1 Solve the equation $3x + 2 = \frac{2}{x-1}$. [3]

$$3x + 2 = \frac{2}{x-1}$$

$\times(x-1)$ $\times(x-1)$

$$(3x+2)(x-1) = 2$$

$$3x^2 - 3x + 2x - 2 = 2$$

$$\overset{a}{3x^2} \overset{b}{-x} \overset{c}{-4} = 0$$

factorise: $ac = -12 \rightarrow -4, 3$

$$3x^2 + 3x - 4x - 4 = 0$$

$$3x(x+1) - 4(x+1) = 0$$

$$(3x-4)(x+1) = 0$$

$3x - 4 = 0$ or $x + 1 = 0$

$3x = 4$ $\qquad$ $\underline{\underline{x = -1}}$

$\underline{\underline{x = \frac{4}{3}}}$

2 The equation of a curve is such that $\frac{dy}{dx} = 12(\frac{1}{2}x - 1)^{-4}$. It is given that the curve passes through the point $P(6, 4)$.

(a) Find the equation of the tangent to the curve at P. [2]

gradient at (6,4): $12(\frac{1}{2}(6) - 1)^{-4}$

$= 12(3-1)^{-4}$

$= 12 \times 2^{-4}$

$= \frac{3}{4}$

eqn of tangent:

$y - 4 = \frac{3}{4}(x - 6)$

$y - 4 = \frac{3}{4}x - \frac{9}{2}$

$y = \frac{3}{4}x - \frac{1}{2}$

(b) Find the equation of the curve. [4]

$y = \int 12(\frac{1}{2}x - 1)^{-4} dx$

$= \frac{-1}{3} \times 2 \times 12(\frac{1}{2}x - 1)^{-3} + C$

$= -8(\frac{1}{2}x - 1)^{-3} + C$

sub. (6,4):

$4 = -8(\frac{1}{2} \times 6 - 1)^{-3} + C$

$4 = -8(2)^{-3} + C$

$4 = -8 \times \frac{1}{8} + C$

$4 = -1 + C$

$C = 5$

$\rightarrow y = -8(\frac{1}{2}x - 1)^{-3} + 5$

3 A curve has equation $y = ax^{\frac{1}{2}} - 2x$, where $x > 0$ and a is a constant. The curve has a stationary point at the point P, which has x-coordinate 9.

Find the y-coordinate of P. [5]

$y = ax^{\frac{1}{2}} - 2x$

$\frac{dy}{dx} = \frac{1}{2}ax^{-\frac{1}{2}} - 2$

SP: $\frac{dy}{dx} = 0$: $\frac{1}{2}ax^{-\frac{1}{2}} - 2 = 0$

$\frac{1}{2}ax^{-\frac{1}{2}} = 2$

$ax^{-\frac{1}{2}} = 4$

$\frac{a}{\sqrt{x}} = 4$

sub. $x = 9$: $\frac{a}{\sqrt{9}} = 4$

$\frac{a}{3} = 4$

$a = 12$

$\rightarrow$ $y = 12x^{\frac{1}{2}} - 2x$

sub $x = 9$:

$y = 12(9)^{\frac{1}{2}} - 2(9)$

$= 12 \times 3 - 18$

$= 36 - 18$

$= 18$

4 The coefficient of x^2 in the expansion of $\left(1+\frac{2}{p}x\right)^5+(1+px)^6$ is 70.

Find the possible values of the constant p. [6]

$\left(1+\frac{2}{p}x\right)^5 = 1^5 + 5\times1^4\times\left(\frac{2}{p}x\right) + \underline{10\times1^3\times\left(\frac{2}{p}x\right)^2}$

$\rightarrow 10\times\frac{4}{p^2}x^2$

$= \frac{40}{p^2}x^2$

$(1+px)^6 = 1^6 + 6\times1^5\times(px) + \underline{15\times1^4\times(px)^2}$

$\rightarrow 15p^2x^2$

coeff of x^2: $\frac{40}{p^2}x^2 + 15p^2x^2$

coeff $= \frac{40}{p^2} + 15p^2$

$\rightarrow \frac{40}{p^2} + 15p^2 = 70$

$\times p^2 \quad \times p^2 \quad \times p^2$

$40 + 15p^4 = 70p^2$

$15p^4 - 70p^2 + 40 = 0$

$3p^4 - 14p^2 + 8 = 0$

Let $Y = p^2$:

