up the probability distribution table for X, in terms of k. [2]

$x=1$: $k(5-1) = 4k$

$x=2$: $2k(5-2) = 6k$

$x=3$: $3k(5-3) = 6k$

$x=4$: $4k(5-4) = 4k$

x	1	2	3	4
$P(X=x)$	$4k$	$6k$	$6k$	$4k$

(b) Show that Var(X) = 1.05. [4]

$\sum P = 1$: $4k + 6k + 6k + 4k = 1$

$20k = 1$

$k = 0.05$

$E(x)$: $E(X) = 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.3 + 4 \times 0.2$

$4k = 4 \times 0.05$

$= 2.5$

$Var(x)$: $Var(X) = 1^2 \times 0.2 + 2^2 \times 0.3 + 3^2 \times 0.3 + 4^2 \times 0.2 - (E(x))^2$

$= 0.2 + 1.2 + 2.7 + 3.2 - (2.5)^2$

$= 7.3 - 6.25$

$= 1.05$ QED

1 The random variable X takes the values -2, 2 and 3. It is given that

$$P(X = x) = k(x^2 - 1),$$

where k is a constant.

(a) Draw up the probability distribution table for X, giving the probabilities as numerical fractions. [3]

$x = -2$: $k((-2)^2 - 1) = 3k$

$x = 2$: $k(2^2 - 1) = 3k$

$x = 3$: $k(3^2 - 1) = 8k$

$\Sigma P = 1$: $3k + 3k + 8k = 1$

$14k = 1$

$k = \frac{1}{14}$

x	-2	2	3
$P(X=x)$	$3/14$	$3/14$	$8/14$

(b) Find $E(X)$ and $Var(X)$. [3]

$E(x) = -2 \times \frac{3}{14} + 2 \times \frac{3}{14} + 3 \times \frac{8}{14}$

$= \frac{-3}{7} + \frac{3}{7} + \frac{12}{7}$

$= \frac{12}{7}$

$Var(x) = (-2)^2 \times \frac{3}{14} + 2^2 \times \frac{3}{14} + 3^2 \times \frac{8}{14} - (E(x))^2$

$= \frac{6}{7} + \frac{6}{7} + \frac{36}{7} - \left(\frac{12}{7}\right)^2$

$= \frac{48}{7} - \frac{144}{49} = \frac{192}{49}$

1 The probability distribution table for a random variable X is shown below.

x	-2	-1	0.5	1	2
$P(X = x)$	0.12	p	q	0.16	0.3

Given that $E(X) = 0.28$, find the value of p and the value of q. [4]

$\Sigma P = 1$:

$0.12 + p + q + 0.16 + 0.3 = 1$

$p + q + 0.58 = 1$

$p + q = 0.42$ ①

$E(x) = 0.28$:

$-2\times 0.12 + -1\times p + 0.5\times q + 1\times 0.16 + 2\times 0.3 = 0.28$

$-0.24 - p + 0.5q + 0.16 + 0.6 = 0.28$

$-p + 0.5q + 0.52 = 0.28$

$-p + 0.5q = -0.24$ ②

① + ②: $1.5q = 0.18$

$q = 0.12$

→ ①: $p + 0.12 = 0.42$

$p = 0.3$

2 A fair 6-sided die has the numbers 1, 2, 2, 3, 3, 3 on its faces. The die is rolled twice. The random variable X denotes the sum of the two numbers obtained.

(a) Draw up the probability distribution table for X. [3]

	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	5	5	5
2	3	4	4	5	5	5
3	4	5	5	6	6	6
3	4	5	5	6	6	6
3	4	5	5	6	6	6

x	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

(b) Find E(X) and Var(X). [3]

$$E(X) = 2\times\frac{1}{36} + 3\times\frac{4}{36} + 4\times\frac{10}{36} + 5\times\frac{12}{36} + 6\times\frac{9}{36}$$

$$= \frac{2}{36} + \frac{12}{36} + \frac{40}{36} + \frac{60}{36} + \frac{54}{36}$$

$$= \underline{\frac{14}{3}}$$

$$Var(X) = 2^2\times\frac{1}{36} + 3^2\times\frac{4}{36} + 4^2\times\frac{10}{36} + 5^2\times\frac{12}{36} + 6^2\times\frac{9}{36} - (E(X))^2$$

$$= \frac{4}{36} + \frac{36}{36} + \frac{160}{36} + \frac{300}{36} + \frac{324}{36} - \left(\frac{14}{3}\right)^2$$

$$= \frac{206}{9} - \frac{196}{9} = \underline{\frac{10}{9}}$$

3 The random variable X takes the values 1, 2, 3, 4. It is given that $P(X = x) = kx(x + a)$, where k and a are constants.