$3Y^2 - 14Y + 8 = 0$

$(3Y-2)(Y-4) = 0$

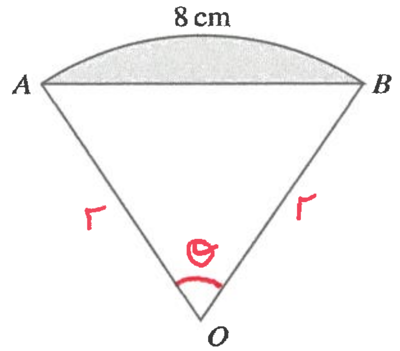
$3Y-2=0$ or $Y-4=0$

$Y=\frac{2}{3}$ or $Y=4$

$\rightarrow p^2 = \frac{2}{3}$ or $p^2 = 4$

$\underline{\underline{p = \pm\sqrt{\frac{2}{3}}}}$ or $\underline{\underline{p = \pm 2}}$

5



The diagram shows a sector OAB of a circle with centre O. The length of the arc AB is 8 cm. It is given that the perimeter of the sector is 20 cm.

(a) Find the perimeter of the shaded segment. [4]

Perimeter of sector OAB:

perimeter $= r + r + 8$

$= 2r + 8$

$2r + 8 = 20$

$2r = 12$

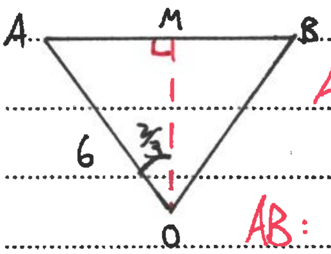
$r = 6$

Arc AB:

arc length $= r\theta$

$8 = 6\theta$

$\theta = \frac{4}{3}$



AM: $\sin\left(\frac{2}{3}\right) = \frac{AM}{6}$

$AM = 6\sin\left(\frac{2}{3}\right)$

AB: $AB = 2 \times 6\sin\left(\frac{2}{3}\right)$

$= 12\sin\left(\frac{2}{3}\right)$

Perimeter $= 8 + 12\sin\left(\frac{2}{3}\right)$

$= 15.4$ cm

(b) Find the area of the shaded segment. [2]

Area of sector OAB:

$$\text{Area} = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2} \times 6^2 \times \frac{4}{3}$$

$$= 24$$

Area of triangle OAB:

$$\text{Area} = \frac{1}{2} \times 6 \times 6 \times \sin\left(\frac{4}{3}\right)$$

$$= 17.5_{\text{STO}}$$

Area of shaded segment:

$$\text{Area} = 24 - 17.5$$

$$= \underline{6.51}\ \text{cm}^2$$

6 **(a)** Show that the equation

$$\frac{1}{\sin\theta + \cos\theta} + \frac{1}{\sin\theta - \cos\theta} = 1$$

may be expressed in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are constants to be found. [3]

$$\frac{1}{\sin\theta + \cos\theta} \frac{\times(\sin\theta - \cos\theta)}{\times(\sin\theta - \cos\theta)} + \frac{1}{\sin\theta - \cos\theta} \frac{\times(\sin\theta + \cos\theta)}{\times(\sin\theta + \cos\theta)} = 1$$

$$\frac{\sin\theta - \cos\theta}{\sin^2\theta - \cos^2\theta} + \frac{\sin\theta + \cos\theta}{\sin^2\theta - \cos^2\theta} = 1$$

$$\frac{\sin\theta - \cancel{\cos\theta} + \sin\theta + \cancel{\cos\theta}}{\sin^2\theta - \cos^2\theta} = 1$$

$$\frac{2\sin\theta}{\sin^2\theta - \cos^2\theta} = 1$$

$$2\sin\theta = \sin^2\theta - \cos^2\theta$$

$1 - \sin^2\theta$

$$2\sin\theta = \sin^2\theta - (1 - \sin^2\theta)$$

$$2\sin\theta = \sin^2\theta - 1 + \sin^2\theta$$

$$2\sin\theta = 2\sin^2\theta - 1$$

$$2\sin^2\theta - 2\sin\theta - 1 = 0$$ QED

(b) Hence solve the equation $\dfrac{1}{\sin\theta + \cos\theta} + \dfrac{1}{\sin\theta - \cos\theta} = 1$ for $0° \leqslant \theta \leqslant 360°$. [3]

from (a): $2\sin^2\theta - 2\sin\theta - 1 = 0$

Let $Y = \sin\theta$:

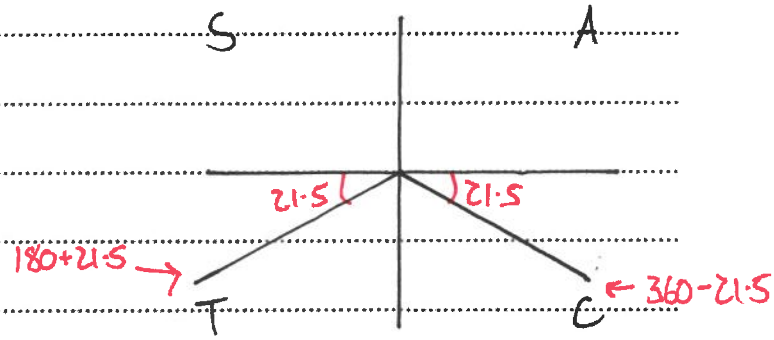
$$2Y^2 - 2Y - 1 = 0$$

$$Y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$Y = \frac{1+\sqrt{3}}{2} \quad \text{or} \quad Y = \frac{1-\sqrt{3}}{2}$$

$$\sin\theta = \frac{1+\sqrt{3}}{2} \qquad \sin\theta = \frac{1-\sqrt{3}}{2}$$

no solutions $\qquad \theta = -21.5°$



$$\theta = 201.5°, 338.5°$$

7 A tool for putting fence posts into the ground is called a 'post-rammer'. The distances in millimetres that the post sinks into the ground on each impact of the post-rammer follow a geometric progression. The first three impacts cause the post to sink into the ground by 50 mm, 40 mm and 32 mm respectively.

(a) Verify that the 9th impact is the first in which the post sinks less than 10 mm into the ground. [3]

Sequence: $50, 40, 32 \ldots$

$r = \frac{40}{50} = \underline{0.8}$

$u_n = 50 \times 0.8^{n-1}$

8th impact: $u_8 = 50 \times 0.8^7$
$= 10.5\text{ mm}$

9th impact: $u_9 = 50 \times 0.8^8$
$= 8.39\text{ mm}$

$\rightarrow$ 9th impact is the first to be less than 10 mm.

(b) Find, to the nearest millimetre, the total depth of the post in the ground after 20 impacts. [2]

$$S_{20} = \frac{50(1-0.8^{20})}{1-0.8}$$

$$= 247\text{mm}$$

(c) Find the greatest total depth in the ground which could theoretically be achieved. [2]

$$S_{\infty} = \frac{50}{1-0.8}$$

$$= 250\text{mm}$$

8 The function f is defined by $f(x) = 2 - \frac{3}{4x - p}$ for $x > \frac{p}{4}$, where p is a constant.

(a) Find $f'(x)$ and hence determine whether f is an increasing function, a decreasing function or neither. [3]

$f(x) = 2 - 3(4x-p)^{-1}$

$f'(x) = 3 \times 4(4x-p)^{-2}$

$= 12(4x-p)^{-2}$

$= \frac{12}{\underbrace{(4x-p)^2}_{\text{always positive}}}$ positive

so $f'(x) > 0$

therefore increasing function

8 The function f is defined by $f(x) = 2 - \frac{3}{4x - p}$ for $x > \frac{p}{4}$, where p is a constant.

(b) Express $f^{-1}(x)$ in the form $\frac{p}{a} - \frac{b}{cx - d}$, where a, b, c and d are integers. [4]

$$y = 2 - \frac{3}{4x - p}$$

$$y - 2 = -\frac{3}{4x - p}$$

$$2 - y = \frac{3}{4x - p}$$

$$(2-y)(4x-p) = 3$$

$$8x - 2p - 4xy + py = 3$$

$$8x - 4xy = 3 + 2p - py$$

$$x(8 - 4y) = 3 + 2p - py$$

$$x = \frac{3 + 2p - py}{8 - 4y}$$ factorise

$$= \frac{3 + p(2-y)}{4(2-y)}$$

$$= \frac{3}{4(2-y)} + \frac{p\cancel{(2-y)}}{4\cancel{(2-y)}}$$

$$= \frac{p}{4} + \frac{3}{4(2-y)}$$

$$f^{-1}(x) = \frac{p}{4} + \frac{3}{8 - 4x}$$

(c) Hence state the value of p for which $f^{-1}(x) \equiv f(x)$. [1]

$$\frac{p}{4} + \frac{3}{8 - 4x} = 2 - \frac{3}{4x - p}$$

$$\frac{p}{4} + \frac{3}{8 - 4x} = 2 + \frac{3}{p - 4x}$$

if $p = 8$, these are the same

Not enough room or marks to do anything more complicated

9709/11/O/N/22

9 Functions f and g are both defined for $x \in \mathbb{R}$ and are given by

$$f(x) = x^2 - 4x + 9,$$
$$g(x) = 2x^2 + 4x + 12.$$

(a) Express f(x) in the form $(x - a)^2 + b$. [1]

$x^2 - 4x + 9 = (x-2)^2 - 4 + 9$

$= (x-2)^2 + 5$

(b) Express g(x) in the form $2[(x + c)^2 + d]$. [2]

$2x^2 + 4x + 12 = 2[x^2 + 2x + 6]$

$= 2[(x+1)^2 - 1 + 6]$

$= 2[(x+1)^2 + 5]$

(c) Express $g(x)$ in the form $kf(x+h)$, where k and h are integers. [1]

$(x-2)^2+5 \rightarrow 2[(x+1)^2+5]$

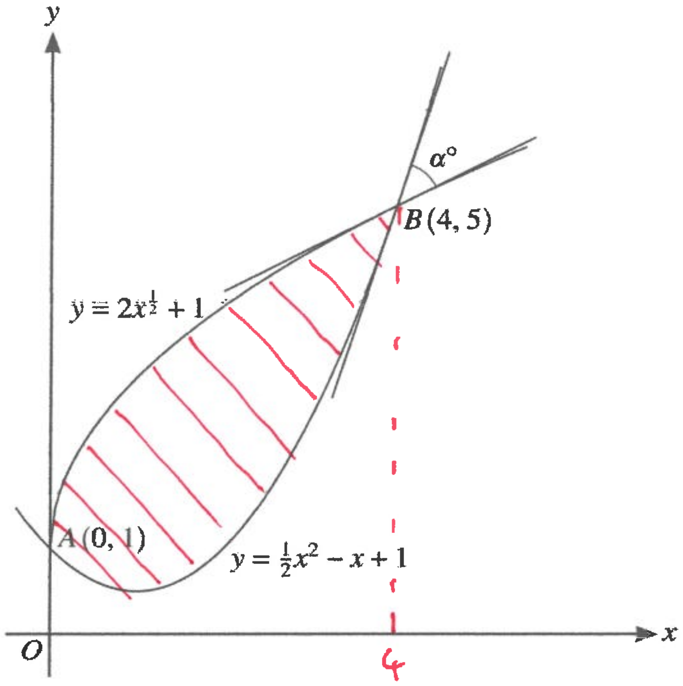
$f(x) \rightarrow 2f(x+3)$

(d) Describe fully the two transformations that have been combined to transform the graph of $y = f(x)$ to the graph of $y = g(x)$. [4]

① Translation by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

② Stretch parallel to y-axis, scale factor 2.

10



Curves with equations $y = 2x^{\frac{1}{2}} + 1$ and $y = \frac{1}{2}x^2 - x + 1$ intersect at $A(0, 1)$ and $B(4, 5)$, as shown in the diagram.