(a) Given that $P(X = 4) = 3P(X = 2)$, find the value of a and the value of k. [4]

$x=1$: $k(1+a) = k + ak$

$x=2$: $2k(2+a) = 4k + 2ak$

$x=3$: $3k(3+a) = 9k + 3ak$

$x=4$: $4k(4+a) = 16k + 4ak$

$P(X=4) = 3 \times P(X=2)$:

$$16k + 4ak = 3(4k + 2ak)$$
$$16k + 4ak = 12k + 6ak$$
$$4k = 2ak$$
$$4 = 2a$$
$$\underline{a = 2}$$

→ $x=1$: $k + 2k = 3k$

$x=2$: $4k + 4k = 8k$

$x=3$: $9k + 6k = 15k$

$x=4$: $16k + 8k = 24k$

$\sum P = 1$: $3k + 8k + 15k + 24k = 1$

$$50k = 1$$
$$\underline{k = 0.02}$$

(b) Draw up the probability distribution table for X, giving the probabilities as numerical fractions. [1]

x	1	2	3	4
$P(X=x)$	$\frac{3}{50}$	$\frac{8}{50}$	$\frac{15}{50}$	$\frac{24}{50}$

(c) Given that $E(X) = 3.2$, find $Var(X)$. [2]

$Var(X) = 1^2 \times \frac{3}{50} + 2^2 \times \frac{8}{50} + 3^2 \times \frac{15}{50} + 4^2 \times \frac{24}{50} - (E(X))^2$

$= 0.06 + 0.64 + 2.7 + 7.68 - 3.2^2$

$= 11.08 - 10.24$

$= \underline{0.84}$

4 A fair four-sided spinner has edges numbered 1, 2, 2, 3. A fair three-sided spinner has edges numbered −2, −1, 1. Each spinner is spun and the number on the edge on which it comes to rest is noted. The random variable X is the sum of the two numbers that have been noted.

(a) Draw up the probability distribution table for X. [3]

	1	2	2	3
-2	-1	0	0	1
-1	0	1	1	2
1	2	3	3	4

x	-1	0	1	2	3	4
$P(X=x)$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

(b) Find Var(X). [3]

$$\text{Var}(X) = (-1)^2 \times \frac{1}{12} + 0^2 \times \frac{3}{12} + 1^2 \times \frac{3}{12} + 2^2 \times \frac{2}{12} + 3^2 \times \frac{2}{12} + 4^2 \times \frac{1}{12} - (E(X))^2$$

$$= \frac{1}{12} + 0 + \frac{3}{12} + \frac{8}{12} + \frac{18}{12} + \frac{16}{12} - (E(X))^2$$

$$= \frac{46}{12} - (E(X))^2$$

$$E(X) = -1 \times \frac{1}{12} + 0 \times \frac{3}{12} + 1 \times \frac{3}{12} + 2 \times \frac{2}{12} + 3 \times \frac{2}{12} + 4 \times \frac{1}{12}$$

$$= \frac{-1}{12} + 0 + \frac{3}{12} + \frac{4}{12} + \frac{6}{12} + \frac{4}{12}$$

$$= \frac{16}{12}$$

$$\text{Var}(X) = \frac{46}{12} - \left(\frac{16}{12}\right)^2 = \underline{\frac{37}{18}}$$

4 A fair spinner has edges numbered 0, 1, 2, 2. Another fair spinner has edges numbered −1, 0, 1. Each spinner is spun. The number on the edge on which a spinner comes to rest is noted. The random variable X is the sum of the numbers for the two spinners.

(a) Draw up the probability distribution table for X. [3]

	0	1	2	2
-1	-1	0	1	1
0	0	1	2	2
1	1	2	3	3

x	-1	0	1	2	3
$P(X=x)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{3}{12}$	$\frac{2}{12}$

(b) Find Var(X). [3]

$$E(X) = -1\times\frac{1}{12} + 0\times\frac{2}{12} + 1\times\frac{4}{12} + 2\times\frac{3}{12} \times 3\times\frac{2}{12}$$

$$= \frac{-1}{12} + 0 + \frac{4}{12} + \frac{6}{12} + \frac{6}{12}$$

$$= \frac{15}{12}$$

$$Var(X) = (-1)^2\times\frac{1}{12} + 0^2\times\frac{2}{12} + 1^2\times\frac{4}{12} + 2^2\times\frac{3}{12} + 3^2\times\frac{2}{12} - (E(X))^2$$

$$= \frac{1}{12} + 0 + \frac{4}{12} + \frac{12}{12} + \frac{18}{12} - \left(\frac{15}{12}\right)^2$$

$$= \frac{35}{12} - \frac{225}{144} = \underline{\underline{\frac{65}{48}}}$$

3 A bag contains 5 yellow and 4 green marbles. Three marbles are selected at random from the bag, without replacement.

(a) Show that the probability that exactly one of the marbles is yellow is $\frac{5}{14}$. [3]

one yellow from 5 $\rightarrow$ $\dfrac{{}^{5}C_{1} \times {}^{4}C_{2}}{{}^{9}C_{3}}$ $\leftarrow$ two greens from 4

$\uparrow$ total number of possibilities

$= \dfrac{5}{14}$ QED

The random variable X is the number of yellow marbles selected.

(b) Draw up the probability distribution table for X. [3]

0Y: $\dfrac{{}^{5}C_{0} \times {}^{4}C_{3}}{{}^{9}C_{3}} = \dfrac{1}{21}$

1Y: $\frac{5}{14}$ (part (a))

2Y: $\dfrac{{}^{5}C_{2} \times {}^{4}C_{1}}{{}^{9}C_{3}} = \dfrac{10}{21}$

3Y: $\dfrac{{}^{5}C_{3} \times {}^{4}C_{0}}{{}^{9}C_{3}} = \dfrac{5}{42}$

x	0	1	2	3
$P(X=x)$	$\frac{1}{21}$	$\frac{5}{14}$	$\frac{10}{21}$	$\frac{5}{42}$

(c) Find E(X). [1]

$E(x) = 0 \times \frac{1}{21} + 1 \times \frac{5}{14} + 2 \times \frac{10}{21} + 3 \times \frac{5}{42}$

$= 0 + \frac{5}{14} + \frac{20}{21} + \frac{15}{42}$

$= \frac{5}{3}$

5 Eric has three coins. One of the coins is fair. The other two coins are each biased so that the probability of obtaining a head on any throw is $\frac{1}{4}$, independently of all other throws. Eric throws all three coins at the same time.

Events A and B are defined as follows.

A: all three coins show the same result

B: at least one of the biased coins shows a head

(a) Show that $P(B) = \frac{7}{16}$. [2]

F	B_1	B_2	
H	H	H	$\frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{32}$
H	H	T	$\frac{1}{2} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{32}$
H	T	H	$\frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{32}$
T	H	H	$\frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{32}$
T	T	H	$\frac{1}{2} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{32}$
T	H	T	$\frac{1}{2} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{32}$

$+ = \frac{14}{32}$

$= \frac{7}{16}$ QED

(b) Find $P(A \mid B)$. [2]

$P(A \cap B) = P(A \mid B) \times P(B)$

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$P(A \cap B) = HHH$

$= \frac{1}{32}$

$P(B) = \frac{7}{16}$ (part (a))

$P(A \mid B) = \dfrac{\frac{1}{32}}{\frac{7}{16}}$

$= \frac{1}{14}$

The random variable X is the number of heads obtained when Eric throws the three coins.

(c) Draw up the probability distribution table for X. [3]

F B$_1$ B$_2$

3H: H H H $\frac{1}{32}$

2H: H H T $\frac{3}{32}$

H T H $\frac{3}{32}$

T H H $\frac{1}{32}$

1H: T T H $\frac{3}{32}$

T H T $\frac{3}{32}$

H T T $\frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{32}$

0H: T T T $\frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{32}$

x	0	1	2	3
$P(X=x)$	$\frac{9}{32}$	$\frac{15}{32}$	$\frac{7}{32}$	$\frac{1}{32}$

7 Sharma knows that she has 3 tins of carrots, 2 tins of peas and 2 tins of sweetcorn in her cupboard. All the tins are the same shape and size, but the labels have all been removed, so Sharma does not know what each tin contains.