(a) Find the area of the region between the two curves. [5]

$$\int_0^4 \left[\left(2x^{\frac{1}{2}} + 1\right) - \left(\frac{1}{2}x^2 - x + 1\right)\right] dx$$

$$\int_0^4 \left(2x^{\frac{1}{2}} - \frac{1}{2}x^2 + x\right) dx$$

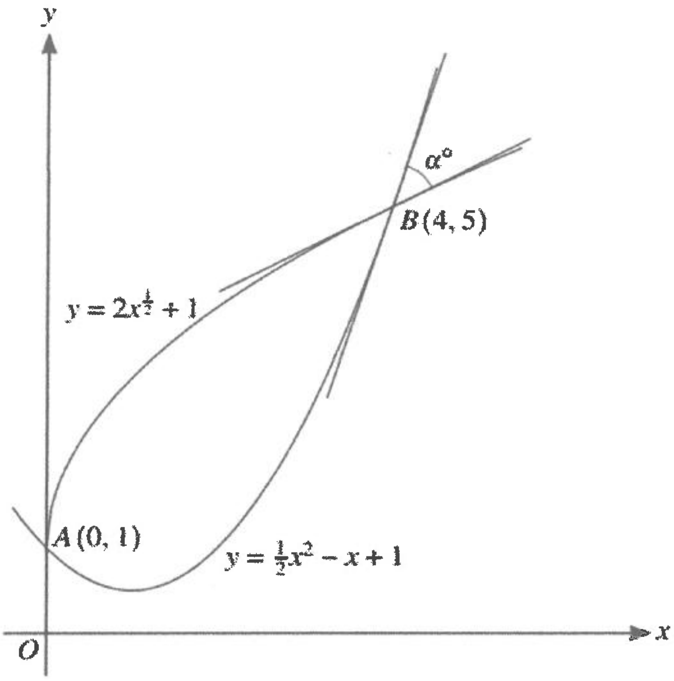
$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{1}{6}x^3 + \frac{1}{2}x^2\right]_0^4$$

$$= \left[\frac{4}{3}(4)^{\frac{3}{2}} - \frac{1}{6}(4)^3 + \frac{1}{2}(4)^2\right] - [0]$$

$$= \frac{4}{3} \times 8 - \frac{1}{6} \times 64 + \frac{1}{2} \times 16$$

$$= \frac{32}{3} - \frac{32}{3} + 8$$

$$= \underline{8}$$



Curves with equations $y = 2x^{\frac{1}{2}} + 1$ and $y = \frac{1}{2}x^2 - x + 1$ intersect at $A\,(0,\ 1)$ and $B\,(4,\ 5)$, as shown in the diagram.

The acute angle between the two tangents at B is denoted by $\alpha°$, and the scales on the axes are the same.

(b) Find α. **[5]**

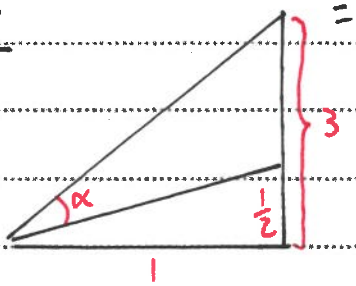
①: $y = 2x^{\frac{1}{2}} + 1$ ②: $y = \frac{1}{2}x^2 - x + 1$

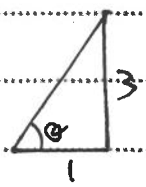
$\frac{dy}{dx} = x^{-\frac{1}{2}}$ $\frac{dy}{dx} = x - 1$

$x = 4$: $x = 4$:

①: $m = 4^{-\frac{1}{2}}$ ②: $m = 4 - 1$

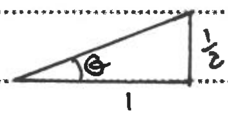
$= \frac{1}{2}$ $= 3$





$\tan\theta = \frac{3}{1}$

$\theta = 71.6°$ STO



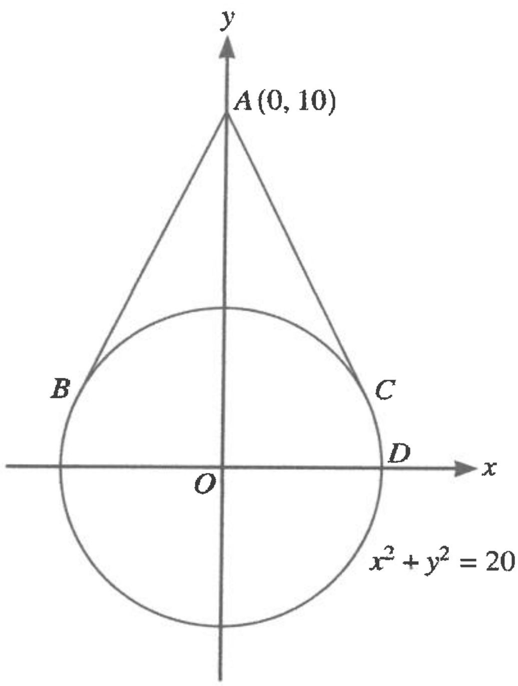
$\tan\theta = \frac{\frac{1}{2}}{1}$

$\theta = 26.6°$ STO

$\alpha = 71.6 - 26.6$

$= \underline{45°}$

11



The diagram shows the circle with equation $x^2 + y^2 = 20$. Tangents touching the circle at points B and C pass through the point A (0, 10).

(a) By letting the equation of a tangent be $y = mx + 10$, find the two possible values of m. [4]

Sub. $y = mx + 10$ into eqⁿ of circle:

$$x^2 + (mx + 10)^2 = 20$$

$$x^2 + m^2x^2 + 20mx + 100 = 20$$

$$x^2 + m^2x^2 + 20mx + 80 = 0$$

$$\underset{a}{(1 + m^2)}x^2 + \underset{b}{20}mx + \underset{c}{80} = 0$$

line is tangent, so $b^2 - 4ac = 0$:

$$(20m)^2 - 4(1 + m^2)(80) = 0$$

$$400m^2 - 320 - 320m^2 = 0$$

$$80m^2 - 320 = 0$$

$$80m^2 = 320$$

$$m^2 = 4$$

$$\underline{m = \pm 2}$$

(b) Find the coordinates of B and C. [3]

When $m=2$, $y=2x+10$. Sub into circle:

$x^2+(2x+10)^2=20$

$x^2+4x^2+40x+100=20$

$5x^2+40x+80=0$

$x^2+8x+16=0$

$(x+4)^2=0$

$x=-4$

$y=2(-4)+10$

$=-8+10$

$=2$

$(-4,2)$ ← B

When $m=-2$, $y=-2x+10$. Sub into circle:

$x^2+(-2x+10)^2=20$

$x^2+4x^2-40x+100=20$

$5x^2-40x+80=0$

$x^2-8x+16=0$

$(x-4)^2=0$

$x=4$

$y=-2(4)+10$

$=-8+10$

$=2$

$(4,2)$ ← C

(can also find second point using symmetry)

The point D is where the circle crosses the positive x-axis.

(c) Find angle BDC in degrees. [3]

at D, $y=0$: $x^2+0^2=20$

$x=\sqrt{20}$

length BC: $BC=8$

length CD: $CD=\sqrt{(\sqrt{20}-4)^2+(0-2)^2}$

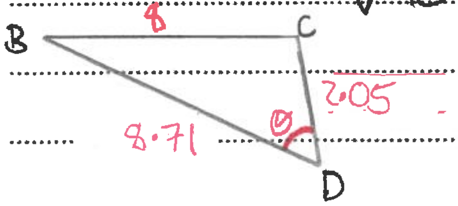
$=\sqrt{20-8\sqrt{20}+16+4}$

$=\sqrt{40-8\sqrt{20}}=2.05$ $\boxed{\text{STO}}$

length BD: $BD=\sqrt{(\sqrt{20}--4)^2+(0-2)^2}$

$=\sqrt{20+8\sqrt{20}+16+4}$

$=\sqrt{40+8\sqrt{20}}=8.71$ $\boxed{\text{STO}}$



Cosine rule:

$8^2=(2.05)^2+(8.71)^2-2(2.05)(8.71)\cos\theta$

$\cos\theta=\dfrac{64-(2.05)^2-(8.71)^2}{-2(2.05)(8.71)}$

$\theta=63.4°$