Sharma wants carrots for her meal, and she starts opening the tins one at a time, chosen randomly, until she opens a tin of carrots. The random variable X is the number of tins that she needs to open.

(a) Show that $P(X = 3) = \frac{6}{35}$. [2]

3 tins of carrots, 4 other tins:

$$\frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} = \frac{6}{35} \quad \text{QED}$$

not carrot not carrot carrot

(b) Draw up the probability distribution table for X. [4]

$P(X=1) = \frac{3}{7} \left(= \frac{15}{35}\right)$

$P(X=2) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7} \left(= \frac{10}{35}\right)$

$P(X=3) = \frac{6}{35}$

$P(X=4) = \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{3}{4} = \frac{3}{35}$

$P(X=5) = \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} = \frac{1}{35}$

x	1	2	3	4	5
$P(X=x)$	$\frac{15}{35}$	$\frac{10}{35}$	$\frac{6}{35}$	$\frac{3}{35}$	$\frac{1}{35}$

(c) Find Var(X). [3]

$E(X) = 1 \times \frac{15}{35} + 2 \times \frac{10}{35} + 3 \times \frac{6}{35} + 4 \times \frac{3}{35} + 5 \times \frac{1}{35}$

$= \frac{15}{35} + \frac{20}{35} + \frac{18}{35} + \frac{12}{35} + \frac{5}{35}$

$= \frac{70}{35} = \underline{2}$

$Var(x) = 1^2 \times \frac{15}{35} + 2^2 \times \frac{10}{35} + 3^2 \times \frac{6}{35} + 4^2 \times \frac{3}{35} + 5^2 \times \frac{1}{35} - (E(x))^2$

$= \frac{15}{35} + \frac{40}{35} + \frac{54}{35} + \frac{48}{35} + \frac{25}{35} - (2)^2$

$= \frac{182}{35} - 4$

$= \underline{\frac{6}{5}}$

5 A red spinner has four sides labelled 1, 2, 3, 4. When the spinner is spun, the score is the number on the side on which it lands. The random variable X denotes this score. The probability distribution table for X is given below.

x	1	2	3	4
$P(X = x)$	0.28	p	$2p$	$3p$

(a) Show that $p = 0.12$. [1]

$\Sigma P = 1$: $0.28 + p + 2p + 3p = 1$

$0.28 + 6p = 1$

$6p = 0.72$

$p = 0.12$ QED

A fair blue spinner and a fair green spinner each have four sides labelled 1, 2, 3, 4. All three spinners (red, blue and green) are spun at the same time.

(b) Find the probability that the sum of the three scores is 4 or less. [3]

R	B	G	Probability
1	1	1	$0.28 \times \frac{1}{4} \times \frac{1}{4} = 0.0175$
1	1	2	$0.28 \times \frac{1}{4} \times \frac{1}{4} = 0.0175$
1	2	1	$0.28 \times \frac{1}{4} \times \frac{1}{4} = 0.0175$
2	1	1	$0.12 \times \frac{1}{4} \times \frac{1}{4} = 0.0075$

$0.0175 + 0.0175 + 0.0175 + 0.0075 = 0.06$

(c) Find the probability that the product of the three scores is 4 or less given that X is odd. [5]

$P(\text{product} \leq 4 \cap X \text{ is odd}) = P(\text{product} \leq 4 \mid X \text{ is odd}) \times P(X \text{ is odd})$

$$P(\text{product} \leq 4 \mid X \text{ is odd}) = \frac{P(\text{product} \leq 4 \cap X \text{ is odd})}{P(X \text{ is odd})}$$

$P(X \text{ is odd})$ (using probability distribution table):

$P(X \text{ is odd}) = 0.28 + 0.24$
$= \underline{0.52}$

$P(\text{product} \leq 4 \cap X \text{ is odd})$:

R	B	G	Probability
1	1	1	
1	1	2	
1	1	3	
1	1	4	$0.28 \times \frac{1}{4} \times \frac{1}{4} \times 8 = 0.14$
1	2	1	
1	3	1	
1	4	1	
1	2	2	
3	1	1	$0.24 \times \frac{1}{4} \times \frac{1}{4} = 0.015$

$P(\text{product} \leq 4 \cap X \text{ is odd}) = 0.14 + 0.015$
$= \underline{0.155}$

$$P(\text{product} \leq 4 \mid X \text{ is odd}) = \frac{0.155}{0.52}$$
$= \underline{0.298}$